

Multidimensional Intensive Steel Quenching and Wave Power Models for Cylindrical Sample

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Abstract: - In this paper we develop mathematical models for 3-D, 2-D and one-dimensional hyperbolic heat equations (wave equation or telegraph equation) with inner source power and construct their analytical solutions for the determination of the initial heat flux for cylindrical sample. In some cases we give expression of wave energy. Some solutions of time inverse problems are obtained in the form of first kind Fredholm integral equation, but others has been obtained in closed analytical form as series. We viewed both direct and inverse problems at the time.

Key-Words: - Hyperbolic Equation, Ocean Energy, Steel Quenching, Green Function, Exact Solution, Inverse Problem, Fredholm integral equation, Series.

1 Introduction

Contrary to traditional method the intensive quenching process uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions [1]-[6], [33]. Traditionally for the mathematical description of the intensive quenching process, classical heat conduction equation is used. We have proposed to use hyperbolic heat equation [10]-[24], [40]-[42] for more realistic description of the intensive quenching (IQ) process (especially for the initial stage of the process). Models of systematic hyperbolic heat equation, their mathematical research and solutions are discussed in monograph [28].

The idea of the usage of hyperbolic heat equation can be easily transferred to completely different sector of application - to the generation of electricity in sea or ocean by usage of wave energy [7]-[9], [29] and [30]. It is important to note, that Ekergard and his co-authors [29] examine the development of the system in time, describing the equipment with ordinary differential equation. Here we describe the equipment in development of both - in time as well as in spatial arrangement of equipment using the multi-dimensional hyperbolic heat equation. Wave power plant has to work for long time period in moving environment – waves, see [30]. Therefore it is important to examine not only the development of equipment in time, but also the movement of its different components [20]-[24]. Wave energy generator models can be viewed both Cartesian

coordinate and cylindrical co-ordinates. In papers [11]-[14], [20]-[27] we investigate the rectangular models. Generators of cylindrical form with fin we investigate in papers [10], [17] and [18]. For three, two and one dimensional cylinder we dedicate this paper.

In our previous papers we have constructed various one and two dimensional analytical exact and approximate [10]-[16], [19]-[24] solutions for IQ processes. Here are both - approximate (on the basis of conservative averaging method, see [10], [19], [24], [25], [31], [32] and exact (on the basis of Green function method, see [11]-[16], [21]-[23]). We consider three-dimensional, two-dimensional and one-dimensional statements for non-homogeneous equation with non-homogeneous boundary conditions. Such statements allow constructing mathematical models for wave power plants in connection with other equipment, for example, with wind power. Boundary conditions could be different types, thus they allow us to use Green function method.

In recent years, we have been able to generalize the Green's function method to areas, which consist of several canonical connected sub-areas, and thus we have obtained the exact solutions for the L-, T- and Π -type areas [10], [11], [21], [24] - [26]. We have constructed of two cylinders [17], [18] and two-layer sphere [15], [19]. For the cylinder with fin the solution was obtained for stationary case and hyperbolic heat transfer equation.

2 Mathematical Formulation of 3-D Problem for IQP or Wave Power

Already in the introduction we noted that Professor M. Leijon, see [29] examined the development of system in time. Here we offer to consider the description of system in time and space. For this purpose instead of the ordinary differential equation, we consider the following partial differential equation:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2} \right] - CU + F(r, \varphi, z, t), r \in [0, R], \varphi \in [0, 2\pi], \quad (1)$$

$$z \in [0, l], t \in [0, T], C \geq 0, a_\tau^2 = \frac{a^2}{\tau_r}, a^2 = \frac{k}{c\rho}.$$

Here c is specific heat capacity, k - heat conductivity coefficient, ρ - density, τ_r - relaxation time. The source term $F(r, \varphi, z, t)$ can be from different parts of the same device or outer source, for example, wind source.

In the case of wave energy we can assume different non-homogeneous boundary conditions. Important is to formulate boundary conditions (3), (4) and (5) in the heat energy transfer form [15], [17], [27]:

$$r \frac{\partial U}{\partial r} \Big|_{r=0} = 0, \quad (2)$$

$$\left(R \frac{\partial U}{\partial r} + k_1 U \right) \Big|_{r=R} = Rg_1(\varphi, z, t), k_1 = \frac{Rh_3}{k}. \quad (3)$$

$$\left(\frac{\partial U}{\partial z} - k_2 U \right) \Big|_{z=0} = g_2(r, \varphi, t), k_i = \frac{h_i}{k}, i = 2, 3, \quad (4)$$

$$\left(\frac{\partial U}{\partial z} + k_3 U \right) \Big|_{z=l} = g_3(r, \varphi, t). \quad (5)$$

Here h_i is heat exchange coefficient. On all the other sides of device we have heat exchange with environment. In fact it is possible to look at other types of boundary conditions: first (Dirichlet) and second (Neumann) type. The initial conditions for the function $U(r, \varphi, z, t)$ are assumed in following form:

$$U \Big|_{t=0} = U_0(r, \varphi, z), \quad (6)$$

$$\frac{\partial U}{\partial t} \Big|_{t=0} = U_1(r, \varphi, z). \quad (7)$$

From the practical point of view in the steel quenching the condition (7) can be unrealistic. The initial heat flux must be determined theoretically. As additional condition we assume that either the temperature distribution or the heat fluxes distribution at the end of process is given (known):

$$U \Big|_{t=T} = U_T(r, \varphi, z), \quad (8)$$

$$\frac{\partial U}{\partial t} \Big|_{t=T} = U_T^1(r, \varphi, z). \quad (9)$$

The formulation of the three dimensional mathematical model is important for wave energy generator [8]. It is good to see from the above point on the fig. 1:

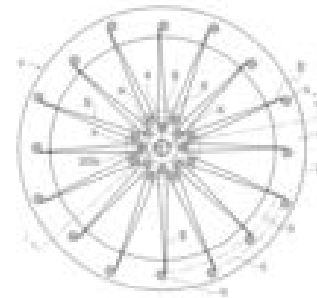


FIG. 1

Fig. 1. The view from the above point of cylindrical piezoelectric generator from patent [8].

For 3-D mathematical model is important that solution in φ -direction is continuous and smooth. These 2 conditions are important for the reduction of 3-D model to 2-D model by conservative averaging method [10], [31] and [32] (see later in the section 5):

$$U \Big|_{\varphi=0} = U \Big|_{\varphi=2\pi}, \quad (10)$$

$$\frac{\partial U}{\partial \varphi} \Big|_{\varphi=0} = \frac{\partial U}{\partial \varphi} \Big|_{\varphi=2\pi}. \quad (11)$$

3 Solution of 3-D Problem

Firstly we assume that we have non-homogeneous Klein-Gordon equation-with source term: $C \geq 0$.

The solution in three-dimensional problem is in following form:

$$U(r, \varphi, z, t) = H(r, \varphi, z, t) + \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \times \int_0^l U_1(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) d\eta + \int_0^{2\pi} d\zeta \times \int_0^R \xi d\xi \int_0^l U_0(\xi, \eta, \zeta) \frac{\partial}{\partial t} G(r, \varphi, z, \xi, \eta, \zeta, t) d\eta. \quad (12)$$

Here are source term and boundary conditions:

$$\begin{aligned}
 H(r, \varphi, z, t) = & a_\tau^2 R^2 \int_0^t d\tau \int_0^{2\pi} d\zeta \times \\
 & \int_0^l g_1(\eta, \zeta, \tau) G(r, \varphi, z, R, \eta, \zeta, t - \tau) d\eta - a_\tau^2 \int_0^t d\tau \times \\
 & \int_0^{2\pi} d\zeta \int_0^R \xi g_2(\xi, \zeta, \tau) G(r, \varphi, z, \xi, 0, \zeta, t - \tau) d\xi + a_\tau^2 \\
 & \times \int_0^t d\tau \int_0^{2\pi} d\zeta \int_0^R \xi g_3(\xi, \zeta, \tau) G(r, \varphi, z, \xi, l, \zeta, t - \tau) d\xi \\
 & + \int_0^t d\tau \int_0^{2\pi} d\zeta \times \\
 & \int_0^R \xi d\xi \int_0^l F(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) d\eta.
 \end{aligned} \tag{13}$$

The Green function [34] - [36] for initial-boundary problem for Klein-Gordon equation is known; see [37]:

$$\begin{aligned}
 G(r, \varphi, z, \xi, \eta, \zeta, t) = & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{\pi} \times \\
 & \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm} \xi)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2) [J_n(\mu_{nm} R)]^2} \times \\
 & \frac{\cos[n(\varphi - \eta)] h_s(z) h_s(\zeta) \sin(\lambda_{nms} t)}{\|h_s\|^2 \lambda_{nms}}.
 \end{aligned} \tag{14}$$

Here $J_n(\xi)$ – is Bessel’s function and

$$\lambda_{nms} = \sqrt{a_\tau^2 (\mu_{nm}^2 + \beta_s^2) + C},$$

$$A_n = \begin{cases} 1, & \text{if } n = 0, \\ 2, & \text{if } n > 0; \end{cases}$$

$$h_s(z) = \cos(\beta_s z) + \frac{k_2}{\beta_s} \sin(\beta_s z),$$

$$\|h_s\|^2 = \frac{k_3(\beta_s^2 + k_2^2)}{2\beta_s^2(\beta_s^2 + k_3^2)} + \frac{k_2}{2\beta_s^2} + \frac{l}{2} \left(1 + \frac{k_2^2}{\beta_s^2} \right).$$

The eigenvalues μ_{nm}, β_s are positive roots of the transcendental equations:

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0, \quad \frac{tg(\beta l)}{\beta} = \frac{k_2 + k_3}{\beta^2 - k_2 k_3}.$$

We assume that at final moment $t = T$ is known only one boundary condition (8). Then from solution (12) we easy obtain Fredholm first type integral equation with respect to function

$$\begin{aligned}
 U_1(r, \varphi, z): \\
 \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \int_0^l U_1(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, T) d\eta \\
 = \Phi(r, \varphi, z).
 \end{aligned} \tag{15}$$

The unknown right side function $\Phi(r, \varphi, z)$ is in the following form:

$$\begin{aligned}
 \Phi(r, \varphi, z) = & U_T(r, \varphi, z) - H(r, \varphi, z, T) - \\
 & \int_0^{2\pi} d\zeta \int_0^R \xi d\xi \int_0^l U_0(\xi, \eta, \zeta) \frac{\partial}{\partial t} G(r, \varphi, z, \xi, \eta, \zeta, t) \Big|_{t=T} d\eta.
 \end{aligned}$$

Similar situation is, if second boundary condition (9) is done. We differentiate solution (12) regarding time:

$$\begin{aligned}
 \frac{\partial}{\partial t} U(r, \varphi, z, t) = & \frac{\partial}{\partial t} H(r, \varphi, z, t) + \\
 & \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \int_0^l U_1(\xi, \eta, \zeta) \frac{\partial}{\partial t} G(r, \varphi, z, \xi, \eta, \zeta, t) d\eta + \\
 & \int_0^{2\pi} d\zeta \int_0^R \xi d\xi \int_0^l U_0(\xi, \eta, \zeta) \frac{\partial^2}{\partial t^2} G(r, \varphi, z, \xi, \eta, \zeta, t) d\eta.
 \end{aligned}$$

We again obtain 1st kind Fredholm integral equation for the determination of unknown initial heat flux:

$$\begin{aligned}
 \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \int_0^l U_1(\xi, \eta, \zeta) \frac{\partial}{\partial t} G(r, \varphi, z, \xi, \eta, \zeta, t) \Big|_{t=T} d\eta \\
 = \Phi_1(r, \varphi, z).
 \end{aligned} \tag{16}$$

Here

$$\begin{aligned}
 \Phi_1(r, \varphi, z) = & U_T^1(r, \varphi, z) - \frac{\partial}{\partial t} H(r, \varphi, z, t) \Big|_{t=T} - \\
 & \int_0^{2\pi} d\zeta \int_0^R \xi d\xi \int_0^l U_0(\xi, \eta, \zeta) \frac{\partial^2}{\partial t^2} G(r, \varphi, z, \xi, \eta, \zeta, t) \Big|_{t=T} d\eta.
 \end{aligned}$$

There is an interesting situation, if both additional conditions (8), (9) are known. In this case we introduce new time argument by formula

$$\tilde{t} = T - t. \tag{17}$$

The formulation for new function $V(r, \varphi, z, \tilde{t})$ with time variable \tilde{t} is following:

$$\begin{aligned}
 \frac{\partial^2 V}{\partial \tilde{t}^2} = & a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} \right] - \\
 & - CV + F(r, \varphi, z, T - \tilde{t}), \\
 V|_{\tilde{t}=0} = & U_T(r, \varphi, z), \quad \frac{\partial V}{\partial \tilde{t}} \Big|_{\tilde{t}=0} = -U_T^1(r, \varphi, z), \\
 \left(R \frac{\partial V}{\partial r} + k_1 V \right) \Big|_{r=R} = & Rg_1(\varphi, z, T - \tilde{t}),
 \end{aligned} \tag{18}$$

$$\left(\frac{\partial V}{\partial z} - k_2 V\right)\Big|_{z=0} = g_2(r, \varphi, T - \tilde{t}),$$

$$\left(\frac{\partial V}{\partial z} + k_3 V\right)\Big|_{z=l} = g_3(r, \varphi, T - \tilde{t}).$$

Similar to (12) the solution of inverse problem looks like the formulae (12):

$$V(x, y, z, \tilde{t}) = H(x, y, z, \tilde{t}) - \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \times \int_0^l U_T^1(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, \tilde{t}) d\eta + \int_0^{2\pi} d\zeta \times \int_0^R \xi d\xi \int_0^l U_T(\xi, \eta, \zeta) \frac{\partial}{\partial \tilde{t}} G(r, \varphi, z, \xi, \eta, \zeta, \tilde{t}) d\eta. \quad (19)$$

There it is easy to transform the expression for $H(x, y, z, \tilde{t})$ in following form:

$$H(x, y, z, \tilde{t}) = a_\tau^2 R \int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\zeta \times \int_0^l g_1(\eta, \zeta, \tau) G(r, \varphi, z, R, \eta, \zeta, T - \tau) d\eta - a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\zeta \int_0^R g_2(\xi, \zeta, \tau) G(x, y, z, \xi, 0, \zeta, T - \tau) d\xi + a_\tau^2 \times \int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\zeta \int_0^R g_3(\xi, \zeta, \tau) G(x, y, z, \xi, b, \zeta, T - \tau) d\xi + \int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\zeta \times \int_0^R d\xi \int_0^l F(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, T - \tau) d\eta.$$

For the heat flux in time from (17) we have the expression:

$$\frac{\partial}{\partial \tilde{t}} V(r, \varphi, z, \tilde{t}) = \frac{\partial}{\partial \tilde{t}} H(r, \varphi, z, \tilde{t}) + \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \int_0^l V_1(\xi, \eta, \zeta) \frac{\partial}{\partial \tilde{t}} G(r, \varphi, z, \xi, \eta, \zeta, \tilde{t}) d\eta + \int_0^{2\pi} d\zeta \int_0^R \xi d\xi \int_0^l V_0(\xi, \eta, \zeta) \frac{\partial^2}{\partial \tilde{t}^2} G(r, \varphi, z, \xi, \eta, \zeta, \tilde{t}) d\eta.$$

From last expression at $\tilde{t} = T$ and equality (18) we have solution for the time inverse problem:

$$U_T^1(r, \varphi, z) = -\frac{\partial}{\partial \tilde{t}} H(r, \varphi, z, \tilde{t})\Big|_{\tilde{t}=T} - \quad (20)$$

$$\int_0^R \xi d\xi \int_0^{2\pi} d\zeta \int_0^l V_1(\xi, \eta, \zeta) \frac{\partial}{\partial \tilde{t}} G(r, \varphi, z, \xi, \eta, \zeta, \tilde{t})\Big|_{\tilde{t}=T} d\eta - \int_0^{2\pi} d\zeta \int_0^R \xi d\xi \int_0^l V_0(\xi, \eta, \zeta) \frac{\partial^2}{\partial \tilde{t}^2} G(r, \varphi, z, \xi, \eta, \zeta, \tilde{t})\Big|_{\tilde{t}=T} d\eta.$$

Very interesting is wave energy [38] as you can see in [21]:

$$I_0(t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{\sin^2(\lambda_{nms} t)}{\lambda_{nms}^2}.$$

4 Solution of 3-D problem with constant initial conditions

In the previous section we have constructed some three dimensional solutions for direct and time inverse problems for hyperbolic heat equation. Often enough initial conditions are constant functions [21], [24]. In this case we have to solve the solutions in the form of series. For simplicity we look the homogeneous boundary conditions:

$$U(r, \varphi, z, t) = U_1 \int_0^R \xi d\xi \int_0^{2\pi} d\zeta \times \int_0^l G(r, \varphi, z, \xi, \eta, \zeta, t) d\eta + U_0 \int_0^{2\pi} d\zeta \times \int_0^R \xi d\xi \int_0^l \frac{\partial}{\partial t} G(r, \varphi, z, \xi, \eta, \zeta, t) d\eta = U_0 G_0 + U_1 G_1. \quad (21)$$

We use the Green function form (14) in the little different form:

$$G(r, \varphi, z, \xi, \eta, \zeta, t) = \frac{1}{\pi} \times \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm} \xi)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2) [J_n(\mu_{nm} R)]^2} \times \frac{[\cos(n\varphi) \cos(n\eta) + \sin(n\varphi) \sin(n\eta)]}{\|h_s\|^2} \times \frac{h_s(z) h_s(\zeta) \sin(\lambda_{nms} t)}{\lambda_{nms}}. \quad (22)$$

The function G_0 after integration can be obtained in following form:

$$G_0 = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) \cos(\lambda_{nms} t)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2) [J_n(\mu_{nm} R)]^2} \times$$

$$\frac{[\cos(n\varphi)\sin(nl) + \sin(n\varphi)(1 - \cos(n\eta))]}{n\|h_s\|^2} h_s(z) \times \frac{\sin(2\pi\beta_s) + \frac{k_2}{\beta_s}(1 - \cos(2\pi\beta_s))}{\beta_s} \int_0^R \xi J_n(\mu_{nm}\xi) d\xi.$$

Similarly we can transform the function G_1 :

$$G_1 = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm}r) \sin(\lambda_{nm}t)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2) [J_n(\mu_{nm}R)]^2} \times \frac{[\cos(n\varphi)\sin(nl) + \sin(n\varphi)(1 - \cos(n\eta))]}{n\|h_s\|^2} h_s(z) \times \frac{\sin(2\pi\beta_s) + \frac{k_2}{\beta_s}(1 - \cos(2\pi\beta_s))}{\beta_s} \int_0^R \xi J_n(\mu_{nm}\xi) d\xi.$$

In this paper we can show that time reverse problem with two final time conditions is not ill-posed problem and can be solved similarly as time direct problem. It was shown in our paper [21] that for rectangular sample time reverse problem can be solved without some numerical problem. It is good known that for inverse problem is not easy to calculate the solution [39] - [42].

5 Solution of Two Dimensional Problem

Two dimensional problem can be obtained in two ways. First way is standard: we use monograph [37] for the two-dimensional solution and Green function. The mathematical model is in the form:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} \right] - CU + F(r, z, t),$$

$$r \in [0, R], z \in [0, l], t \in [0, T], C \geq 0,$$

$$r \frac{\partial U}{\partial r} \Big|_{r=0} = 0, \left(\frac{\partial U}{\partial r} + k_1 U \right) \Big|_{r=R} = g_1(z, t),$$

$$\left(\frac{\partial U}{\partial z} - k_2 U \right) \Big|_{z=0} = g_2(r, t), \tag{23}$$

$$\left(\frac{\partial U}{\partial z} + k_3 U \right) \Big|_{z=l} = g_3(r, t),$$

$$U|_{t=0} = U_0(r, z), \frac{\partial U}{\partial t} \Big|_{t=0} = U_1(r, z).$$

Of course, the temperature distribution and the heat fluxes distribution at the end of process is given:

$$U|_{t=T} = U_T(r, z), \frac{\partial U}{\partial t} \Big|_{t=T} = U_T^1(r, z).$$

The solution of two dimensional problem is in following form:

$$U(r, z, t) = H(r, z, t) + \int_0^R \xi d\xi \times \int_0^l U_1(\xi, \eta) G(r, z, \xi, \eta, t) d\eta + \tag{24}$$

$$\int_0^R \xi d\xi \int_0^l U_0(\xi, \eta) \frac{\partial}{\partial t} G(r, z, \xi, \eta, t) d\eta.$$

The known boundary conditions and source term are in the function $H(r, z, t)$:

$$H(r, z, t) = -a_\tau^2 \int_0^t d\tau \int_0^R g_2(\xi, \tau) G(r, z, \xi, 0, t - \tau) d\xi$$

$$+ a_\tau^2 \int_0^t d\tau \int_0^l g_1(\eta, \tau) G(r, z, R, \eta, t - \tau) d\eta + \tag{25}$$

$$+ a_\tau^2 \int_0^t d\tau \int_0^R g_3(\xi, \tau) G(r, z, \xi, l, t - \tau) d\xi +$$

$$\int_0^t d\tau \int_0^R d\xi \int_0^l F(\xi, \eta, \tau) G(r, z, \xi, \eta, t - \tau) d\eta.$$

The Green function for two-dimensional problem is in the form [37]:

$$G(r, z, \xi, \eta, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\mu_n^2 J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right)}{(k_1^2 R^2 + \mu_n^2) J_0^2(\mu_n)}$$

$$\frac{\varphi_m(z) \varphi_m(\eta)}{\|\varphi_m\|^2 \sqrt{\lambda_m}} \exp \left[- \left(a_\tau^2 \lambda_m^2 + \frac{a_\tau^2 \mu_n^2}{R^2} \right) t \right], \tag{26}$$

$$\varphi_m(z) = \cos(\lambda_m z) + \frac{k_2}{\lambda_m} \sin(\lambda_m z),$$

$$\|\varphi_m\|^2 = \frac{k_3(\lambda_m^2 + k_2^2)}{2\lambda_m^2(\lambda_m^2 + k_3^2)} + \frac{k_2}{2\lambda_m^2} + \frac{l}{2} \left(1 + \frac{k_2^2}{\lambda_m^2} \right).$$

The eigenvalues μ_n, λ_m are positive roots of the transcendental equations:

$$\mu J_1(\mu) + k_1 R J_0(\mu) = 0, \frac{tg(\lambda l)}{\lambda} = \frac{k_2 + k_3}{\lambda^2 - k_2 k_3}.$$

Here we will obtain the solution for two-dimensional problem as it was done in our papers [10], [32], [43] and [44] by method of conservative averaging:

$$V(r, z, t) = \frac{1}{2\pi} \int_0^{2\pi} U(r, \varphi, z, t) d\varphi.$$

We integrate the main differential equation (1) in the direction $\varphi \in [0, 2\pi]$:

$$\frac{\partial^2 V}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} \right] +$$

$$\frac{1}{2\pi r^2} \frac{\partial U}{\partial \varphi} \Big|_{\varphi=0}^{\varphi=2\pi} - CV + f(r, z, t),$$

$$f(r, z, t) = \frac{1}{2\pi} \int_0^{2\pi} F(r, \varphi, z, t) d\varphi.$$

The equality (11) gives the two dimensional equation:

$$\frac{\partial^2 V}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} \right] -$$

$$CV + f(r, z, t).$$

For this equation as initial and boundary conditions is formula (23).

6 Solution of One Dimensional Problem

We will start with a formulation of the mathematical model of the steel cylinder which is relatively thin in z directions: $l \ll R$. In accordance with the conservative averaging method [31], [32] we introduce for two-dimensional formulation from the two-dimensional function the following integral averaged value (one space-dimensional function):

$$u(r, t) = (l)^{-1} \int_0^l U(r, z, t) dz, \tag{28}$$

$$\bar{f}(r, t) = (l)^{-1} \int_0^l f(r, z, t) dz.$$

We integrate equation (27) in the direction z :

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] +$$

$$+ \frac{1}{l} \frac{\partial U}{\partial z} \Big|_{z=0}^{z=l} - Cu + \bar{f}(r, t). \tag{29}$$

The boundary conditions (23) for new function $u(r, t)$ are:

$$\frac{1}{l} \frac{\partial U}{\partial z} \Big|_{z=0} = k_2 U(r, 0, t) + g_2(r, t), \frac{1}{l} \frac{\partial U}{\partial z} \Big|_{z=l} =$$

$$-k_3 U(r, l, t) + g_3(r, t).$$

We look for thin cylinder, it means that we have:

$$U(r, 0, t) = U(r, l, t) \cong u(r, t).$$

Finally we transform the equation (29) in Klein-Gordon equation form:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] - cu + \tilde{f}(r, t),$$

$$c = C + k_2 + k_3, \tag{30}$$

$$\tilde{f}(r, t) = \bar{f}(r, t) - g_2(r, t) + g_3(r, t).$$

The main differential equation together with boundary conditions and initial conditions from (25) are in following form:

$$\left(\frac{\partial u}{\partial r} + k_1 u \right) \Big|_{r=R} = \bar{g}_1(t),$$

$$\bar{g}_1(t) = (l)^{-1} \int_0^l g_1(z, t) dz,$$

$$u \Big|_{t=0} = u_0(r), \frac{\partial u}{\partial t} \Big|_{t=0} = u_1(r), \tag{31}$$

$$u_0(t) = (l)^{-1} \int_0^l U_0(z, t) dz,$$

$$u_1(t) = (l)^{-1} \int_0^l U_1(z, t) dz.$$

Solution of this problem is with Green function (see [35], [37]):

$$u(r, t) = \int_0^R u_0(\xi) \frac{\partial}{\partial t} G(r, \xi, t) d\xi + \int_0^R u_1(\xi) G(r, \xi, t) d\xi +$$

$$a_\tau^2 \int_0^t \bar{g}_1(\tau) G(r, R, t - \tau) d\tau +$$

$$\int_0^t d\tau \int_0^R \tilde{f}(\xi, \tau) G(r, \xi, t - \tau) d\xi. \tag{32}$$

Green function from [37] is in the form:

$$G(r, \xi, t) = \frac{2\xi}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 J_0 \left(\frac{\mu_n r}{R} \right)}{(k_1^2 R^2 + \mu_n^2) J_0^2(\mu_n)} \times$$

$$J_0 \left(\frac{\mu_n \xi}{R} \right) \frac{\sin(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}}, \lambda_n = \frac{a_\tau^2 \mu_n^2}{R^2} + c. \tag{33}$$

The eigenvalues μ_n are positive roots of the transcendental equation:

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0.$$

Other situation is for cylinder with small diameter: $R \ll l$. We define from (27) new function $v(z, t)$:

$$v(z, t) = \frac{1}{R^2} \int_0^R r V(r, z, t) dr,$$

$$\hat{f}(z, t) = \frac{1}{R^2} \int_0^R r f(r, z, t) dr.$$

We integrate the modified differential equation (27) in r direction:

$$\frac{\partial^2 r V}{\partial t^2} = a_\tau^2 \left[\frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 r V}{\partial z^2} \right] - CrV + rf(r, z, t).$$

This gives:

$$\frac{\partial^2 v}{\partial t^2} = a_\tau^2 \frac{\partial^2 v}{\partial z^2} + r \frac{\partial V}{\partial r} \Big|_{r=0}^{r=R} - Cv + \hat{f}(z, t).$$

The boundary condition in the r direction gives:

$$r \frac{\partial V}{\partial r} \Big|_{r=R} = -k_1 R V(R, t) + R g_1(z, t),$$

$$r \frac{\partial V}{\partial r} \Big|_{r=0} = 0.$$

Finally we have the one dimensional Klein-Gordon partial differential equation:

$$\frac{\partial^2 v}{\partial t^2} = a_\tau^2 \frac{\partial^2 v}{\partial z^2} - dv + g(z, t), \tag{34}$$

$$d = C + k_1 R, g(z, t) = \hat{f}(z, t) + R g_1(z, t).$$

The boundary conditions and initial conditions from (23) can be rewritten in following form:

$$\left(\frac{\partial v}{\partial z} - k_2 v \right) \Big|_{z=0} = g_2(t),$$

$$\left(\frac{\partial v}{\partial z} + k_3 v \right) \Big|_{z=l} = g_3(t), \tag{35}$$

$$v \Big|_{t=0} = v_0(z), \frac{\partial v}{\partial t} \Big|_{t=0} = v_1(z).$$

Here the new averaged functions are:

$$v_0(z) = \frac{1}{R^2} \int_0^R r U_0(r, z) dr, v_1(z) = \frac{1}{R^2} \int_0^R r U_1(r, z) dr,$$

$$g_2(t) = \frac{1}{R^2} \int_0^R r g_2(r, t) dr, g_3(t) = \frac{1}{R^2} \int_0^R r g_3(r, t) dr.$$

We have solution in following form:

$$v(z, t) = \int_0^l v_0(\eta) \frac{\partial}{\partial t} G(z, \eta, t) d\eta + \int_0^l v_1(\eta) G(z, \eta, t) d\eta - a_\tau^2 \int_0^t g_2(\tau) G(z, 0, t - \tau) d\tau + a_\tau^2 \int_0^t g_3(\tau) G(z, l, t - \tau) d\tau + \int_0^t d\tau \int_0^l g(\eta, \tau) G(z, \eta, t - \tau) d\eta. \tag{36}$$

Green function in this case is [35], [37]:

$$G(z, \eta, t) = \sum_{n=1}^{\infty} \frac{y_n(z) y_n(\zeta) \sin\left(t \sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\|y_n\|^2 \sqrt{a_\tau^2 \lambda_n^2 + d}},$$

$$y_n(z) = \cos(\lambda_n z) + \frac{k_2}{\lambda_n} \sin(\lambda_n z), \tag{37}$$

$$\|y_n\| = \frac{k_2 \lambda_n^2 + k_2^2}{2 \lambda_n^2 \lambda_n^2 + k_3^2} + \frac{k_2}{2 \lambda_n^2} + \frac{l}{2} \left(1 + \frac{k_2^2}{\lambda_n^2} \right).$$

The eigenvalues λ_n are positive roots of the transcendental equations:

$$\frac{tg(\lambda l)}{\lambda} = \frac{k_2 + k_3}{\lambda^2 - k_2 k_3}.$$

For both one dimensional problems we have two final conditions. For the problem (30), (31) the additional conditions are:

$$u \Big|_{t=T} = u_T(r), \frac{\partial u}{\partial t} \Big|_{t=T} = u_T^1(r). \tag{38}$$

And for the problem (34), (35) the additional conditions are:

$$v \Big|_{t=T} = v_T(z), \frac{\partial v}{\partial t} \Big|_{t=T} = v_T^1(z). \tag{39}$$

7 Time Inverse One Dimensional Problem

We would like to continue with the one dimensional problem (30)-(32) with time inverse formulation (17) for $\bar{u}(r, \tilde{t})$:

$$\frac{\partial^2 \bar{u}}{\partial \tilde{t}^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}}{\partial r} \right) \right] - c \bar{u} + \tilde{f}(r, T - \tilde{t}), \tag{40}$$

$$r \in (0, R), \tilde{t} \in (0, T], \bar{u} \Big|_{\tilde{t}=0} = u_T(r), \frac{\partial \bar{u}}{\partial \tilde{t}} \Big|_{\tilde{t}=0} = -u_T^1(r),$$

$$\left(\frac{\partial \bar{u}}{\partial r} + k_1 \bar{u} \right) \Big|_{r=R} = \bar{g}_1(T - \tilde{t}),$$

$$\bar{u} \Big|_{\tilde{t}=T} = u_0(r), \quad \frac{\partial \bar{u}}{\partial \tilde{t}} \Big|_{\tilde{t}=T} = -u_1(r).$$

Solution is similar with formula (32):

$$\begin{aligned} \bar{u}(r, \tilde{t}) = & \int_0^R u_T(\xi) \frac{\partial}{\partial \tilde{t}} G(r, \xi, \tilde{t}) d\xi - \\ & \int_0^R u_T^1(\xi) G(r, \xi, \tilde{t}) d\xi + \\ & a_\tau^2 \int_0^{\tilde{t}} \bar{g}_1(T - \tilde{t} + \tau) G(r, R, \tilde{t} - \tau) d\tau + \\ & \int_0^{\tilde{t}} d\tau \int_0^l \tilde{f}(\xi, T - \tilde{t} + \tau) G(r, \xi, \tilde{t} - \tau) d\xi. \end{aligned}$$

The solution can be rewritten in following form:

$$\begin{aligned} \bar{u}(r, \tilde{t}) = & \int_0^R u_T(\xi) \frac{\partial}{\partial \tilde{t}} G(r, \xi, \tilde{t}) d\xi - \\ & \int_0^R u_T^1(\xi) G(r, \xi, \tilde{t}) d\xi + \\ & a_\tau^2 \int_0^{\tilde{t}} \bar{g}_1(T - \tau_1) G(r, R, \tau_1) d\tau_1 + \\ & \int_0^{\tilde{t}} d\tau_1 \int_0^l \tilde{f}(\xi, T - \tau_1) G(r, \xi, \tau_1) d\xi. \end{aligned} \tag{41}$$

For the heat flux we have an expression:

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} \bar{u}(r, \tilde{t}) = & \int_0^l u_T(\xi) \frac{\partial^2}{\partial \tilde{t}^2} G(r, \xi, \tilde{t}) d\xi \\ & - \int_0^l u_T^1(\xi) \frac{\partial}{\partial \tilde{t}} G(r, \xi, \tilde{t}) d\xi + \\ & a_\tau^2 \frac{\partial}{\partial \tilde{t}} \int_0^{\tilde{t}} \bar{g}_1(T - \tau_1) G(r, R, \tau_1) d\tau_1 + \\ & \frac{\partial}{\partial \tilde{t}} \int_0^{\tilde{t}} d\tau_1 \int_0^l \tilde{f}(\xi, T - \tau_1) G(r, \xi, \tau_1) d\xi. \end{aligned}$$

From here, a nice explicit representation of the necessary initial heat flux immediately follows:

$$\begin{aligned} u_1(r) = & - \int_0^l v_T(\xi) \frac{\partial}{\partial \tilde{t}} G(r, \xi, \tilde{t}) \Big|_{\tilde{t}=T} d\xi + \\ & \int_0^l u_T(\xi) \frac{\partial^2}{\partial \tilde{t}^2} G(r, \xi, \tilde{t}) \Big|_{\tilde{t}=T} d\xi + \bar{g}_1(0) G(r, R, T) \\ & + \int_0^l \tilde{f}(\xi, 0) G(r, \xi, T) d\xi. \end{aligned} \tag{42}$$

8 Solution of 1-D problem with constant initial conditions

We would like to finish with the one dimensional solution with a simplification for constant initial conditions in the formulation (34)-(35):

$$\begin{aligned} v \Big|_{t=0} = v_0(z) = v_0 = const, \\ \frac{\partial v}{\partial t} \Big|_{t=0} = v_1(z) = v_1 = const. \end{aligned} \tag{43}$$

The solution of the time direct problem is the following. We assume that $g(z, t) = g_2(t) = g_3(t) = 0$:

$$\begin{aligned} u(z, t) = v_0 \int_0^l \frac{\partial}{\partial t} G(z, \xi, t) d\xi + \\ v_1 \int_0^l G(z, \xi, t) d\xi = v_0 I_0 + v_1 I_1. \end{aligned} \tag{44}$$

Intensive steel quenching process with initial conditions (43) is very natural [10]-[14]. We have the homogeneous equation (34) and the homogeneous boundary conditions. As next step we integrate Green functions in the formula (44):

$$\begin{aligned} I_0 = & \int_0^l \frac{\partial}{\partial t} G(z, \xi, t) d\xi = \\ & \sum_{n=1}^{\infty} \frac{y_n(z) y_n(l) \cos\left(t\sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\lambda_n \|y_n^2\|}, \\ y_n(l) = & \cos(\lambda_n l) + \frac{k_2}{\lambda_n} \sin(\lambda_n l); \\ I_1 = & \int_0^l G(z, \xi, t) d\xi = \\ & \sum_{n=1}^{\infty} \frac{y_n(z) y_n(l) \sin\left(t\sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\lambda_n \|y_n^2\|}. \end{aligned}$$

Finally it means that we have expression for temperature in the form of series:

$$\begin{aligned} u(x, t) = v_0 I_0 + v_1 I_1 = \sum_{n=1}^{\infty} \frac{y_n(z) y_n(l)}{\lambda_n \|y_n^2\|} \times \\ \left[v_0 \cos\left(t\sqrt{a_\tau^2 \lambda_n^2 + d}\right) + v_1 \sin\left(t\sqrt{a_\tau^2 \lambda_n^2 + d}\right) \right]. \end{aligned} \tag{45}$$

Similarly we can transform the derivative for heat flux in the form of series:

$$\frac{\partial u}{\partial t} = v_0 \int_0^l \frac{\partial^2}{\partial t^2} G(z, \eta, t) d\eta +$$

$$v_1 \int_0^R \frac{\partial}{\partial t} G(z, \eta, t) d\eta = v_0 J_0 + v_1 J_1.$$

$$J_0 = - \sum_{n=1}^{\infty} \frac{y_n(z) y_n(l) \sin\left(t \sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\left(\sqrt{a_\tau^2 \lambda_n^2 + d}\right)^{-1} \lambda_n \|y_n^2\|},$$

$$J_1 = \sum_{n=1}^{\infty} \frac{y_n(z) y_n(l) \cos\left(t \sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\left(\sqrt{a_\tau^2 \lambda_n^2 + d}\right)^{-1} \lambda_n \|y_n^2\|}. \quad (46)$$

$$\frac{\partial u}{\partial t} = v_0 J_0 + v_1 J_1 = -v_0 \times$$

$$\sum_{n=1}^{\infty} \frac{y_n(z) y_n(l) \sin\left(t \sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\left(\sqrt{a_\tau^2 \lambda_n^2 + d}\right)^{-1} \lambda_n \|y_n^2\|} +$$

$$+ v_1 \sum_{n=1}^{\infty} \frac{y_n(z) y_n(l) \cos\left(t \sqrt{a_\tau^2 \lambda_n^2 + d}\right)}{\left(\sqrt{a_\tau^2 \lambda_n^2 + d}\right)^{-1} \lambda_n \|y_n^2\|}.$$

Numerical results for the formulation (34)-(35) are the same as in the paper [23].

9 Conclusion

We have constructed some solutions for direct and time inverse problems for hyperbolic heat equation. The solutions for determination of initial heat flux are obtained either in the form of Fredholm integral equation of 1st kind with continuous kernel or in the closed analytical form – in the form of series or ordinary integrals.

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References:

- [1] Kobasko N.I. *Steel Quenching in Liquid Media under Pressure*. – Kyiv, Naukova Dumka, 1980.
- [2] Kobasko, N. I. Intensive Steel Quenching Methods, *Handbook "Theory and Technology of Quenching"*, Springer-Verlag, 1992.
- [3] Totten G.E., Bates C.E., and Clinton N.A. *Handbook of Quenchants and Quenching Technology*. ASM International, 1993.
- [4] Aronov M.A., Kobasko N., Powell J.A. Intensive Quenching of Carburized Steel Parts, *IASME Transactions*, Issue 9, Vol. 2, November 2005, p. 1841-1845.
- [5] Kobasko N.I. Self-regulated thermal processes during quenching of steels in liquid media. – *International Journal of Microstructure and Materials Properties*, Vol. 1, No 1, 2005, p. 110-125.
- [6] Kobasko N.I. Transient Nucleate Boiling Process to Be Widely for Super Strengthening of Materials and Obtaining Other Benefits in Heat Treating Industry. – *UA Patent No. 109935*, Published on Oct. 26, 2015, Bulletin No. 20.
- [7] Salter S.H. Apparatus for use in the extraction of energy from waves on water. *US Patent 4,134,023*. January 14, 1977.
- [8] Carroll C.B. Piezoelectric rotary electrical energy generator. *US Patent 6194815 B1*. February 27, 2001.
- [9] Carroll C.B., Bell M. Wave energy convertors utilizing pressure differences. *US Patent 20040217597 A1*. November 4, 2004.
- [10] Buike M., Buikis A. Approximate Solutions of Heat Conduction Problems in Multi- Dimensional Cylinder Type Domain by Conservative Averaging Method, Part 1. *Proceedings of the 5th IASME/WSEAS Int. Conf. on Heat Transfer, Thermal Engineering and Environment*, Vouliagmeni, Athens, August 25 -27, 2007, p. 15 – 20.
- [11] Bobinska T., Buike M., Buikis A. Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-Shape Samples, Part 2 (Inverse Problem). *Proceedings of 5th IASME/WSEAS International Conference on Continuum Mechanics (CM'10)*, University of Cambridge, UK, February 23-25, 2010. p. 21-26.
- [12] Buike M., Buikis A. Hyperbolic heat equation as mathematical model for steel quenching of L-shape samples, Part 1 (Direct Problem). *Applied and Computational Mathematics*. Proceedings of the 13th WSEAS International Conference on Applied Mathematics (MATH'08), Puerto De La Cruz, Tenerife, Canary Islands, Spain, December 15-17, 2008. WSEAS Press, 2008. p. 198-203.
- [13] Buike M., Buikis A. Several Intensive Steel Quenching Models for Rectangular Samples. *Proceedings of NAUN/WSEAS International*

- Conference on Fluid Mechanics and Heat & Mass Transfer*, Corfu Island, Greece, July 22-24, 2010. p.88-93.
- [14] Bobinska T., Buike M., Buikis A. Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-and T-Shape Samples, Direct and Inverse Problems. *Transactions of Heat and Mass Transfer*. Vol.5, Issue 3, July 2010. p. 63-72.
- [15] Blomkalna S., Buikis A. Heat conduction problem for double-layered ball. *Progress in Industrial Mathematics at ECMI 2012*. Springer, 2014. p. 417-426.
- [16] Bobinska T., Buike M., Buikis A. Comparing solutions of hyperbolic and parabolic heat conduction equations for L-shape samples. *Recent Advances in Fluid Mechanics and Heat @Mass Transfer*. Proceedings of the 9th IASME/WSEAS International Conference on THE'11. Florence, Italy, August 23-25, 2011. p. 384-389.
- [17] A. Piliksere, A. Buikis. Analytical solution for intensive quenching of cylindrical sample. *Proceedings of 6th International Scientific Colloquium "Modelling for Material Processing"*, Riga, September 16-17, 2010, p. 181-186.
- [18] A. Piliksere, M. Buike, A. Buikis. Steel quenching process as hyperbolic heat equation for cylinder. *Proceedings of 6th Baltic Heat Conference – BHTC2011*, 2011, ISBN 978-952-15-2640-4 (CD-ROM)
- [19] Blomkalna S., Buike M., Buikis A. Several intensive steel quenching models for rectangular and spherical samples. *Recent Advances in Fluid Mechanics and Heat & Mass Transfer*. Proceedings of the 9th IASME/WSEAS International Conference on THE'11. Florence, Italy, August 23-25, 2011. p. 390-395.
- [20] Buikis A., Kalis H. Hyperbolic type approximation for the solutions of the hyperbolic heat conduction equation in 3-D domain. „*Mathematical and Computational Methods in Applied Sciences*.” Proceedings of the 3rd International Conference on Applied, Numerical and Computational Mathematics (*ICANCM'15*). Sliema, Malta, August 17-19, 2015. pp. 42-51.
- [21] Buike M., Buikis A., Kalis H. Wave energy and steel quenching models, which are solved exactly and approximately. *Mathematical and Computational Methods in Applied Sciences*. Proceedings of the 3rd International Conference on Applied, Numerical and Computational Mathematics (*ICANCM'15*). Sliema, Malta, August 17-19, 2015. pp. 72-81.
- [22] A. Buikis, H. Kalis. Hyperbolic Heat Equation in Bar and Finite Difference Schemes of Exact Spectrum. *Latest Trends on Theoretical and Applied Mechanics, Fluid Mechanics and Heat & Mass Transfer*. WSEAS Press, 2010. pp. 142-147.
- [23] M. Buike, A. Buikis, H. Kalis. Time Direct and Time Inverse Problems for Wave Energy and Steel Quenching Models, Solved Exactly and Approximately. *WSEAS Transactions on Heat and Mass Transfer*. Vol. 10, 2015. p. 30-43.
- [24] Buikis A., Buike M. Some analytical 3-D steady-state solutions for systems with rectangular fin. *IASME Transactions*. Issue 7, Vol. 2, September 2005, pp. 1112-1119.
- [25] M. Lencmane and A. Buikis. Analytical solution of a two-dimensional double-fin assembly. *Recent Advances in Fluid Mechanics and Heat & Mass Transfer*, Proceedings of the 9th IASME/WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment (HTE'11), Florence, Italy, August 23 – 25, 2011, pp. 396 – 401.
- [26] M. Lencmane and A. Buikis. Analytical solution for steady stable and transient heat process in a double-fin assembly. *International Journal of Mathematical Models and Methods in Applied Sciences* 6 (2012), No. 1. p.81 – 89.
- [27] M. Lencmane, A. Buikis. Some new mathematical models for the Transient Hot Strip method with thin interlayer. *Proc. of the 10th WSEAS Int. Conf. on Heat Transfer, Thermal*

- engineering and environment* (HTE '12), WSEAS Pres, 2012. p. 283-288.
- [28] Wang L., Zhou X., Wei X. *Heat Conduction. Mathematical Models and Analytical Solutions*. Springer, 2008.
- [29] Ekergard B., Castellucci V., Savin A., Leijon M. Axial Force Damper in a Linear Wave Energy Convertor. *Development and Applications of Oceanic Engineering*. Vol. 2, Issue 2, May 2013.
- [30] Wikipedia:
http://en.wikipedia.org/wiki/Wave_power.
- [31] Buikis, A. Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems. *Advanced Computational Methods in Heat Transfer*, IX. WIT Press, 2006. p. 311-320.
- [32] Vilums, R., Buikis, A. Conservative averaging method for partial differential equations with discontinuous coefficients. *WSEAS Transactions on Heat and Mass Transfer*. Vol. 1, Issue 4, 2006, p. 383-390.
- [33] Kobasko N.I. Real and effective heat transfer coefficients (HTCs) used for computer simulation of transient nuclear boiling processes during quenching. *Materials Performance and Characteristics*, Vol.1, No. 1, June 2012. p. 1-20.
- [34] Roach G.F. *Green's Functions*. Cambridge University Press, 1999.
- [35] Carslaw, H.S., Jaeger, C.J. *Conduction of Heat in Solids*. Oxford, Clarendon Press, 1959.
- [36] Stakgold I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, Inc., 1998.
- [37] Polyanin A.D. *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. Chapman & Hall/CRC, 2002. (Russian edition, 2001).
- [38] Debnath L. *Nonlinear Partial Differential Equations for Scientists and Engineers*. 2nd ed. Birkhäuser, 2005.
- [39] Buikis A., Guseinov S. Some one-dimensional coefficients inverse model problems of the heat transfer. – *Proceedings of the Latvian Academy of Sciences*, Sec. B, **57**, No 3/4 (626), 2003. p. 133-137.
- [40] Buikis A., Guseinov Sh. Solution of Reverse Hyperbolic Heat equation for intensive carburized steel quenching. *Proceedings of ICCES'05 (Advances in Computational and Experimental Engineering and Sciences)*. December 1-6, IIT Madras, 2005. p. 741-746.
- [41] Buikis A., Guseinov Sh., Buike M. Modelling of Intensive Steel Quenching Process by Time Inverse Hyperbolic Heat Conduction. *Proceedings of the 4th International Scientific Colloquium "Modelling for Material Processing"*. Riga, June 8-9, 2006. p. 169-172.
- [42] Buikis A., Guseinov Sh. Conservative averaging method for solutions of inverse problems of mathematical physics. *Progress in Industrial Mathematics at ECMI 2002*. A. Buikis, R. Ciegis, A. D. Fitt (Eds.), Springer, 2004. p. 241-246.
- [43] Buikis, A. "Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems". *Advanced Computational Methods in Heat Transfer*, IX. WIT Press, 2006. p. 311-320.
- [44] Buikis A. „Aufgabenstellung und Lösung einer Klasse von Problemen der mathematischer Physik mit nichtklassischen Zusatzbedingungen“. *Rostock Math. Colloquium*, 1984, 25, S. 53-62. (In Germany).