

Three-dimensional velocity field for blood flow using the power-law viscosity function

FERNANDO CARAPAU

Universidade de Évora
Departamento de Matemática and CIMA-UE
Rua Romão Ramalho, 59, 7001-671 Évora
PORTUGAL
flc@uevora.pt

RICARDO CONCEICAO

Universidade de Évora
Renewable Energies Chair and IES-UE
Palácio Vimioso, 7002-554 Évora
PORTUGAL
rfc@uevora.pt

Abstract: The three-dimensional model associated with blood flow where viscosity depends on shear-rate, such power-law type dependence, is a complex model to study in terms of computational optimization, which in many relevant situations becomes infeasible. In order to simplify the three-dimensional model and as an alternative to classic one-dimensional models, we will use the Cosserat theory related with fluid dynamics to approximate the velocity field and thus obtain a one-dimensional system consisting of an ordinary differential equation depending only on time and on a single spatial variable, the flow axis. From this reduced system, we obtain the unsteady equation for the mean pressure gradient depending on the volume flow rate, Womersley number and the flow index over a finite section of the tube geometry. Attention is focused on some numerical simulations for constant and non-constant mean pressure gradient using a Runge-Kutta method and on the analysis of perturbed flows. In particular, given a specific data we can get information about the volume flow rate and consequently we can illustrate the three-dimensional velocity field on the constant circular cross-section of the tube. Moreover, we compare the three-dimensional exact solution for steady volume flow rate with the corresponding one-dimensional solution obtained by the Cosserat theory.

Key-Words: Cosserat theory, blood flow, shear-thinning fluid one-dimensional model, power-law model, volume flow rate, mean pressure gradient.

1 Introduction

The study of blood flow in the cardiovascular system has always fascinated scientists, particularly mathematicians, physicists, biologists, chemists and engineers. Also, the fascination for the subject demonstrated by the artists is relevant, where the anatomical drawings of the genius of Leonardo Da Vinci are highlighted. The earliest records of this study date back to 3500 BC in the Egyptian period, later becoming somewhat more elaborate in the Greek, Roman and Islamic periods. After that, in the seventeenth century, the European scientist William Harvey was the first to present an accurate explanation on blood flow in the cardiovascular system in his work entitled *Exercitatio anatomica de motu cordis et sanguinis in animalibus*, see [1]. In our days, we know that blood is a very complex fluid and the mathematical modelling of blood flow is a difficult and challenging problem. The human circulatory system is a closed network of vessels carrying blood. Blood is a suspension of particles (mainly red blood cells, white blood cells and platelets) in a fluid called plasma and the vessels can be regarded as hollow tubes with differ-

ent scales. Generally, blood may be considered as a homogeneous fluid. In large arteries, blood may exhibit a standard behavior of a Newtonian fluid and the wall may be considered elastic (or mildly viscoelastic). The small arteries are characterized by a strong branching and may, in general, be considered rigid. Here, blood exhibits non-Newtonian phenomenon due to shear-thinning viscosity (see e.g. Chien et al. [2, 3]) and viscoelasticity effects, mainly stress relaxation and normal stress difference effects, see Thurston [4]. Moreover, when we consider arterioles, capillaries and venules, the microstructure and rheological behavior of blood cannot be avoided since the dimension of the blood particles are now of the same order of that of the vessel. Here, phenomena like aggregation and deformability of red blood cells have great influence on the rheological behavior of blood, especially on its viscosity at low shear-rates, and blood must be considered as a shear-thinning and viscoelastic fluid see e.g. Chien et al. [2, 3] and, Baskurt and Meiselman [5]. Modelling blood flow through the human circulatory system is certainly a very complex problem. Among others, we refer to the following difficult

ties: the complex geometry of the vessels, complex rheological behavior of blood, pulsatility of the flow field and consequently pulsatility of the walls, complex inelastic permeable walls, different deformability of the red cells at different shear-rates and the lack of boundary data to close the corresponding mathematical models. As a consequence of all these complex aspects related to the three-dimensional model to study blood flow the idea in this article is to present an alternative theory to reduce the three-dimensional model to a one-dimensional model. In historical point of view, Euler in the eighteenth century was the first to introduce one-dimensional models to study blood flow in the human circulatory system, see [6]. The classical one-dimensional models governing equations can be obtained by first integrating the incompressibility condition and the axial component of linear momentum for the blood flow field over the circular cross-section of the tube and introducing some assumptions related with the nonlinear convective acceleration and the viscous dissipation terms. These closure approximations are typically based on assuming a purely axial flow with a fixed dependence on axial variables, for more detail, see e.g. [7, 8, 9]. As an alternative, we consider one-dimensional models obtained by the Cosserat theory (also called director theory) associated with fluid dynamics (see Caulk and Naghdi [10]), where the only approximation in account is related with the three-dimensional velocity field. The reduced model is obtained by integrating the linear momentum equation over the circular cross-section of the tube, taking a velocity field approximation provided by the Cosserat theory. This procedure yields a one-dimensional system, depending only on time and a single spatial variable. The velocity field approximation satisfies exactly both the incompressibility condition and the kinematic boundary condition. A detailed discussion about this director theory can be found in the work of Naghdi et al. [11, 12]. The relevance of using a director theory is not in regarding it as an approximation to three-dimensional equations, but rather in their use as independent theories to predict some of the main properties of the three-dimensional problems. Advantages of the director theory include: the theory incorporates all components of the linear momentum; it is a hierarchical theory, making it possible to increase the accuracy of the model; there is no need for closure approximations; the flow field is not assumed to be uni-directional; invariance under superposed rigid body motions is satisfied at each order and the wall shear stress enters directly in the formulation as a dependent variable. Here we are interested in studying the initial boundary value problem for incompressible homogeneous power-law fluid to model blood flow in a straight circular rigid and im-

permeable walled vessel with constant radius. Using this director theory, we can intend to predict the main properties of a three-dimensional given problem, where the fluid three-dimensional velocity field¹

$$\boldsymbol{\vartheta} = \boldsymbol{\vartheta}(\boldsymbol{x}, t) = \vartheta_i \boldsymbol{e}_i$$

is approximated by² (see [10]):

$$\boldsymbol{\vartheta} = \boldsymbol{v} + \sum_{N=1}^k x_{\alpha_1} \dots x_{\alpha_N} \boldsymbol{W}_{\alpha_1 \dots \alpha_N}, \quad (1)$$

with

$$\boldsymbol{v} = v_i(z, t) \boldsymbol{e}_i, \quad \boldsymbol{W}_{\alpha_1 \dots \alpha_N} = W_{\alpha_1 \dots \alpha_N}^i(z, t) \boldsymbol{e}_i. \quad (2)$$

This velocity field approximation (1) satisfies both the incompressibility condition and the kinematic boundary condition exactly. In condition (1), \boldsymbol{v} represents the velocity along the axis of symmetry z at time t , $x_{\alpha_1} \dots x_{\alpha_N}$ are the polynomial weighting functions with order k (this number identifies the order of the hierarchical theory and is related to the number of directors), the vectors $\boldsymbol{W}_{\alpha_1 \dots \alpha_N}$ are the director velocities which are symmetric with respect to their indices and \boldsymbol{e}_i are the associated unit basis vectors. The selection of such weighting functions represents an important aspect of the formulation of our problem. A good choice of these weighting functions can reduce the complexity of the system of ordinary differential equations in the director formulation of the theory. This choice should be consistent with the hierarchical structure of the basic theory so that the equations for each level of the hierarchy include the equations of all lower orders. The vectors $\boldsymbol{W}_{\alpha_1 \dots \alpha_N}$ are related to physical features of the fluid. Using this approach with nine directors (i.e., $k = 3$ at condition (1)) and integrating the equations for the conservation of linear momentum over a circular cross-section of the fluid domain, we obtain the unsteady equation for the mean pressure gradient depending on the volume flow rate, Womersley number and the flow index over a finite section of the tube geometry. Attention is focused on some numerical simulations for constant and non-constant mean pressure gradient using a Runge-Kutta method and on the analysis of perturbed flows. In particular, given a specific data we can get information about the volume flow rate and consequently we can illustrate the three-dimensional velocity field on

¹Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be the rectangular space cartesian coordinates (for convenience we set $x_3 = z$) and t is the time variable.

²In the sequel, latin indices subscript take the values 1, 2, 3; greek indices subscript 1, 2, and the usual summation convention is employed over a repeated index.

the constant circular cross-section of the tube. Moreover, we compare the three-dimensional exact solution for steady volume flow rate with the corresponding one-dimensional solution obtained by the Cosserat theory. Recently, a director theory approach for modeling blood flow in the arterial system, as an alternative to the classical one-dimensional models, has been introduced by Sequeira et al. [13, 14]. At work [13] blood is considered as a Newtonian fluid and in the work [14], blood is considered as a non-Newtonian fluid where viscosity depends on the shear-rate. This article is a review work based on [14] where new results will be presented. This new results are concerning to the behavior of the three-dimensional velocity field in the circular cross-section of a tube with constant radius along the axis of the flow and on the analysis of perturbed flows.

2 Governing equations

Let us model blood as a homogeneous shear-thinning fluid moving within a straight and impermeable rigid tube of constant circular cross-section, the vessel domain Ω , see Figure 1. The boundary $\partial\Omega$ is composed

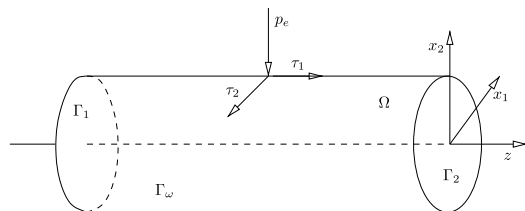


Figure 1: Fluid domain Ω with normal and tangential components of the surface traction vector p_e and τ_1 , τ_2 with constant circular cross-section along the axis of symmetry z .

by the proximal cross-section Γ_1 , by the distal cross-section Γ_2 and by the lateral wall of the tube Γ_w , define by the constant scalar function³ ϕ , which is related to the circular cross-section of the tube by the following relationship

$$\phi^2 = x_1^2 + x_2^2. \quad (3)$$

For our haemodynamics problem, the equations of axisymmetric motion, stating the conservation of linear momentum without body forces and mass are

³In the general case when the scalar function ϕ depends on the spatial variable z and time t we have a fluid-structure interaction problem.

given, in $\Omega \times (0, T)$, by

$$\begin{cases} \rho \left(\frac{\partial \vartheta}{\partial t} + \vartheta \cdot \nabla \vartheta \right) = \nabla \cdot \mathbf{T}, \\ \nabla \cdot \vartheta = 0, \\ \mathbf{T} = -p\mathbf{I} + \mu(|\dot{\gamma}|) \mathbf{A}_1, \quad \mathbf{t}_w = \mathbf{T} \cdot \boldsymbol{\eta}, \end{cases} \quad (4)$$

with the initial condition

$$\vartheta(\mathbf{x}, 0) = \vartheta_0(\mathbf{x}) \quad \text{in } \Omega, \quad (5)$$

and the homogeneous Dirichlet boundary condition

$$\vartheta(\mathbf{x}, t) = 0 \quad \text{on } \Gamma_w \times (0, T), \quad (6)$$

where ρ is the constant fluid density, p is the pressure, $-p\mathbf{I}$ is the spherical part of the stress due to the constraint of incompressibility, $\mu(|\dot{\gamma}|)$ is the shear-dependent viscosity function and

$$\mathbf{A}_1 = \nabla \vartheta + (\nabla \vartheta)^T \quad (7)$$

is the first Rivlin-Ericksen tensor, $\nabla \vartheta$ is the spatial velocity gradient and $(\nabla \vartheta)^T$ is the transpose of $\nabla \vartheta$. Equation (4)₁ represents the balance of linear momentum without body forces and (4)₂ is the incompressibility condition. In equation (4)₃, \mathbf{T} is the constitutive equation associated with a generalized Newtonian fluid and \mathbf{t}_w denotes the stress vector on the surface whose outward unit normal vector is $\boldsymbol{\eta}(\mathbf{x}, t) = \eta_i(\mathbf{x}, t)\mathbf{e}_i$. The components of the outward unit normal vector to the surface of the vessel domain Ω are given by

$$\eta_1 = \frac{x_1}{\phi}, \quad \eta_2 = \frac{x_2}{\phi}, \quad \eta_3 = 0. \quad (8)$$

Concerning the shear-dependent viscosity function

$$\mu(|\dot{\gamma}|) : \mathbb{R}^+ \rightarrow \mathbb{R}^+,$$

$\dot{\gamma}$ is a scalar measure of the rate of shear defined by $|\dot{\gamma}| = \sqrt{2\mathbf{D} : \mathbf{D}}$ with

$$\mathbf{D} := \frac{1}{2}(\nabla \vartheta + (\nabla \vartheta)^T)$$

being the rate of deformation tensor. The particular functional dependence on shear-rate is generally chosen in order to fit experimental data. In this work, we consider a power-law fluid model, i.e.,

$$\mu(|\dot{\gamma}|) = k|\dot{\gamma}|^{n-1} \quad (9)$$

where the parameters k and n are positive constants called the consistency and the flow index, respectively. If $n = 1$ in (9), the viscosity is a constant k

and blood is modeled as a Newtonian fluid. If $n < 1$ at (9) then

$$\lim_{|\dot{\gamma}| \rightarrow +\infty} \mu(|\dot{\gamma}|) = 0, \quad \lim_{|\dot{\gamma}| \rightarrow 0} \mu(|\dot{\gamma}|) = +\infty,$$

and we have a shear-thinning fluid behavior, i.e., the viscosity decreases monotonically with shear-rate. For $n > 1$ at (9), we get

$$\lim_{|\dot{\gamma}| \rightarrow +\infty} \mu(|\dot{\gamma}|) = +\infty, \quad \lim_{|\dot{\gamma}| \rightarrow 0} \mu(|\dot{\gamma}|) = 0,$$

and the fluid shows a shear-thickening behavior, i.e., the viscosity increases with shear-rate. This theoretical model has limited applications to real fluid due to the unboundedness of the viscosity function, but is widely used and can be accurate for specific flow regimes. The theoretical study of the model (4) – (6) with (9), namely existence, uniqueness and regularity of classical and weak solutions still poses some difficulties. In this work we are interested in the numerical study of the model (4) – (6) with (9), using the director approach related to fluid dynamics. Since equation (3) defines a material surface, the three-dimensional velocity field ϑ must satisfy the kinematic condition⁴

$$\frac{d}{dt}(\phi^2 - x_1^2 - x_2^2) = 0$$

i.e.,

$$-x_1\vartheta_1 - x_2\vartheta_2 = 0 \quad (10)$$

on the boundary defined by (3). Averaged quantities such as volume flow rate and pressure are needed to study one-dimensional models. Consider $S = S(z, t)$ a generic axial section of the domain Ω at time t defined by the spatial variable z , bounded by the circle defined by (3), and let $A(z, t)$ be the area of this section $S(z, t)$. Then, the volume flow rate Q is defined by

$$Q(z, t) = \int_{S(z,t)} \vartheta_3(\mathbf{x}, t) da, \quad (11)$$

and the average pressure \bar{p} , by

$$\bar{p}(z, t) = \frac{1}{A(z, t)} \int_{S(z,t)} p(\mathbf{x}, t) da. \quad (12)$$

Starting with representation (1) it follows (see [10]) that the approximation of the three-dimensional velocity field $\vartheta = \vartheta_i(\mathbf{x}, t)\mathbf{e}_i$ using nine directors, is given by

$$\begin{aligned} \vartheta = & \left[x_1(\xi + \sigma(x_1^2 + x_2^2)) - x_2(\omega + \eta(x_1^2 + x_2^2)) \right] \mathbf{e}_1 \\ & + \left[x_1(\omega + \eta(x_1^2 + x_2^2)) + x_2(\xi + \sigma(x_1^2 + x_2^2)) \right] \mathbf{e}_2 \\ & + \left[v_3 + \gamma(x_1^2 + x_2^2) \right] \mathbf{e}_3, \end{aligned} \quad (13)$$

⁴The material time derivative is given by $\frac{d}{dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \vartheta \cdot \nabla(\cdot)$.

where $\xi, \omega, \gamma, \sigma, \eta$ are scalar functions of the spatial variable z and time t . The physical significance of these scalar functions in (13) is the following: γ is related to transverse shearing motion, ω and η are related to rotational motion (also called swirling motion) about \mathbf{e}_3 , while ξ and σ are related to transverse elongation. We use nine directors because it is the minimum number for which the incompressibility condition and the kinematic boundary conditions on the lateral surface of the tube are satisfied pointwise. Using the velocity approach (13), the kinematic conditions (10) on the lateral boundary reduce to

$$-\phi^2(\xi + \phi^2\sigma) = 0 \quad (14)$$

and the incompressibility condition given by equation (4)₂ becomes

$$(v_3)_z + 2\xi + (x_1^2 + x_2^2)(\gamma_z + 4\sigma) = 0, \quad (15)$$

where the subscripted variable denotes partial differentiation. For equation (15) to hold at every point in the fluid the velocity coefficient must satisfy the separate conditions

$$(v_3)_z + 2\xi = 0, \quad \gamma_z + 4\sigma = 0. \quad (16)$$

Hence the boundary condition (10) and the incompressibility condition given by equation (4)₂ are satisfied exactly by the velocity field (13) if we impose the conditions (14) and (16). On the wall boundary of the rigid tube we impose the no-slip boundary condition requiring that the velocity field (13) vanishes identically on the surface (3), i.e., condition (6) is satisfied. Thus, it follows that

$$\xi + \phi^2\sigma = 0, \quad \omega + \phi^2\eta = 0, \quad v_3 + \phi^2\gamma = 0. \quad (17)$$

Therefore, equation (14) is satisfied identically and the two incompressibility conditions (16) reduce to

$$(v_3)_z + 2\xi = 0, \quad (\phi^2 v_3)_z = 0. \quad (18)$$

Considering the flow in a rigid tube with constant circular cross-section given by surface (3) without swirling motion (i.e., $\omega = \eta = 0$), conditions (11), (13), (17) and (18) then, the volume flow rate Q is just a function of time t , given by

$$Q(t) = \frac{\pi}{2} \phi^2 v_3(z, t) \quad (19)$$

and, consequently, the velocity field (13) can be rewritten as

$$\vartheta(\mathbf{x}, t) = \frac{2Q(t)}{\pi\phi^2} \left(1 - \frac{x_1^2 + x_2^2}{\phi^2} \right) \mathbf{e}_3, \quad (20)$$

and the initial condition (5) is satisfied when we consider in applications $Q(0) = cts$. To simplify our study, it is convenient to resolve the stress vector t_w on the lateral surface in terms of it is outward unit normal $\boldsymbol{\eta}$ and in terms of the components of the surface traction vector τ_1, τ_2 and p_e in the form

$$t_w = \tau_1 \boldsymbol{\lambda} - p_e \boldsymbol{\eta} + \tau_2 \mathbf{e}_\theta, \quad (21)$$

where τ_1 is the wall shear stress, while $\boldsymbol{\lambda}$ and \mathbf{e}_θ are the unit tangent vectors define by

$$\boldsymbol{\lambda} = \boldsymbol{\eta} \times \mathbf{e}_\theta, \quad \mathbf{e}_\theta = (x_\alpha / \phi) \mathbf{e}_{\alpha\beta} \mathbf{e}_\beta, \quad (22)$$

with $e_{11} = e_{22} = 0$ and $e_{12} = -e_{21} = 1$ (see [10]). Using conditions (8) and (22), the expression for the stress vector (21) can be rewritten in terms of it is rectangular Cartesian components as

$$t_w = \frac{1}{\phi} (-p_e x_1 - \tau_2 x_2) \mathbf{e}_1 + \frac{1}{\phi} (-p_e x_2 + \tau_2 x_1) \mathbf{e}_2 + \tau_1 \mathbf{e}_3. \quad (23)$$

Now, instead of the momentum equation (4)₁ be verified pointwise in the fluid we impose the following integral conditions (see [10])

$$\int_S \left[\nabla \cdot \mathbf{T} - \rho \left(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta} \right) \right] da = 0, \quad (24)$$

$$\int_S \left[\nabla \cdot \mathbf{T} - \rho \left(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta} \right) \right] x_{\alpha_1} \dots x_{\alpha_N} da = 0, \quad (25)$$

where $N = 1, 2, 3$. Using the divergence theorem and a form of Liebnitz rule, equations (24) and (25) for nine directors, can be reduced to the following vector equations:

$$\frac{\partial \mathbf{n}}{\partial z} + \mathbf{f} = \mathbf{a} \quad (26)$$

and

$$\frac{\partial \mathbf{m}^{\alpha_1 \dots \alpha_N}}{\partial z} + \mathbf{l}^{\alpha_1 \dots \alpha_N} = \mathbf{k}^{\alpha_1 \dots \alpha_N} + \mathbf{b}^{\alpha_1 \dots \alpha_N}, \quad (27)$$

where \mathbf{n} , $\mathbf{k}^{\alpha_1 \dots \alpha_N}$, $\mathbf{m}^{\alpha_1 \dots \alpha_N}$ are resultant forces define by

$$\mathbf{n} = \int_S \mathbf{T}_3 da, \quad \mathbf{k}^\alpha = \int_S \mathbf{T}_\alpha da, \quad (28)$$

$$\mathbf{k}^{\alpha\beta} = \int_S \left(\mathbf{T}_\alpha x_\beta + \mathbf{T}_\beta x_\alpha \right) da, \quad (29)$$

$$\mathbf{k}^{\alpha\beta\gamma} = \int_S \left(\mathbf{T}_\alpha x_\beta x_\gamma + \mathbf{T}_\beta x_\alpha x_\gamma + \mathbf{T}_\gamma x_\alpha x_\beta \right) da \quad (30)$$

and

$$\mathbf{m}^{\alpha_1 \dots \alpha_N} = \int_S \mathbf{T}_3 x_{\alpha_1} \dots x_{\alpha_N} da. \quad (31)$$

The quantities \mathbf{a} and $\mathbf{b}^{\alpha_1 \dots \alpha_N}$ are inertia terms define by

$$\mathbf{a} = \int_S \rho \left(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta} \right) da, \quad (32)$$

$$\mathbf{b}^{\alpha_1 \dots \alpha_N} = \int_S \rho \left(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta} \right) x_{\alpha_1} \dots x_{\alpha_N} da \quad (33)$$

and \mathbf{f} , $\mathbf{l}^{\alpha_1 \dots \alpha_N}$, which arise due to surface traction on the lateral boundary, are define by

$$\mathbf{f} = \int_{\partial S} t_w ds, \quad (34)$$

$$\mathbf{l}^{\alpha_1 \dots \alpha_N} = \int_{\partial S} t_w x_{\alpha_1} \dots x_{\alpha_N} ds. \quad (35)$$

The equation for the mean pressure gradient will be obtained using the resulting quantities from (28) to (35) on equations (26) – (27). On equations (34) – (35) we will apply the stress vector t_w given by (23). Now, using the velocity field (20), the surface (3), the volume flow rate (19), the incompressibility constraints (16), no-slip conditions (17)_{1,3} and the stress vector (23) in equations (28) to (35), we can explicitly calculate the forces \mathbf{n} , $\mathbf{k}^{\alpha_1 \dots \alpha_N}$, $\mathbf{m}^{\alpha_1 \dots \alpha_N}$, the inertia terms \mathbf{a} , $\mathbf{b}^{\alpha_1 \dots \alpha_N}$ and the surface tractions \mathbf{f} , $\mathbf{l}^{\alpha_1 \dots \alpha_N}$. Hence, plugging these solutions into equations (26) – (27) and using equation (12), we get by solving a linear system the unsteady equation for the average pressure gradient, given by

$$\bar{p}_z(z, t) = -\frac{4\rho Q_t(t)}{3\pi\phi^2} - \frac{4k(2^{\frac{5n+1}{2}})Q^n(t)}{(n+3)\pi^n\phi^{3n+1}}. \quad (36)$$

Integrating equation (36), over a finite section of the tube with $z_1 < z_2$, we get the mean pressure gradient over the interval $[z_1, z_2]$ at time t , given by

$$G(t) = \frac{\bar{p}(z_1, t) - \bar{p}(z_2, t)}{z_2 - z_1} = \frac{4\rho Q_t(t)}{3\pi\phi^2} + \frac{4k(2^{\frac{5n+1}{2}})Q^n(t)}{(n+3)\pi^n\phi^{3n+1}}. \quad (37)$$

Now, let us consider the following dimensionless variables

$$\hat{t} = \omega_0 t, \quad \hat{Q} = \frac{2\rho}{\pi\phi k} Q, \quad \hat{G} = \frac{\rho^n \phi^{2n+1}}{k^{n+1}} G, \quad (38)$$

where ω_0 is the characteristic frequency for unsteady flow. In the cases where a steady volume flow rate is specified the nondimensional volume flow rate \hat{Q} is identical to the classical \mathcal{R}_e (Reynolds number) used for flow in tubes. Substituting the new variables (38)

in equation (37), we obtain the nondimensional mean pressure gradient

$$\hat{G}(\hat{t}) = \frac{2}{3} \mathcal{W}_o^2 \hat{Q}_{\hat{t}}(\hat{t}) + \frac{2^{\frac{3n+5}{2}}}{n+3} \hat{Q}^n(\hat{t}), \quad (39)$$

where $\mathcal{W}_o = \phi^n \sqrt{\rho^n \omega_0 / k^n}$ is the Womersley number, which is the most commonly used parameter to reflect the pulsatility of the flow. Using (38)₂ and

$$\hat{x}_1 = \frac{x_1}{\phi}, \quad \hat{x}_2 = \frac{x_2}{\phi}, \quad \hat{z} = \frac{z}{\phi}, \quad \hat{\vartheta} = \frac{\phi \rho}{k} \vartheta \quad (40)$$

at the velocity equation (20), we get the nondimensional three-dimensional velocity field

$$\hat{\vartheta}(\hat{x}, \hat{t}) = \hat{Q}(\hat{t}) \left(1 - (\hat{x}_1^2 + \hat{x}_2^2) \right) e_3. \quad (41)$$

From equation (39), the volume flow rate in the steady case is given by

$$\hat{Q} = \sqrt[n]{\frac{n+3}{2^{\frac{3n+5}{2}}} \hat{G}}. \quad (42)$$

In order to evaluate the flow predictions of the one-dimensional theory developed here, we next consider the exact three-dimensional volume flow rate of an axisymmetric steady flow through a straight tube with constant circular cross-section, given by (see Bird et al. [15])

$$\tilde{Q} = \frac{n}{3n+1} \sqrt[n]{2^{n-1} \hat{G}}. \quad (43)$$

Next, we present numerical simulations associated with equations (39), (41), (42) and (43) for specific flow regimes.

3 Numerical results

In this session we will present numerical results associated to equation (39) where the mean pressure gradient will be given and we will check the evolution of the volume flow rate for specific flow regimes. Consequently, we present the behavior of the three-dimensional velocity field (41) in the circular cross-section of the tube. Finally, considering the steady case, we will compare the exact solution (43) with the approximate solution (42), validating in this way and for specific data the Cosserat theory as a valid alternative for the three-dimensional study of a specific physical model associated to the flow of a fluid

3.1 Steady problem

In the steady case, let us compare the exact volume flow rate solution (43) with the volume flow rate approximate solution (42). In Figure 2, we can see the steady volume flow rate (42) behavior for a fixed value of the mean pressure gradient as a function of the flow index, concluding that the solution converges to a certain value as we increase the flow index for a given mean pressure gradient. Comparing the ap-

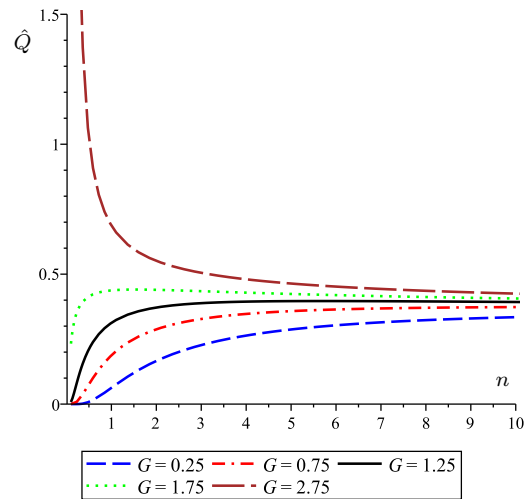


Figure 2: Variation of the steady volume flow rate (42) as a function of flow index with fixed mean pressure gradient.

proximate solution for volume flow rate with the exact solution we can conclude that the approximation is excellent for the case of shear-thinning fluid which is the relevant case in this work, see Figure 3. Considering this comparison, we can state that in this case the Cosserat theory is valid for flow index values such that $0 \ll n < 1$ with small range of mean pressure gradient, the situation $n \rightarrow 0$ is neglected because it has no physical meaning in this work. In the case of shear-thickening fluid the comparison is no longer relevant as the flow index increases, see Figure 3. Also, we can conclude that the approximation solution (42) is not relevant when we increase the mean pressure gradient, see Figure 3 and Figure 4. For blood flow the range of the mean pressure gradient is very small.

Next, we will illustrate the behavior of the three-dimensional steady velocity field (41) where the steady volume flow rate (42) is given with \hat{G} and flow index fixed. In the case of $\hat{G} = 1.75$, Figure 5 and Figure 6 shown us very small variation on the intensity of the steady velocity (41) for shear-thinning fluid and shear-thickening fluid respectively. Next, we consider the behavior of the steady velocity (41) when we increase the constant mean pressure gradient

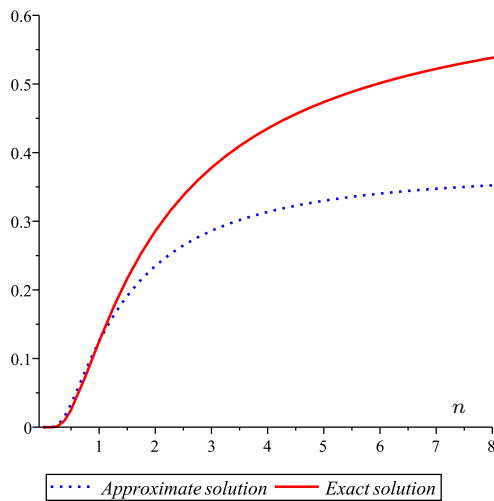


Figure 3: Comparison between the exact solution (43) and the approximate solution (42) for the steady volume flow rate as a function of flow index with small mean pressure gradient $\hat{G} = 0.5$.

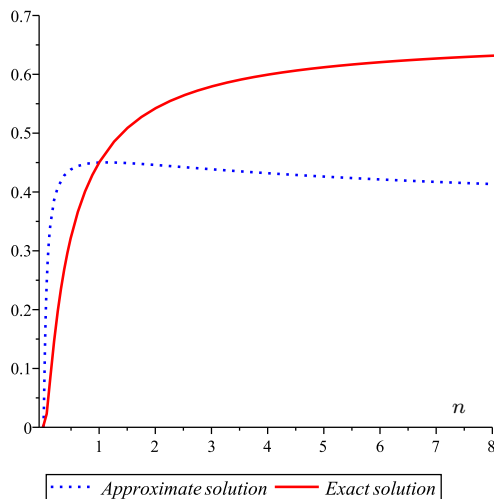


Figure 4: Comparison between the exact solution (43) and the approximate solution (42) for the steady volume flow rate as a function of flow index with mean pressure gradient $\hat{G} = 1.8$.

$\hat{G} = 2.75$, see Figure 7 and Figure 8. This results are in excellent agreement with the behavior of the steady volume flow rate shown in Figure 2.

3.2 Unsteady problem

In Figure 9, we can observe the behavior of the unsteady volume flow rate solution given by (39) obtained using a Runge-Kutta method with constant mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$ in the case of

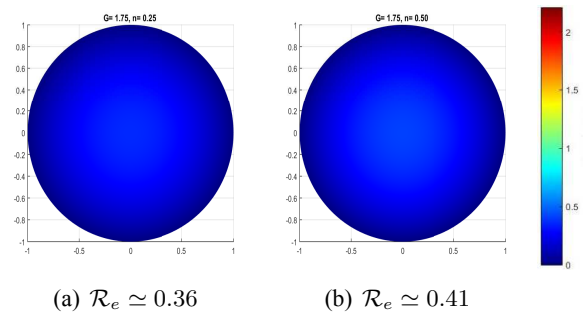


Figure 5: Three-dimensional steady velocity field (41) with steady volume flow rate (42), where $\mathcal{R}_e \simeq 0.36$ ($\hat{G} = 1.75, n = 0.25$) and $\mathcal{R}_e \simeq 0.41$ ($\hat{G} = 1.75, n = 0.5$)

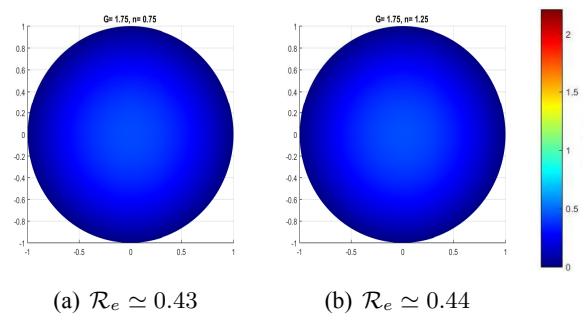


Figure 6: Three-dimensional steady velocity field (41) with steady volume flow rate (42), where $\mathcal{R}_e \simeq 0.43$ ($\hat{G} = 1.75, n = 0.75$) and $\mathcal{R}_e \simeq 0.44$ ($\hat{G} = 1.75, n = 1.25$).

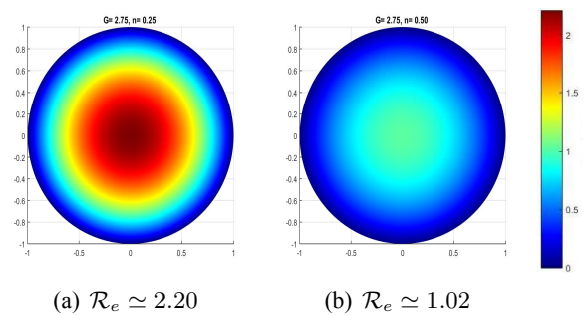


Figure 7: Three-dimensional steady velocity field (41) with steady volume flow rate (42), where $\mathcal{R}_e \simeq 2.20$ ($\hat{G} = 2.75, n = 0.25$) and $\mathcal{R}_e \simeq 1.02$ ($\hat{G} = 2.75, n = 0.5$)

shear-thinning fluid when we increase the Womersley number. We note that while the flow index n increases the amplitude of the solution in the initial transient phase increases and becomes less pronounced as the Womersley number increases. In this particular case of a constant mean pressure gradient, the system (39) converges toward a steady state solution. When we

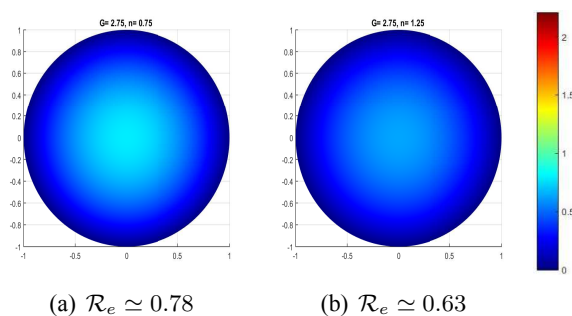


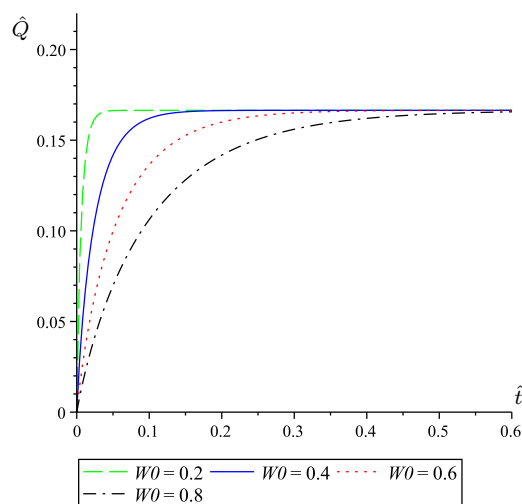
Figure 8: Three-dimensional steady velocity field (41) with steady volume flow rate (42), where $\mathcal{R}_e \simeq 0.78$ ($\hat{G} = 2.75, n = 0.75$) and $\mathcal{R}_e \simeq 0.63$ ($\hat{G} = 2.75, n = 1.25$).

move from a situation of shear-thinning fluid to shear-thickening fluid the behavior of the volume flow rate solution is similar, i.e., as we increase the flow index the amplitude of the solution in the initial transient phase increases, see Figure 10. Consequently, we can see that the behavior of the volume flow rate solution is similar for shear-thinning fluid and shear-thickening fluid respectively. But, in the case shear-thinning fluid the volume flow rate solution presents amplitudes of lower value in comparison with the case of a shear-thickening fluid such situation may be relevant in certain physical applications. Now, with the information of the volume flow rate (39) obtained for certain flow regimes we can return to the three-dimensional situation to obtain the behavior of the velocity field (41) in time in the circular cross-section of the tube.

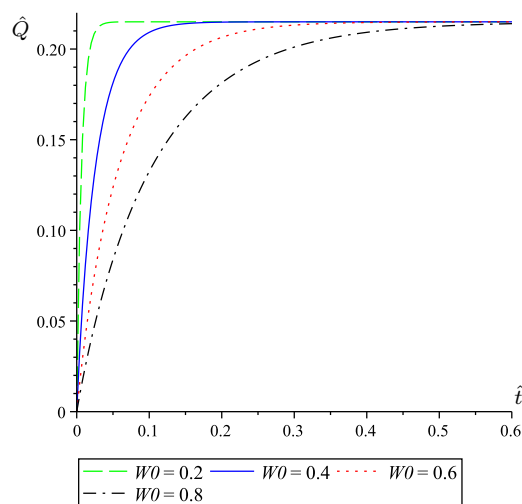
The Figure 11 and Figure 12, illustrate the three-dimensional velocity field (41) behavior in the circular cross-section of the tube in the initial transition phase, and we can see the increase of the velocity intensity as we increase the flow index in a situation of shear-thinning fluid. Also, we can see that the velocity field needs a very short time to stabilize at a constant intensity, and this is due to the initial transition phase. In our study, the shear-thinning fluid situation is taken into account for blood flow. In the case of shear-thickening fluid see Figure 13 and Figure 14, we can verify that the increase of the flow index from shear-thinning fluid to the shear-thickening fluid increases the velocity field intensity for the same initial transition phase with the same flow regimes. Next, we consider non-constant mean pressure gradient, given by equation (44)

$$\hat{G}(\hat{t}) = 1 + \frac{\sin^2(\hat{t})}{e^{\hat{t}}}, \quad (44)$$

which shows an interesting behavior, see Figure 15. More specifically it shows a strong variation in the



(a) Flow index $n = 0.6$.

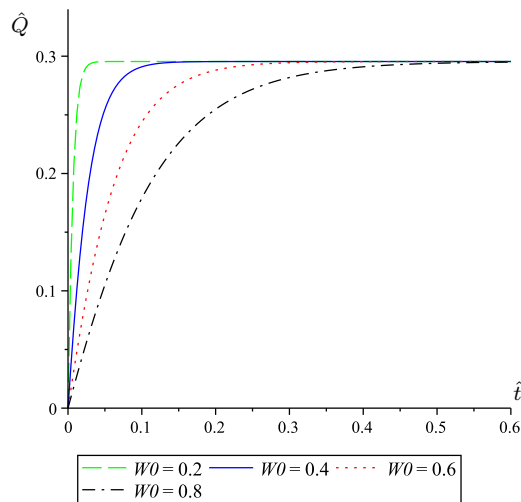


(b) Flow index $n = 0.8$.

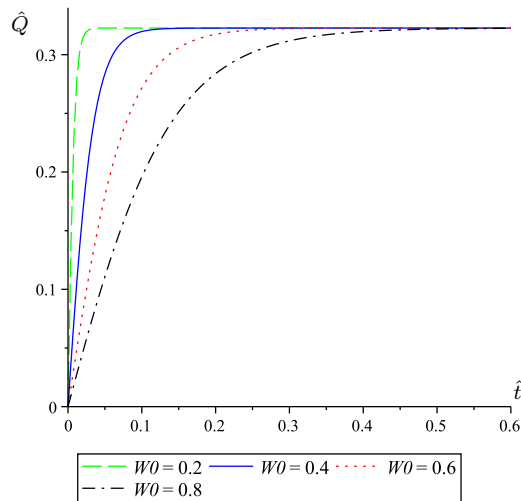
Figure 9: Unsteady volume flow rate (39) with constant mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$ where $\hat{Q}(0) = 0, \mathcal{W}_o = (0.2, 0.4, 0.6, 0.8)$ for shear-thinning fluid

initial stage and after the initial transient phase has small fluctuations which tend to decrease with time. Considering the mean pressure gradient (44) on equation (39) and using a Runge-Kutta method for specific flow regimes we can get information about the volume flow rate behavior. Shown in Figure 16, results for volume flow rate in the case of shear-thinning fluid and we can see the amplitude of the volume flow rate increase with the flow index in the initial transition phase, which tends to follow the behavior of the mean pressure gradient function (44), the fluctuation in the solution decreases in time.

Figure 17, shows us that in the initial transition



(a) Flow index $n = 1.4$.



(b) Flow index $n = 1.8$.

Figure 10: Unsteady volume fl w rate (39) with constant mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$ where $\hat{Q}(0) = 0$, $\mathcal{W}_o = (0.2, 0.4, 0.6, 0.8)$ for shear-thickening fluid

phase the amplitude of the volume fl w rate tries to increase with the increase of the fl w index. Therefore, Figure 16 and Figure 17 shows us the evolution of the volume fl w rate as we move from a shear-thinning fluid situation to a shear-thickening fluid situation. Next, we will see the behavior of the three-dimensional velocity field

The Figure 18 and Figure 19, shows us the three-dimensional velocity fiel (41) intensity for specifi fl w regimes in the case of shear-thinning fluid during the initial transition phase. In comparison with the constant mean pressure gradient case, we can see that in this case where the mean pressure gradient is non-

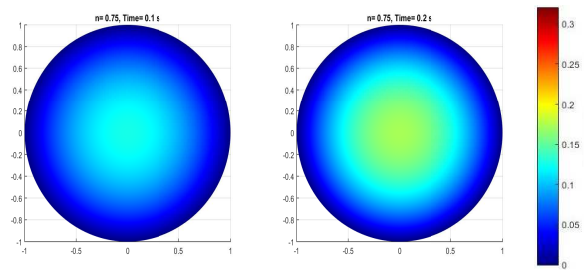


Figure 11: Three-dimensional velocity fiel (41) where the volume fl w rate is obtained by (39) with mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$, $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 0.75$ (shear-thinning fluid) Time parameters: $\hat{t} = 0.1, \hat{t} = 0.2$.

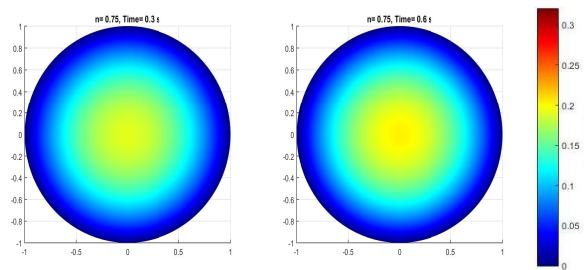


Figure 12: Three-dimensional velocity fiel (41) where the volume fl w rate is obtained by (39) with mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$, $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 0.75$ (shear-thinning fluid) Time parameters: $\hat{t} = 0.3, \hat{t} = 0.6$.

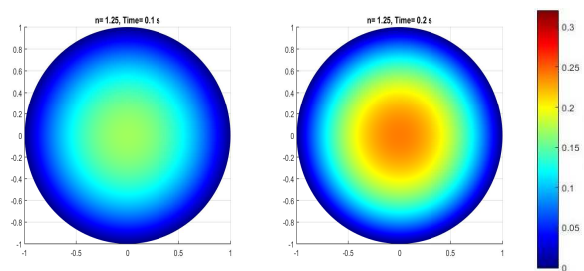


Figure 13: Three-dimensional velocity fiel (41) where the volume fl w rate is obtained by (39) with mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$, $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 1.25$ (shear-thickening fluid) Time parameters: $\hat{t} = 0.1, \hat{t} = 0.2$.

constant, the velocity fiel intensity increases during the initial transition phase. The same conclusions for the case of shear-thickening fluid see Figure 20 and Figure 21.

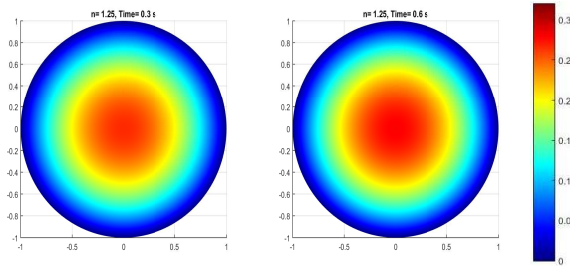


Figure 14: Three-dimensional velocity field (41) where the volume flow rate is obtained by (39) with mean pressure gradient $\hat{G}(\hat{t}) = \hat{G} = 1$, $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 1.25$ (shear-thickening fluid) Time parameters: $\hat{t} = 0.3, \hat{t} = 0.6$.

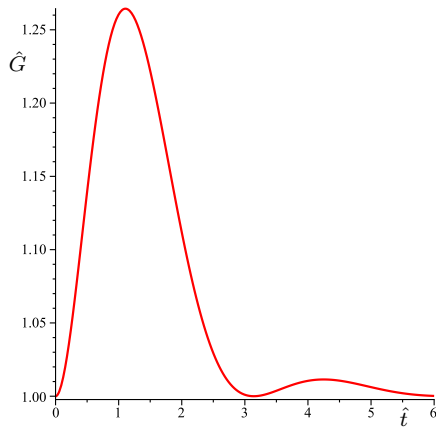


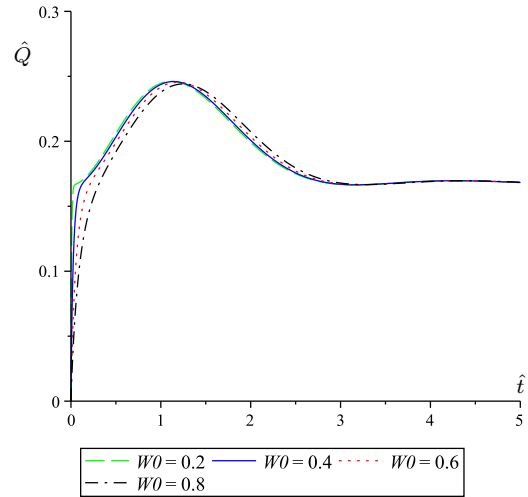
Figure 15: Non-constant mean pressure gradient given by (44).

4 Perturbations flows

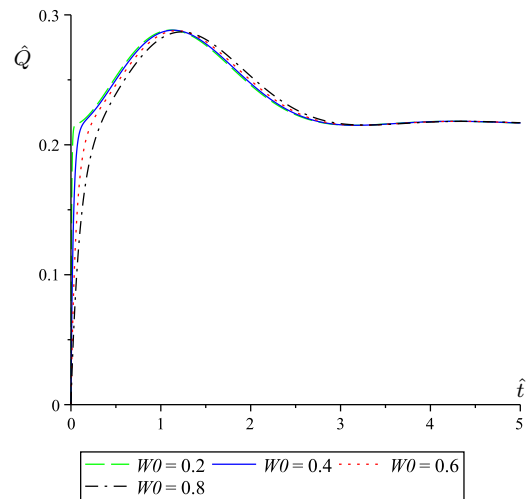
In many physical applications involving fluid flows in specific domains it is important to determine the changes in flow characteristics induced by perturbations in the initial or boundary data, body forces or pressure drop. In fact, since it is virtually impossible to maintain an exactly constant pressure drop, one should be able to predict how much a perturbation of given magnitude in pressure drop will affect the volume flow rate. Therefore, let us consider a uniform perturbation of magnitude ϵ at the function (44) (see Figure 22). For each $\epsilon > 0$, define the quantities,

$$\hat{G}_\epsilon^+(\hat{t}) = (1 + \epsilon)\hat{G}(\hat{t}), \quad \hat{G}_\epsilon^-(\hat{t}) = (1 - \epsilon)\hat{G}(\hat{t}), \quad (45)$$

we denote by \hat{Q}_ϵ^+ and \hat{Q}_ϵ^- the perturbed volume flow rates corresponding to \hat{G}_ϵ^+ and \hat{G}_ϵ^- , respectively. Next, we will consider a perturbation in the approximate solution obtained by the Cosserat theory and we will verify the stability of the one-dimensional solution.



(a) Flow index $n = 0.6$.



(b) Flow index $n = 0.8$.

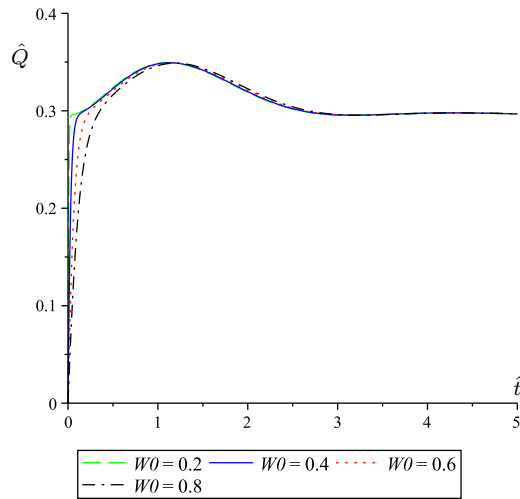
Figure 16: Unsteady volume flow rate (39) with non-constant mean pressure gradient (44) where $\hat{Q}(0) = 0$ and $\mathcal{W}_o = (0.2, 0.4, 0.6, 0.8)$ for shear-thinning fluid

4.1 Steady problem

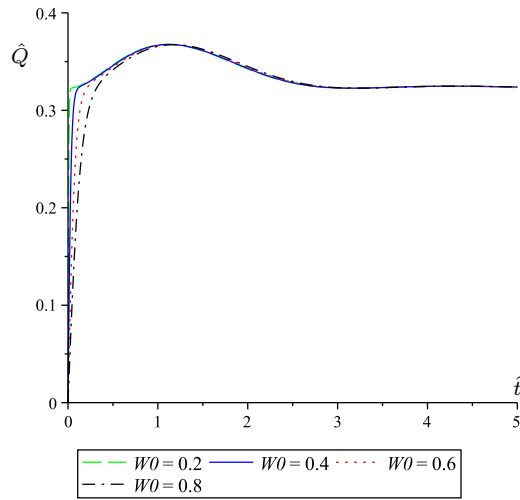
Considering the perturbation

$$\hat{G}_\epsilon^\pm = (1 \pm \epsilon)\hat{G},$$

where \hat{G} is a constant mean pressure gradient, for sufficiently large \hat{t} , after the initial transient phase, we can use the characterization of the steady solution deduced in (42), and explicitly compute the perturbed



(a) Flow index $n = 1.4$.



(b) Flow index $n = 1.8$.

Figure 17: Unsteady volume fl w rate (39) with non-constant mean pressure gradient (44) where $\hat{Q}(0) = 0$ and $\mathcal{W}_o = (0.2, 0.4, 0.6, 0.8)$ for shear-thickening fluid

volume fl w rate, using (45), as follows:

$$\begin{aligned} \hat{Q}_\varepsilon^\pm &= \sqrt[n]{\frac{n+3}{2} \frac{3n+5}{2} \hat{G}_\varepsilon^\pm} = \sqrt[n]{\frac{n+3}{2} \frac{3n+5}{2} (1 \pm \varepsilon) \hat{G}} \\ &= \sqrt[n]{\frac{n+3}{2} \frac{3n+5}{2} \hat{G}} (1 \pm \varepsilon)^{1/n} \\ &= \hat{Q} (1 \pm \varepsilon)^{1/n}. \end{aligned} \quad (46)$$

Normalizing the above perturbed volume fl w rate \hat{Q}_ε^\pm by the unperturbed volume fl w rate \hat{Q} , we get

$$\frac{\hat{Q}_\varepsilon^\pm}{\hat{Q}} = (1 \pm \varepsilon)^{1/n}, \quad (47)$$

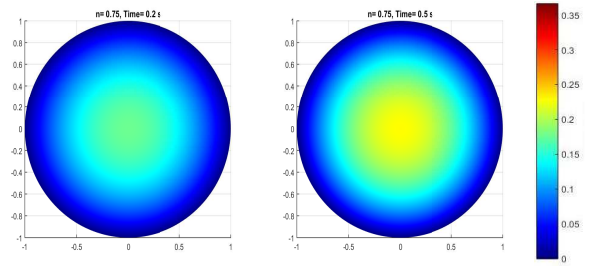


Figure 18: Three-dimensional velocity fiel (41) where the volume fl w rate is obtained by (39) with mean pressure gradient (44), $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 0.75$ (shear-thinning fluid) Time parameters: $\hat{t} = 0.2, \hat{t} = 0.5$.

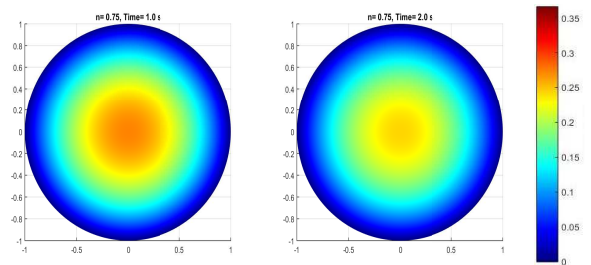


Figure 19: Three-dimensional velocity fiel (41) where the volume fl w rate is obtained by (39) with mean pressure gradient (44), $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 0.75$ (shear-thinning fluid) Time parameters: $\hat{t} = 1, \hat{t} = 2$.

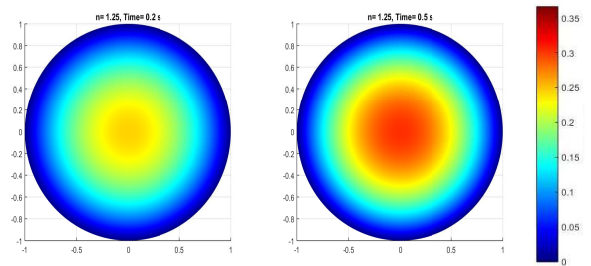


Figure 20: Three-dimensional velocity fiel (41) where the volume fl w rate is obtained by (39) with mean pressure gradient (44), $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 1.25$ (shear-thickening fluid) Time parameters: $\hat{t} = 0.2, \hat{t} = 0.5$.

which means that in the steady case, this kind of multiplicative perturbation acts linearly. Changing the mean pressure gradient by a factor of $(1 \pm \varepsilon)$, we changes the unperturbed volume fl w rate by a factor of $(1 \pm \varepsilon)^{1/n}$. In particular this shows that the steady state solution is linearly stable. Perturbations will be negligible if $(1 \pm \varepsilon)^{1/n} \simeq 1$, which happens when $\varepsilon \rightarrow 0$ (i.e., for small changes in the mean pres-

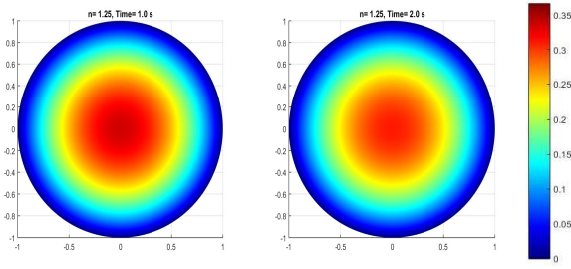


Figure 21: Three-dimensional velocity field (41) where the volume flow rate is obtained by (39) with mean pressure gradient (44), $\hat{Q}(0) = 0$, $\mathcal{W}_o = 0.8$ and $n = 1.25$ (shear-thickening fluid) Time parameters: $\hat{t} = 1, \hat{t} = 2$.

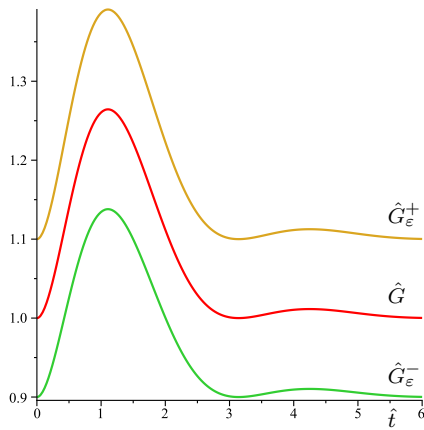


Figure 22: Multiplicative perturbation of the mean pressure gradient (44), with magnitude $\varepsilon = 0.1$.

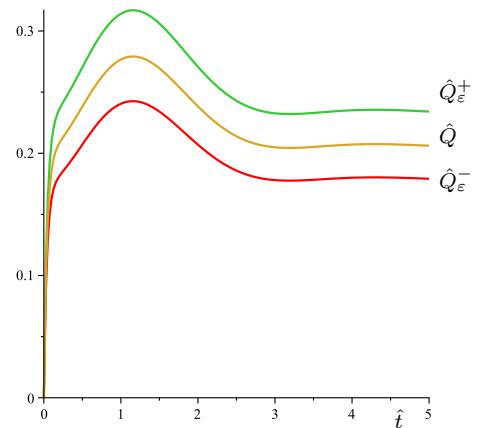
sure gradient) or when the flow index related with the viscosity goes to infinity, i.e., shear-thickening fluid with high flow index.

4.2 Unsteady problem

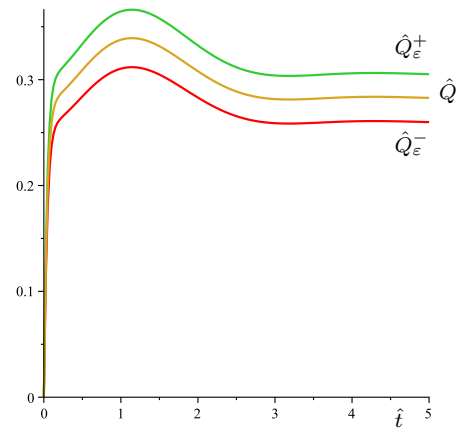
In the case of non-constant mean pressure gradient the same ideas hold, apart from the fact that it is no longer possible to deduce exact expressions for the perturbed volume flow rates. However, we can compute the time evolution of the perturbation volume flow rate \hat{Q}_ε^\pm . In Figure 23, we illustrate the time evolution of the volume flow rate with mean pressure gradient (44), together with the perturbed flow rates \hat{Q}_ε^\pm of magnitude $\varepsilon = 0.1$, forming a strip around \hat{Q} containing all perturbations of magnitude less or equal to ε . Figure 24, shows the amplitude of this strip

$$|\hat{Q}_\varepsilon^+ - \hat{Q}_\varepsilon^-| \quad (48)$$

for several values of flow index n with fixed Womersley number, showing that increasing the flow index



(a) Flow index $n = 0.75$.



(b) Flow index $n = 1.25$.

Figure 23: Time evolution of the unperturbed volume flow rate \hat{Q} , and perturbed volume flow rate \hat{Q}_ε^\pm , with magnitude $\varepsilon = 0.1$ and $\mathcal{W}_o = 0.5$ for shear-thinning and shear-thickening fluids respectively.

n reduces sensitivity to the perturbations, as already mentioned in the case of a constant mean pressure gradient.

5 Conclusion

In this work, the Cosserat theory has been used to derive a one-dimensional power-law fluid model in a straight, rigid and impermeable tube with uniform circular cross-section, as an alternative approach, to predict some of the main properties of associated three-dimensional models. Unsteady non-dimensional equation for the mean pressure gradient depending on the volume flow rate, Womersley number and the flow index over a finite section of the tube geometry has been obtained. Taking into account the volume flow rate approximate solution for certain flow regimes we obtained relevant information about the

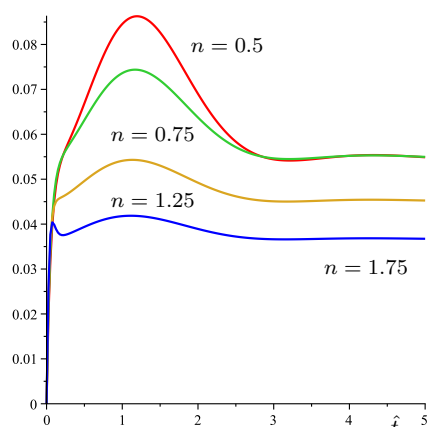


Figure 24: Time evolution of perturbation (48) for different values of flow index ($n = 0.5, n = 0.75, n = 1.25, n = 1.75$), with $W_o = 0.5$ and magnitude $\varepsilon = 0.1$.

behavior of the intensity of the three-dimensional velocity field in the circular cross-section of the tube. The predictive capability of this approach theory to study the unsteady flow behavior has been evaluated by comparing its one-dimensional solution with the three-dimensional exact solution for steady flows. We have a good match of the results for shear-thinning fluid situation, which is related to the study of blood flow in the human circulatory system. This theory has strong limitations for sufficiently low and/or high flow index n to real fluid due to the unboundedness of the viscosity asymptotic limits, but can be widely used and accurate for specific flow regimes. Also, we conducted numerical results for perturbed flows, obtaining an exact expression for the perturbed volume flow rates in the steady case, providing a first step towards stability analysis of the model. One of the possible extensions of this work is the application of this one-dimensional approach theory to study the same power-law flow model in curved tubes, fluid-structure interaction and tubes with branches or bifurcations.

Acknowledgements: This work has been partially supported by Centro de Investigação em Matemática e Aplicações da Universidade de Évora (CIMA-UE), Portugal, through the grant UID/MAT/04674/2013 of FCT-Fundação para a Ciência e a Tecnologia. The author, Ricardo Conceição acknowledges the Institute of Earth Sciences, University of Évora (IES-UE) and also FCT because the scholarship SFRH/BD/116344/2016.

References:

- [1] W. Harvey, *Exercitatio anatomica de motu cordis et sanguinis in animalibus*, Frankfurt, W. Fitzeri, 1628.
- [2] S. Chien, S. Usami, R.J. Delenback and M.L. Gregersen, Blood Viscosity: Influence of erythrocyte deformation, *Science* 157(3790), 1967, pp. 827–829.
- [3] S. Chien, S. Usami, R.J. Delenback and M.L. Gregersen, Blood Viscosity: Influence of erythrocyte aggregation, *Science* 157(3790), 1967, pp. 829–831.
- [4] G.B. Thurston, Viscoelasticity of human blood, *Biophys. J.* 12, 1972, pp. 1205–1217.
- [5] O.K. Baskurt and H.J. Meiselman, Blood rheology and hemodynamics, *Seminars in Thrombosis and Hemostasis* 29, 2003, pp. 435–450.
- [6] L. Euler, Principia pro motu sanguinis per arterias determinando, *Opera posthuma mathematica et physica anno 1844 detecta* 2, 1775, pp. 814–823.
- [7] T. Hughes and J. Lubliner, On the one-dimensional theory of blood flow in the larger vessels, *Mathematical Biosciences* 18, 1973, pp. 161–170.
- [8] S.J. Sherwin, V. Franke, J. Peiró and K. Parker, One-dimensional modelling of a vascular network in space-time variables, *Journal of Engineering Mathematics* 47, 2003, pp. 217–250.
- [9] L. Formaggia, D. Lamponi and A. Quarteroni, One-dimensional models for blood flow in arteries, *Journal of Engineering Mathematics* 47, 2003, pp. 251–276.
- [10] D.A. Caulk and P.M. Naghdi, Axisymmetric motion of a viscous fluid inside a slender surface of revolution, *Journal of Applied Mechanics* 54, 1987, pp. 190–196.
- [11] A.E. Green and P.M. Naghdi, A direct theory of viscous fluid flow in pipes I: Basic general developments, *Phil. Trans. R. Soc. Lond. A* 342(1), 1993, pp. 525–542.
- [12] A.E. Green, P.M. Naghdi and M.J. Stallard, A direct theory of viscous fluid flow in pipes II: Flow of incompressible viscous fluid in curved pipes, *Phil. Trans. R. Soc. Lond. A* 342(1), 1993, pp. 543–572.
- [13] A.M. Robertson and A. Sequeira, A director theory approach for modeling blood flow in the arterial system: an alternative to classical 1d models, *Math. Models Methods Appl. Sci.* 15(6), 2005, pp. 871–906.

- [14] F. Carapau and A. Sequeira, 1D Models for Blood Flow in Small Vessels Using the Cosserat Theory, *WSEAS Transactions on Mathematics* 5(1), 2006, pp. 54–62.
- [15] B.R. Bird, R.C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, 2nd edition, John Wiley & Sons, 1987.