Several Intensive Steel Quenching and Wave Power Models

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Abstract: - In this paper we develop mathematical models for 3-D, 2-D and one-dimensional hyperbolic heat equations (wave equation or telegraph equation) and construct their analytical solutions for the determination of the initial heat flux for rectangular samples. In some cases we give expression of wave energy. Some solutions of time inverse problems are obtained in the form of first kind Fredholm integral equation, but others has been obtained in closed analytical form. Finally, writes in one dimension intensive steel quenching model numerical results. We viewed both direct and inverse problems at the time. Are given some of the wave energy results.

Key-Words: - Hyperbolic Equation, Ocean Energy, Steel Quenching, Green Function, Exact Solution, Inverse Problem, Fredholm integral equation,Series.

1 Introduction

The conventional steel quenching is usually performed in environmentally unfriendly oil or water/polymer solutions. Contrary to traditional method the intensive quenching process uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions [1]-[5]. Traditionally for the mathematical description of the intensive quenching process, classical heat conduction equation is used. We have proposed to use hyperbolic heat equation [6]-[12], [21]-[23] for more realistic description of the intensive quenching (IQ) process (especially for the initial stage of the process).

Complete bibliography (till 1989) for hyperbolic heat equation is published in review articles [13], [14]. Models of systematic hyperbolic heat equation, their mathematical research and solutions are discussed in monograph [15]. In our previous papers we have constructed various one and two dimensional analytical exact and approximate [6]-[8] solutions for IQ processes. Here are both - approximate (on the basis of conservative averaging method, see [18]-[20] and exact (on the basis of Green function method, see [6], [10]-[12], [21]-[27]).

The idea of the usage of hyperbolic heat equation can be easily transferred to completely different sector of application - to the generation of electricity in sea or ocean by usage of wave energy [16]. It is important to note, that Ekergard and his co-authors [16] examine the development of the system in time, describing the equipment with ordinary differential equation. Here we describe the equipment in development of both - in time as well as in spatial arrangement of equipment using the three-dimensional hyperbolic heat equation. Wave power plant has to work for long time period in moving environment – waves, see [17]. Therefore it is important to examine not only the development of equipment in time, but also the movement of its different components [32]-[34]. Wave energy generator models can be viewed both Cartesian coordinate and cylindrical co-ordinates. Generators of cylindrical form [33], [34] we will investigate in separate paper. In this article we investigate the rectangular model. For this purpose we dedicate our paper.

We consider three-dimensional, two-dimensional and one-dimensional statements for non-homogeneous equation with non-homogeneous boundary conditions. Such statements allow constructing mathematical models for wave power plants in connection with other equipment, for example, with wind power. Boundary conditions could be different types, thus they allow us to use Green function method. This topic is similar to our paper [9], but the content differs greatly.

In recent years, we have been able to generalize the Green's function method to areas, which consist of several canonical connected sub-areas, and thus we have obtained the exact solutions for the L-, T- and Π-type areas [10], [11], [21], [24], [25]; an area consisting of two cylinders [22], two-layer sphere [12], [23] and layered system [26].

2 Mathematical Formulation of 3-D Problem for IQP or Wave Power

Already in the introduction we noted that Professor M. Leijon, see [16] examined the development of system in time. Here we offer to consider the description of system in time and space. For this purpose instead of the ordinary differential equation, we consider the following partial differential equation:

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right)$$

$$-DV + \Phi(x,y,z,t), a^2 = \frac{k}{c\rho}, \tag{1}$$

$x \in (0,l), y \in (0,b), z \in (0,w), t \in (0,T]$.

Here c is specific heat capacity, k - heat conductivity coefficient, ρ - density, τ_r - relaxation time. The source term $\Phi(x,y,z,t)$ can be from different parts of the same device or outer source, for example, wind source. As the first step we use well known substitution:

$$V(x,y,z,t) = \exp\left(-\frac{t}{2\tau_r}\right) U(x,y,z,t). \tag{2}$$

After transformation (2) equation (1) can be written in the following form:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) - CU +$$

$$F(x,y,z,t),\ C = \left(D - \frac{1}{4\tau_r} \right) \frac{1}{\tau_r}, a_\tau^2 = \frac{a^2}{\tau_r}, \tag{3}$$

$$F(x,y,z,t) = \exp\left(\frac{t}{2\tau_r}\right) \frac{1}{\tau_r} \Phi(x,y,z,t).$$

It is natural to assumption that planes $x=0, y=0, z=0$ are symmetry surfaces of the sample for the case of intensive steel quenching. In the case of wave energy we can assume different non-homogeneous boundary conditions:

$$\left. \left(\frac{\partial U}{\partial x} - \alpha_1 U \right) \right|_{x=0} = g_1(y,z,t), \alpha_i = \frac{h_i}{k}, \tag{4}$$

$$\left. \left(\frac{\partial U}{\partial y} - \alpha_2 U \right) \right|_{y=0} = g_2(x,z,t), \tag{5}$$

$$\left. \left(\frac{\partial U}{\partial z} - \alpha_3 U \right) \right|_{z=0} = g_3(x,y,t),\ i = 1,2,3. \tag{6}$$

Here h_i is heat exchange coefficient. On all the other sides of device we have heat exchange with environment. For generalizing we assume following non-homogeneous third type boundary conditions (Robin conditions) on all the three outer sides:

$$\left. \left(\frac{\partial U}{\partial x} + \beta_1 U \right) \right|_{x=l} = g_4(y,z,t), \beta_i = \frac{h_i}{k}, \tag{7}$$

$$\left. \left(\frac{\partial U}{\partial y} + \beta_2 U \right) \right|_{y=b} = g_5(x,z,t), \tag{8}$$

$$\left. \left(\frac{\partial U}{\partial z} + \beta_3 U \right) \right|_{z=w} = g_6(x,y,t), i = 1,2,3. \tag{9}$$

In fact it is possible to look at other types of boundary conditions: first (Dirichlet) and second (Neumann) type. The initial conditions for the function $V(x,y,z,t)$ are assumed in following form:

$$V|_{t=0} = V_0(x,y,z), \tag{10}$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=0} = W_0(x,y,z). \tag{11}$$

From the practical point of view in the steel quenching the condition (11) can be unrealistic. The initial heat flux must be determined theoretically. As additional condition we assume that either the temperature distribution or the heat fluxes distribution at the end of process is given (known):

$$V|_{t=T} = V_T(x,y,z), \tag{12}$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=T} = W_T(x,y,z). \tag{13}$$

After the transformation (2) then the differential equation (1) transforms into partial differential equation (3) without first time derivative. The initial conditions (10), (11) take the form:

$$U|_{t=0} = V_0(x,y,z), \tag{14}$$

$$\left. \frac{\partial U}{\partial t} \right|_{t=0} = V_1(x,y,z),$$

$$V_1(x,y,z) = W_0(x,y,z) + \frac{V_0(x,y,z)}{2\tau_r}. \tag{15}$$

Additional conditions (12), (13) transform as follows:

$$U|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right) V_T(x,y,z) = U_T(x,y,z), \tag{16}$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=T} = U_T^1\left(x,y,z\right), U_T^1\left(x,y,z\right) = \exp\left(\frac{T}{2\tau_r}\right)\left[W_T(x,y,z)+\frac{V_T(x,y,z)}{2\tau_r}\right]. \quad (17)$$

3 Solution of 3-D Problem

Firstly we assume that we have non-homogeneous Klein-Gordon equation-with source term: $c \geq 0$. The solution in three-dimensional problem is in following form:

$$U(x,y,z,t) = H\left(x,y,z,t\right) + \int_0^l d\xi \int_0^w d\varsigma \int_0^b V_1(\xi,\eta,\varsigma) G(x,y,z,\xi,\eta,\varsigma,t) d\eta + \int_0^l d\xi \int_0^w d\varsigma \int_0^b V_0(\xi,\eta,\varsigma) \frac{\partial}{\partial t} G(x,y,z,\xi,\eta,\varsigma,t) d\eta. \quad (18)$$

Here

$$H\left(x,y,z,t\right) = -a_\tau^2 \int_0^t d\tau \int_0^w d\varsigma \int_0^b g_1\left(\eta,\varsigma,\tau\right) G(x,y,z,0,\eta,\varsigma,t-\tau) d\eta$$
$$+a_\tau^2 \int_0^t d\tau \int_0^w d\varsigma \int_0^b g_4\left(\eta,\varsigma,\tau\right) G(x,y,z,l,\eta,\varsigma,t-\tau) d\eta$$
$$-a_\tau^2 \int_0^t d\tau \int_0^w d\varsigma \int_0^l g_2\left(\xi,\varsigma,\tau\right) G(x,y,z,\xi,0,\varsigma,t-\tau) d\xi$$
$$+a_\tau^2 \int_0^t d\tau \int_0^w d\varsigma \int_0^l g_5\left(\xi,\varsigma,\tau\right) G(x,y,z,\xi,b,\varsigma,t-\tau) d\xi$$
$$-a_\tau^2 \int_0^t d\tau \int_0^b d\eta \int_0^l g_3\left(\xi,\eta,\tau\right) G(x,y,z,\xi,\eta,0,t-\tau) d\xi$$
$$+a_\tau^2 \int_0^t d\tau \int_0^b d\eta \int_0^l g_6\left(\xi,\eta,\tau\right) G(x,y,z,\xi,\eta,w,t-\tau) d\xi$$
$$+\int_0^t d\tau \times \int_0^l d\xi \int_0^w d\varsigma \int_0^b F\left(\xi,\eta,\varsigma,\tau\right) G(x,y,z,\xi,\eta,\varsigma,t-\tau) d\eta.$$

The Green function [27] - [29] for initial-boundary problem for Klein-Gordon equation is known; see [30]:

$$G(x,y,z,\xi,\eta,\zeta,t) = 8\times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\sin\left(\lambda_m x+\varepsilon_m\right)\sin\left(\mu_n y+\varepsilon_n\right)\sin\left(\nu_k z+\varepsilon_k\right)}{E_{mnk}\sqrt{\rho_{mnk}}}$$
$$\times \sin\left(\lambda_m \xi+\varepsilon_m\right)\sin\left(\mu_n \eta+\varepsilon_n\right)\sin\left(\nu_k \zeta+\varepsilon_k\right)\times \sin\left(t\sqrt{\rho_{mnk}}\right),\ \varepsilon_m = arctg\left(\frac{\lambda_m}{l}\right),\ \varepsilon_n = arctg\left(\frac{\mu_n}{b}\right),$$
$$\rho_{mnk} = a_\tau^2\left(\lambda_m^2+\mu_n^2+\nu_k^2\right)+C,$$
$$\varepsilon_k = arctg\left(\frac{\nu_k}{w}\right), E_{mnk} = \left[l+\frac{\left(\alpha_1\beta_1+\lambda_m^2\right)\left(\alpha_1+\beta_1\right)}{\left(\alpha_1^2+\lambda_m^2\right)\left(\beta_1^2+\lambda_m^2\right)}\right]\times \left[b+\frac{\left(\alpha_2\beta_2+\mu_n^2\right)\left(\alpha_2+\beta_2\right)}{\left(\alpha_2^2+\mu_n^2\right)\left(\beta_2^2+\mu_n^2\right)}\right]\times \left[w+\frac{\left(\alpha_3\beta_3+\nu_k^2\right)\left(\alpha_3+\beta_3\right)}{\left(\alpha_3^2+\nu_k^2\right)\left(\beta_3^2+\nu_k^2\right)}\right]. \quad (19)$$

The eigenvalues λ_m, μ_n, ν_k are positive roots of the transcendental equations:

$$\lambda = \frac{\lambda^2-\alpha_1\beta_1}{\alpha_1+\beta_1} tg\left(l\lambda\right), \mu = \frac{\mu^2-\alpha_2\beta_2}{\alpha_2+\beta_2} tg\left(b\mu\right),$$
$$\nu = \frac{\nu^2-\alpha_3\beta_3}{\alpha_3+\beta_3} tg\left(w\nu\right).$$

There is an interesting situation, if both additional conditions (16), (17) are known. In this case we introduce new time argument by formula

$$\tilde{t} = T - t. \quad (20)$$

The formulation for new time variable is following:

$$\frac{\partial^2 U}{\partial \tilde{t}^2} = a_\tau^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right) - CU + F(x,y,z,T-\tilde{t}),$$
$$\left.\left(\frac{\partial U}{\partial x}-\alpha_1 U\right)\right|_{x=0} = g_1\left(y,z,T-\tilde{t}\right),$$
$$\left.\left(\frac{\partial U}{\partial x}+\beta_1 U\right)\right|_{x=l} = g_4\left(y,z,T-\tilde{t}\right), \quad (21)$$
$$\left.\left(\frac{\partial U}{\partial y}-\alpha_2 U\right)\right|_{y=0} = g_2\left(x,z,T-\tilde{t}\right),$$
$$\left.\left(\frac{\partial U}{\partial y}+\beta_2 U\right)\right|_{y=b} = g_5\left(x,z,T-\tilde{t}\right),$$

$$\left.\left(\frac{\partial U}{\partial z}-\alpha_3 U\right)\right|_{z=0}=g_3\left(x,y,T-\tilde{t}\right),$$

$$\left.\left(\frac{\partial U}{\partial z}+\beta_3 U\right)\right|_{z=w}=g_6\left(x,y,T-\tilde{t}\right),$$

$$\left.U\right|_{\tilde{t}=0}=U_T(x,y,z),\left.\frac{\partial U}{\partial \tilde{t}}\right|_{\tilde{t}=0}=-U_T^1(x,y,z).$$

Similar to (18) the solution of inverse problem looks like:

$$U(x,y,z,\tilde{t})=H\left(x,y,z,\tilde{t}\right)-$$
$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b U_T^1(\xi,\eta,\varsigma)G(x,y,z,\xi,\eta,\varsigma,\tilde{t})d\eta$$
$$+\int_0^l d\xi\times \tag{22}$$
$$\int_0^w d\varsigma\int_0^b U_T(\xi,\eta,\varsigma)\frac{\partial}{\partial\tilde{t}}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})d\eta.$$

There is no problem to transform the expression for $H\left(x,y,z,\tilde{t}\right)$ in following form:

$$H\left(x,y,z,\tilde{t}\right)=$$
$$-a_\tau^2\int_{T-\tilde{t}}^T d\tau\int_0^w d\varsigma\int_0^b g_1\left(\eta,\varsigma,\tau\right)G(x,y,z,0,\eta,\varsigma,T-\tau)d\eta$$
$$+a_\tau^2\int_{T-\tilde{t}}^T d\tau\int_0^w d\varsigma\int_0^b g_4\left(\eta,\varsigma,\tau\right)G(x,y,z,l,\eta,\varsigma,T-\tau)d\eta$$
$$-a_\tau^2\int_{T-\tilde{t}}^T d\tau\int_0^w d\varsigma\int_0^l g_2\left(\xi,\varsigma,\tau\right)G(x,y,z,\xi,0,\varsigma,T-\tau)d\xi$$
$$+a_\tau^2\int_{T-\tilde{t}}^T d\tau\int_0^w d\varsigma\int_0^l g_5\left(\xi,\varsigma,\tau\right)G(x,y,z,\xi,b,\varsigma,T-\tau)d\xi$$
$$-a_\tau^2\int_{T-\tilde{t}}^T d\tau\int_0^b d\eta\int_0^l g_3\left(\xi,\eta,\tau\right)G(x,y,z,\xi,\eta,0,T-\tau)d\xi$$
$$+a_\tau^2\int_{T-\tilde{t}}^T d\tau\int_0^b d\eta\int_0^l g_6\left(\xi,\eta,\tau\right)G(x,y,z,\xi,\eta,w,T-\tau)d\xi$$
$$+\int_{T-\tilde{t}}^T d\tau\times$$
$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b F\left(\xi,\eta,\varsigma,\tau\right)G(x,y,z,\xi,\eta,\varsigma,T-\tau)d\eta.$$

For the heat flux in time we have the expression:

$$\frac{\partial}{\partial\tilde{t}}U(x,y,z,\tilde{t})=\frac{\partial}{\partial\tilde{t}}H\left(x,y,z,\tilde{t}\right)-$$
$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b U_T^1(\xi,\eta,\varsigma)\frac{\partial}{\partial\tilde{t}}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})d\eta+$$
$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b U_T(\xi,\eta,\varsigma)\frac{\partial^2}{\partial\tilde{t}^2}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})d\eta.$$

From last expression at $\tilde{t}=T$ and equality (15) we have solution for the time inverse problem:

$$W_0(x,y,z)=-\frac{V_0(x,y,z)}{2\tau_r}+\left.\frac{\partial}{\partial\tilde{t}}H\left(x,y,z,\tilde{t}\right)\right|_{\tilde{t}=T}-$$
$$\int_0^l d\xi\times$$
$$\int_0^w d\varsigma\int_0^b U_T^1(\xi,\eta,\varsigma)\left.\frac{\partial}{\partial\tilde{t}}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})\right|_{\tilde{t}=T}d\eta+$$
$$\int_0^l d\xi\times$$
$$\int_0^w d\varsigma\int_0^b U_T(\xi,\eta,\varsigma)\left.\frac{\partial^2}{\partial\tilde{t}^2}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})\right|_{\tilde{t}=T}d\eta.$$

If only one additional condition (16) is given from the solution (22) we obtain 1st kind Fredholm integral equation for the determination of unknown initial heat flux:

$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b W_0(\xi,\eta,\varsigma)G(x,y,z,\xi,\eta,\varsigma,T)d\eta=$$
$$U_T(x,y,z)-H_0\left(x,y,z,T\right)-\int_0^w d\varsigma\times$$
$$\int_0^l d\xi\int_0^b V_0(\xi,\eta,\varsigma)\left.\frac{\partial}{\partial t}G(x,y,z,\xi,\eta,\varsigma,t)\right|_{t=T}d\eta.$$
$$H_0\left(x,y,z,t\right)=H\left(x,y,z,t\right)+$$
$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b\frac{V_0(\xi,\eta,\varsigma)}{2\tau_r}G(x,y,z,\xi,\eta,\varsigma,t)d\eta.$$

On the other hand, if only one additional condition (17) is given again we obtain the 1st kind Fredholm integral equation:

$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b W_0(\xi,\eta,\varsigma)\left.\frac{\partial}{\partial t}G(x,y,z,\xi,\eta,\varsigma,t)\right|_{t=T}d\eta$$
$$=U_T^1(x,y,z)-\left.\frac{\partial}{\partial t}H_1\left(x,y,z,t\right)\right|_{t=T}-$$
$$\int_0^l d\xi\int_0^w d\varsigma\int_0^b V_0(\xi,\eta,\varsigma)\left.\frac{\partial^2}{\partial t^2}G(x,y,z,\xi,\eta,\varsigma,t)\right|_{t=T}d\eta,$$

$$H_1(x,y,z,t)=H(x,y,z,t)+ \int_0^l d\xi \int_0^w d\varsigma \int_0^b \frac{V_0(\xi,\eta,\varsigma)}{2\tau_r}\frac{\partial}{\partial t}G(x,y,z,\xi,\eta,\varsigma,t)d\eta.$$

4 Solutions of Two Dimensional Problems

Here we will obtain the solution for two-dimensional problem as it was done for three-dimensional statement. The mathematical formulation for thin in $z-$direction parallelepiped (two-dimensional mathematical model) means: $w<<l, w<<b.$ Firstly, in accordance with conservative averaging method [18]-[20] we introduce following integral averaged value (two space-dimensional functions):

$$\bar{U}(x,y,t)=w^{-1}\int_0^w U(x,y,z,t)dz,$$

$$f(x,y,t)=w^{-1}\int_0^w F(x,y,z,t)dz.$$

Secondly, in similar way we obtain following 2-D differential equation:

$$\frac{\partial^2\bar{U}}{\partial t^2}=a_\tau^2\left(\frac{\partial^2\bar{U}}{\partial x^2}+\frac{\partial^2\bar{U}}{\partial y^2}\right)-\bar{c}\bar{U}+g(x,y,t), \quad (23)$$
$$x\in(0,l), y\in(0,b), t\in(0,T].$$

Now here:

$$\bar{c}=C+a_\tau^2\frac{\alpha_3+\beta_3}{w}, g(x,y,t)=f(x,y,t)+\frac{g_6-g_3}{w}.$$

We have the following boundary conditions for 2-D problem:

$$\left.\left(\frac{\partial\bar{U}}{\partial x}-\alpha_1\bar{U}\right)\right|_{x=0}=\bar{g}_1(y,t),$$

$$\bar{g}_1(y,t)=w^{-1}\int_0^w g_1(y,z,t)dz,$$

$$\left.\left(\frac{\partial\bar{U}}{\partial x}+\beta_1\bar{U}\right)\right|_{x=l}=\bar{g}_2(y,t), \quad (24)$$

$$\bar{g}_2(y,t)=w^{-1}\int_0^w g_4(y,z,t)dz,$$

$$\left.\left(\frac{\partial\bar{U}}{\partial y}-\alpha_2\bar{U}\right)\right|_{y=0}=\bar{g}_3(x,t),$$

$$\bar{g}_3(x,t)=w^{-1}\int_0^w g_2(x,z,t)dz,$$

$$\left.\left(\frac{\partial\bar{U}}{\partial y}+\beta_2\bar{U}\right)\right|_{y=b}=\bar{g}_4(x,t),$$

$$\bar{g}_4(x,t)=w^{-1}\int_0^w g_5(x,z,t)dz.$$

The initial conditions for the differential equation (23) we assume in the form:

$$\bar{U}\big|_{t=0}=\bar{U}_0(x,y), \quad (25)$$

$$\left.\frac{\partial\bar{U}}{\partial t}\right|_{t=0}=\bar{V}_0(x,y). \quad (26)$$

We have introduced in (25), (26) following notations:

$$\bar{U}_0(x,y)=w^{-1}\int_0^w V_0(x,y,z)dz,$$

$$\bar{V}_0(x,y)=w^{-1}\int_0^w\left(W_0(x,y,z)+\frac{V_0(x,y,z)}{2\tau_r}\right)dz.$$

The solution of 2-D problem for $\bar{c}\geq 0$ has a form similar to formula (22):

$$\bar{U}(x,y,t)=\int_0^l d\xi\int_0^b \bar{V}_0(\xi,\eta)G(x,y,\xi,\eta,t)d\eta$$
$$+\int_0^l d\xi\int_0^b \bar{U}_0(\xi,\eta)\frac{\partial}{\partial t}G(x,y,\xi,\eta,t)d\eta \quad (27)$$
$$+\bar{H}(x,y,t).$$

Here

$$\bar{H}(x,y,t)=$$
$$-a_\tau^2\int_0^t d\tau\int_0^b \bar{g}_1(\eta,\tau)G(x,y,0,\eta,t-\tau)d\eta+$$
$$+a_\tau^2\int_0^t d\tau\int_0^b \bar{g}_2(\eta,\tau)G(x,y,l,\eta,t-\tau)d\eta-$$
$$a_\tau^2\int_0^t d\tau\int_0^l \bar{g}_3(\xi,\tau)G(x,y,\xi,0,t-\tau)d\xi+ \quad (28)$$
$$a_\tau^2\int_0^t d\tau\int_0^l \bar{g}_4(\xi,\tau)G(x,y,\xi,b,t-\tau)d\xi+$$
$$\int_0^t d\tau\int_0^l d\xi\int_0^b g(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)d\eta.$$

The two-dimensional Green function in the formula (27) has the form [30]:

$$G(x,y,\xi,\eta,t) =$$
$$4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)\phi_m(y)\phi_m(\eta)\sin\left(t\sqrt{\omega_{nm}}\right)}{E_{nm}\sqrt{\omega_{nm}}}, \quad (29)$$
$$\omega_{nm} = a_\tau^2\left(\mu_n^2 + \nu_m^2\right) + \overline{c}.$$

The eigenvalues λ_n, μ_m are positive roots of the transcendental equations:

$$\mu = \frac{\mu^2 - \alpha_1\beta_1}{\alpha_1 + \beta_1} tg\left(l\mu\right), \nu = \frac{\nu^2 - \alpha_2\beta_2}{\alpha_2 + \beta_2} tg\left(b\nu\right).$$

The eigenfunctions in (29) are given by expressions:

$$\varphi_n\left(x\right) = \sin\left(\mu_n x + \varepsilon_n\right), \phi_m\left(y\right) = \sin\left(\nu_m y + \sigma_m\right),$$
$$\varepsilon_n = \arctan\left(\frac{\mu_n}{l}\right),\ \sigma_m = \arctan\left(\frac{\nu_m}{b}\right),$$
$$E_{nm} = \left[l + \frac{\left(\alpha_1\beta_1 + \mu_n^2\right)\left(\alpha_1 + \beta_1\right)}{\left(\alpha_1^2 + \mu_n^2\right)\left(\beta_1^2 + \mu_n^2\right)}\right]\times$$
$$\left[b + \frac{\left(\alpha_2\beta_2 + \nu_m^2\right)\left(\alpha_2 + \beta_2\right)}{\left(\alpha_2^2 + \nu_m^2\right)\left(\beta_2^2 + \nu_m^2\right)}\right].$$

The additional conditions (16) and (17) at the end of process regarding the two-dimensional function $\overline{U}\left(x,y,t\right)$ are following:

$$\overline{U}\Big|_{t=T} = \overline{U}_T(x,y),$$
$$\overline{U}_T(x,y) = \exp\left(\frac{T}{2\tau_r}\right)\overline{v}_T(x,y), \quad (30)$$
$$\overline{v}_T(x,y) = w^{-1}\int_0^w V_T(x,y,z)dz.$$

Respectively

$$\frac{\partial\overline{U}}{\partial t}\Big|_{t=T} = \overline{U}_T^1(x,y), \quad (31)$$
$$\overline{U}_T^1(x,y) = \exp\left(\frac{T}{2\tau_r}\right)\left[\frac{\overline{v}_T(x,y)}{2\tau_r} + \overline{w}_T(x,y)\right],$$
$$\overline{w}_T(x,y) = w^{-1}\int_0^w W_T(x,y,z)dz.$$

Here we consider only the statement with both given additional conditions (30), (31) and again we introduce time $\tilde{t}$ by formula (20). Then the new initial conditions are:

$$\overline{U}\Big|_{\tilde{t}=0} = \overline{U}_T(x,y), \frac{\partial\overline{U}}{\partial\tilde{t}}\Big|_{\tilde{t}=0} = -\overline{U}_T^1(x,y).$$

Solution of two-dimensional direct problem regarding the argument $\tilde{t}$ (but again, this problem is time reverse regarding the argument t !) is given by formula:

$$\overline{U}(x,y,\tilde{t}) = \overline{H}\left(x,y,\tilde{t}\right) +$$
$$\int_0^l d\xi\int_0^b \overline{U}_T(\xi,\eta)\frac{\partial}{\partial\tilde{t}}G(x,y,\xi,\eta,\tilde{t})d\eta -$$
$$-\int_0^l d\xi\int_0^b \overline{U}_T^1(\xi,\eta)G(x,y,\xi,\eta,\tilde{t})d\eta.$$

Then the representation for initial heat flux is:

$$\overline{V}_0(x,y) = \frac{\partial}{\partial\tilde{t}}\overline{H}\left(x,y,\tilde{t}\right)\Big|_{\tilde{t}=T} -$$
$$\int_0^l d\xi\int_0^b \overline{U}_T(\xi,\eta)\frac{\partial^2}{\partial\tilde{t}^2}G(x,y,\xi,\eta,\tilde{t})\Big|_{\tilde{t}=T} d\eta \quad (32)$$
$$+\int_0^l d\xi\int_0^b \overline{U}_T^1(\xi,\eta)\frac{\partial}{\partial\tilde{t}}G(x,y,\xi,\eta,\tilde{t})\Big|_{\tilde{t}=T} d\eta.$$

So we have obtained explicit expression for the initial heat flux in closed analytical form.

If only one additional condition (30) is given we obtain the 1st kind Fredholm integral equation

$$\int_0^l d\xi\int_0^b \overline{V}_0(\xi,\eta)K(x,y,\xi,\eta)d\eta = F(x,y),$$
$$F(x,y) = \overline{U}_T(x,y) - \overline{H}\left(x,y,T\right) -$$
$$\int_0^l d\xi\int_0^b \overline{U}_0(\xi,\eta)\frac{\partial}{\partial t}G(x,y,\xi,\eta,t)\Big|_{t=T} d\eta,$$
$$K(x,y,\xi,\eta) = G(x,y,\xi,\eta,T).$$

On the other hand, if only one additional condition (31) is given again we obtain again the 1st kind Fredholm integral equation:

$$\frac{\partial}{\partial t}\overline{U}(x,y,t) = \frac{\partial}{\partial t}\overline{H}\left(x,y,t\right) +$$
$$\int_0^l d\xi\int_0^b \overline{U}_0(\xi,\eta)\frac{\partial^2}{\partial t^2}G(x,y,\xi,\eta,t)d\eta$$
$$+\int_0^l d\xi\int_0^b \overline{V}_0(\xi,\eta)\frac{\partial}{\partial t}G(x,y,\xi,\eta,t)d\eta.$$

The 1st kind Fredholm integral equation is in the form:

$$\int_0^l d\xi\int_0^b \overline{V}_0(\xi,\eta)K_1(x,y,\xi,\eta)d\eta = F_1(x,y).$$

Here

$$F_1(x,y)=\bar{U}_T^1(x,y)-\frac{\partial}{\partial t}\overline{H}\left(x,y,t\right)-$$
$$\int_0^l d\xi\int_0^b \bar{U}_0(\xi,\eta)\frac{\partial^2}{\partial t^2}G(x,y,\xi,\eta,t)d\eta,$$
$$K_1(x,y,\xi,\eta)=\frac{\partial}{\partial t}G(x,y,\xi,\eta,t)\bigg|_{t=T}.$$

5 Simplifications for Homogeneous Initial Conditions

We would like to finish the two dimensional solution with a simplification for constant initial conditions:

$$\bar{U}\Big|_{t=0}=\bar{U}_0,\bar{U}_0=const,$$
$$\frac{\partial\bar{U}}{\partial t}\bigg|_{t=0}=\bar{V}_0,\bar{V}_0=const. \tag{33}$$

Solutions of time direct problem is these, see (27):

$$\bar{U}(x,y,t)=\bar{V}_0\int_0^l d\xi\int_0^b G(x,y,\xi,\eta,t)d\eta$$
$$+\bar{U}_0\int_0^l d\xi\int_0^b \frac{\partial}{\partial t}G(x,y,\xi,\eta,t)d\eta+\overline{H}\left(x,y,t\right). \tag{34}$$

Intensive steel quenching process with initial conditions (33) is very natural [8]-[12], [21]-[24]. We have homogeneous equation (23) and homogeneous boundary conditions:

$$\frac{\partial^2\bar{U}}{\partial t^2}=a_\tau^2\left(\frac{\partial^2\bar{U}}{\partial x^2}+\frac{\partial^2\bar{U}}{\partial y^2}\right)-\bar{c}\bar{U},$$
$$\left(\frac{\partial\bar{U}}{\partial x}-\alpha_1\bar{U}\right)\bigg|_{x=0}=0,$$
$$\left(\frac{\partial\bar{U}}{\partial x}+\beta_1\bar{U}\right)\bigg|_{x=l}=0,$$
$$\left(\frac{\partial\bar{U}}{\partial y}-\alpha_2\bar{U}\right)\bigg|_{y=0}=0,$$
$$\left(\frac{\partial\bar{U}}{\partial y}+\beta_2\bar{U}\right)\bigg|_{y=b}=0.$$

It means that we have:

$$\overline{H}\left(x,y,t\right)=0.$$

Solution (34) can be simplified:

$$\bar{U}(x,y,t)=\bar{V}_0\int_0^l d\xi\int_0^b G(x,y,\xi,\eta,t)d\eta$$
$$+\bar{U}_0\int_0^l d\xi\int_0^b \frac{\partial}{\partial t}G(x,y,\xi,\eta,t)d\eta=I_0+I_1.$$

We can integrate both integrals. For I_0:

$$I_0=4\bar{V}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\sqrt{\omega_{nm}}}\sin\left(t\sqrt{\omega_{nm}}\right)$$
$$\times\int_0^l\varphi_n(\xi)d\xi\int_0^b\phi_m(\eta)d\eta=$$
$$4\bar{V}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\mu_n\nu_m\sqrt{\omega_{nm}}}\sin\left(t\sqrt{\omega_{nm}}\right)\times$$
$$\times\cos\left(\mu_n x+\varepsilon_n\right)\cos\left(\nu_m y+\sigma_m\right).$$

Similarly we can integrate second integral I_1:

$$I_1=4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\sqrt{\omega_{nm}}}\frac{d}{dt}\sin\left(t\sqrt{\omega_{nm}}\right)$$
$$\times\int_0^l\varphi_n(\xi)d\xi\int_0^b\phi_m(\eta)d\eta=$$
$$4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}}\cos\left(t\sqrt{\omega_{nm}}\right)\times$$
$$\int_0^l\varphi_n(\xi)d\xi\int_0^b\phi_m(\eta)d\eta=$$
$$4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\mu_n\nu_m}\cos\left(t\sqrt{\omega_{nm}}\right)\times$$
$$\cos\left(\mu_n x+\varepsilon_n\right)\cos\left(\nu_m y+\sigma_m\right).$$

The solution we have in the form of two double series:

$$\bar{U}(x,y,t)=$$
$$4\bar{V}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\mu_n\nu_m\sqrt{\omega_{nm}}}\sin\left(t\sqrt{\omega_{nm}}\right)\times$$
$$\cos\left(\mu_n x+\varepsilon_n\right)\cos\left(\nu_m y+\sigma_m\right)+ \tag{35}$$
$$4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\mu_n\nu_m}\cos\left(t\sqrt{\omega_{nm}}\right)\times$$
$$\cos\left(\mu_n x+\varepsilon_n\right)\cos\left(\nu_m y+\sigma_m\right).$$

For the heat flux we have an expression:

$$\frac{\partial}{\partial t}\bar{U}(x,y,t)=$$

$$\bar{U}_0\int_0^l d\xi\int_0^b \frac{\partial^2}{\partial t^2}G(x,y,\xi,\eta,t)d\eta$$
$$+\bar{V}_0\int_0^l d\xi\int_0^b \frac{\partial}{\partial t}G(x,y,\xi,\eta,t)d\eta = I_2 + I_1,$$
$$I_2 = \bar{U}_0\int_0^l d\xi\int_0^b \frac{\partial^2}{\partial t^2}G(x,y,\xi,\eta,t)d\eta,$$
$$I_2 = -4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)\sqrt{\omega_{nm}}}{E_{nm}\mu_n\nu_m}\times$$
$$\sin\left(t\sqrt{\omega_{nm}}\right)\cos\left(\mu_n x+\varepsilon_n\right)\cos\left(\nu_m y+\sigma_m\right).$$

Finally for the time derivative we have:

$$\frac{\partial}{\partial t}\bar{U}(x,y,t) = 4\bar{V}_0\times$$
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\mu_n\nu_m}\cos\left(t\sqrt{\omega_{nm}}\right)\times$$
$$\cos\left(\mu_n x+\varepsilon_n\right)\sin\left(\nu_m y+\sigma_m\right)-$$
$$4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)\sqrt{\omega_{nm}}}{E_{nm}\mu_n\nu_m}\times$$
$$\sin\left(t\sqrt{\omega_{nm}}\right)\cos\left(\mu_n x+\varepsilon_n\right)\cos\left(\nu_m y+\sigma_m\right).$$

For example we have the temperature derivative at endpoint:

$$\bar{U}_T^1(x,y) = 4\bar{V}_0\times$$
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)}{E_{nm}\mu_n\nu_m}\cos\left(T\sqrt{\omega_{nm}}\right)$$
$$\times\cos\left(\mu_n x+\varepsilon_n\right)\sin\left(\nu_m y+\sigma_m\right)-$$
$$4\bar{U}_0\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\phi_m(y)\sqrt{\omega_{nm}}}{E_{nm}\mu_n\nu_m}\times \qquad (36)$$
$$\sin\left(T\sqrt{\omega_{nm}}\right)\cos\left(\mu_n x+\varepsilon_n\right)\times$$
$$\cos\left(\nu_m y+\sigma_m\right).$$

6 Solution of One Dimensional Problem for IQP

We will start with a formulation of the mathematical model of the steel plate which is relatively thin in y and z directions: $w << l, b << l.$ They show that rectangle is thin and narrow. In accordance with the conservative averaging method ([7], [18]-[20]), we introduce the following integral averaged value (one space-dimensional function):

$$u(x,t) = (bw)^{-1}\int_0^b dy\int_0^w U(x,y,z,t)dz,$$
$$\hat{f}(x,t) = (bw)^{-1}\int_0^b dy\int_0^w F(x,y,z,t)dz. \qquad (37)$$

Integrating the 3-D equation (3) in the directions y and z, we obtain:

$$\frac{1}{bw}\int_0^b dy\int_0^w\left[a_\tau^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)-CU\right]dz =$$
$$a_\tau^2\frac{\partial^2 u}{\partial x^2}-Cu+\hat{f}(x,t)+\frac{a_\tau^2}{bw}\int_0^b dy\int_0^w\left(\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dz.$$

The last term can be transformed as follows:

$$\frac{a_\tau^2}{bw}\int_0^b dy\int_0^w\left(\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dz =$$
$$\frac{a_\tau^2}{bw}\left(\int_0^w\frac{\partial U}{\partial y}\bigg|_{y=0}^{y=b}dz+\int_0^b\frac{\partial U}{\partial z}\bigg|_{z=0}^{z=w}dy\right).$$

Taking into account the heat exchange with environment, i.e. boundary conditions (5), (6), (8), (9), we obtain:

$$\frac{1}{bw}\left(\int_0^w\frac{\partial U}{\partial y}\bigg|_{y=0}^{y=b}dz+\int_0^b\frac{\partial U}{\partial z}\bigg|_{z=0}^{z=w}dy\right) =$$
$$-\frac{1}{bw}\int_0^w\left(\beta_2 U\big|_{y=b}+\alpha_2 U\big|_{y=0}-g_5\big|_{y=b}+g_2\big|_{y=0}\right)dz$$
$$-\frac{1}{bw}\int_0^b\left(\beta_3 U\big|_{z=w}+\alpha_3 U\big|_{z=0}-g_6\big|_{z=w}+g_3\big|_{z=0}\right)dy.$$

This allows us to use the simplest approximation – by constant – for the function $U(x,y,z,t)$ in the $y,z-$directions:

$$U(x,y,z,t) = u\left(x,t\right)$$

We obtain a 1-D differential equation as a result:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2\frac{\partial^2 u}{\partial x^2}-cu+f\left(x,t\right), x\in(0,l),$$
$$t\in(0,T], \qquad (38)$$
$$c = -\frac{1}{\tau_\tau^2}+a_\tau^2\left(\frac{\alpha_2+\beta_2}{b}+\frac{\alpha_3+\beta_3}{w}\right),$$

where the source term is as follows:

$$f\left(x,t\right) = \hat{f}\left(x,t\right)+a_\tau^2\times$$
$$\left[\frac{\bar{g}_2\left(x,t\right)-\bar{g}_5\left(x,t\right)}{b}+\frac{\bar{g}_3\left(x,t\right)-\bar{g}_6\left(x,t\right)}{w}\right];$$

$$\widehat{g}_i\left(x,t\right)=\frac{1}{w}\int_0^w g_i\left(x,z,t\right)dz,\ i=2,5;$$
$$\widehat{g}_j\left(x,t\right)=\frac{1}{b}\int_0^b g_j\left(x,y,t\right)dy,\ j=3,6. \tag{39}$$

Initial conditions (14), (15) for the differential equation (20) are as follows:

$$u\big|_{t=0}=u_0(x),$$
$$u_0(x)=\left(bw\right)^{-1}\int_0^b dy\int_0^w V_0(x,y,z)dz, \tag{40}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=v_0(x),$$
$$v_0(x)=\left(bw\right)^{-1}\int_0^b dy\int_0^w V_1(x,y,z)dz. \tag{41}$$

The boundary conditions (4), (7) remain in the same form:

$$\left.\left(\frac{\partial u}{\partial x}-\alpha_1 u\right)\right|_{x=0}=g_1\left(t\right), \tag{42}$$

$$\left.\left(\frac{\partial u}{\partial x}+\beta_1 u\right)\right|_{x=l}=g_4\left(t\right), \tag{43}$$

$$g_i(t)=\left(bw\right)^{-1}\int_0^b dy\int_0^w g_i(y,z,t)dz, i=1,4.$$

It is important to mark that one-dimensional approach is exact only if the solution is approximated with a constant in other two directions. We will return to different one-dimensional models at the end of this section. This one-dimensional statement is substantially more realistic in comparison with the statement given in our paper [9] because heat or elasticity losses from flank sides $y=[0,b]$ and $z=[0,w]$ are taken into account.

Foremost, we assume that we have a non-homogeneous Klein-Gordon equation with a source term $c\geq 0$. Solution of this one-dimensional direct problem (38)-(41) is well known, see, e.g. [30]:

$$u(x,t)=\frac{\partial}{\partial t}\int_0^l u_0(\xi)G(x,\xi,t)d\xi$$
$$+\int_0^l v_0(\xi)G(x,\xi,t)d\xi+H\left(x,t\right), \tag{44}$$

$$H\left(x,t\right)=-a_\tau^2\int_0^t g_1\left(\tau\right)G(x,0,t-\tau)d\tau+$$
$$\int_0^t d\tau\int_0^l f\left(\xi,\tau\right)G(x,\xi,t-\tau)d\xi+$$
$$+a_\tau^2\int_0^t g_4\left(\tau\right)G(x,l,t-\tau)d\tau.$$

The Green function has the following representation ([27]-[30]):

$$G(x,\xi,t)=\sum_{i=1}^{\infty}\frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\mu_i\right)}{\left\|\varphi_i\right\|^2\mu_i},$$
$$\varphi_i(x)=\cos\left(\lambda_i x\right)+\frac{\alpha_1}{\lambda_i}\sin\left(\lambda_i x\right), \tag{45}$$
$$\left\|\varphi_i\right\|^2=\frac{\alpha_1}{2\lambda_i^2}+\frac{l}{2}\left(1+\frac{\alpha_1^2}{\lambda_i^2}\right)+\frac{\beta_1}{2\lambda_i^2}\frac{\lambda_i^2+\alpha_1^2}{\left(\lambda_i^2+\beta_1^2\right)},$$
$$\mu_i=\sqrt{a_\tau^2\lambda_i^2+c}.$$

The eigenvalues λ_i are positive $\sqrt{}$roots of the transcendental equation:

$$\lambda=\frac{\lambda^2-\alpha_1\beta_1}{\alpha_1+\beta_1}\tan(\lambda l). \tag{46}$$

It is easy to write out, so called, “wave energy” [28]:

$$I_w\left(t\right)=\sum_{i=1}^{\infty}\frac{\sin^2\left(t\mu_i\right)}{\mu_i}. \tag{47}$$

Right now we look at case $c<0, a_\tau^2\lambda_i^2+c<0.$ In this case the Green function has different form:

$$G(x,\xi,t)=\sum_{i=1}^{m}\frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}\right)}{\left\|\varphi_i\right\|^2\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}}$$
$$+\sum_{i=m+1}^{\infty}\frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\left\|\varphi_i\right\|^2\sqrt{a_\tau^2\lambda_i^2+c}}. \tag{48}$$

Here the natural number m in the both sums is given by inequalities:

$$a_\tau^2\lambda_i^2+c<0, i=\overline{1,m},$$
$$a_\tau^2\lambda_i^2+c\geq 0, i=\overline{m+1,\infty}.$$

In this case “wave energy” has equality:

$$I_0\left(t\right)=\sum_{i=1}^{m}\frac{\sinh^2\left(t\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}\right)}{\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}}+$$

$$+\sum_{i=m+1}^{\infty}\frac{\sin^2\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\sqrt{a_\tau^2\lambda_i^2+c}}. \quad (49)$$

As it was told earlier, initial condition (41) is unrealizable from the experimental point of view, and the $v_0(x)$ must be calculated theoretically. The differentiation of the solution (44) gives:

$$\frac{\partial}{\partial t}u(x,t)=\int_0^l u_0(\xi)\frac{\partial^2}{\partial t^2}G(x,\xi,t)d\xi+$$
$$+\int_0^l v_0(\xi)\frac{\partial}{\partial t}G(x,\xi,t)d\xi+\frac{\partial}{\partial t}H\left(x,t\right). \quad (50)$$

The additional conditions (16) and (17) at the end of the process regarding $u\left(x,t\right)$ are as follows:

$$u\big|_{t=T}=v_T(x),$$
$$v_T(x)=\left(bw\right)^{-1}\int_0^b dy\int_0^w U_T(x,y,z)dz, \quad (51)$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=T}=w_T(x),$$
$$w_T(x)=\left(bw\right)^{-1}\int_0^b dy\int_0^w U_T^1(x,y,z)dz. \quad (52)$$

The solution (44) at the final moment $t=T$ gives:

$$v_T(x)=\int_0^l u_0(\xi)\frac{\partial}{\partial t}G(x,\xi,t)\Big|_{t=T}d\xi+$$
$$+\int_0^l v_0(\xi)G(x,\xi,T)d\xi+H\left(x,T\right)$$

or:

$$\int_0^l K(x,\xi)v_0(\xi)d\xi=f_0(x). \quad (53)$$

Here

$$f_0(x)=v_T(x)-\int_0^l u_0(\xi)\frac{\partial}{\partial t}G(x,\xi,t)\Big|_{t=T}d\xi-$$
$$-H\left(x,T\right),K(x,\xi)=G(x,\xi,T).$$

As in our paper [6], we finally have obtained the 1st kind Fredholm integral equation for determination of the unknown initial heat flux. We have to use regularization method for this ill-posed problem. If second additional condition is given from formula (50), we obtain the following first kind Fredholm integral equation:

$$\int_0^l \tilde{K}(x,\xi)v_0(\xi)d\xi=g_0(x), \quad (54)$$

$$\tilde{K}(x,\xi)=\frac{\partial}{\partial t}G(x,\xi,t)\Big|_{t=T},g_0(x)=w_T(x)-$$
$$\frac{\partial}{\partial t}H\left(x,t\right)\Big|_{t=T}-\int_0^l u_0(\xi)\frac{\partial^2}{\partial t^2}G(x,\xi,t)\Big|_{t=T}d\xi.$$

There is an interesting situation if both additional conditions (51) and (52) are known – we may introduce a new time argument:

$$\tilde{t}=T-t. \quad (55)$$

The main differential equation (38) remains in its form, only the source term changes:

$$\frac{\partial^2 u}{\partial \tilde{t}^2}=a_\tau^2\frac{\partial^2 u}{\partial x^2}-cu+f\left(x,T-\tilde{t}\right), \quad (56)$$
$$x\in(0,l),\tilde{t}\in(0,T].$$

The boundary conditions (42), (43) change similarly:

$$\left(\frac{\partial u}{\partial x}-\alpha_1 u\right)\Big|_{x=0}=g_1\left(T-\tilde{t}\right),$$
$$\left(\frac{\partial u}{\partial x}+\beta_1 u\right)\Big|_{x=l}=g_4\left(T-\tilde{t}\right). \quad (57)$$

Both additional conditions transform to initial conditions for the equation (56):

$$u\big|_{\tilde{t}=0}=v_T(x),$$
$$\left.\frac{\partial u}{\partial \tilde{t}}\right|_{\tilde{t}=0}=-w_T(x). \quad (58)$$

The solution of the direct problem (56)-(58) is similar to the solution (44):

$$u(x,\tilde{t})=\int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})d\xi$$
$$-\int_0^l w_T(\xi)G(x,\xi,\tilde{t})d\xi+H\left(x,\tilde{t}\right). \quad (59)$$

The last term can be written in the following form:

$$H\left(x,\tilde{t}\right)=-a_\tau^2\int_{T-\tilde{t}}^T g_1\left(\tau\right)G(x,0,\tilde{t}-T+\tau)d\tau$$
$$+a_\tau^2\int_{T-\tilde{t}}^T g_4\left(\tau\right)G(x,l,\tilde{t}-T+\tau)d\tau+ \quad (60)$$
$$\int_{T-\tilde{t}}^T d\tau\int_0^l f\left(\xi,\tau\right)G(x,\xi,\tilde{t}-T+\tau)d\xi.$$

For the heat flux we have an expression similar to the formula (50):

$$\frac{\partial}{\partial \tilde{t}}u(x,\tilde{t})=\int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2}G(x,\xi,\tilde{t})d\xi$$
$$-\int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})d\xi+\frac{\partial}{\partial \tilde{t}}H\left(x,\tilde{t}\right).$$

From here, a nice explicit representation of the necessary initial heat flux immediately follows:

$$v_0(x) = -\int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}} G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T} d\xi + \int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2} G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T} d\xi + \frac{\partial}{\partial \tilde{t}} H\left(x,\tilde{t}\right)\Big|_{\tilde{t}=T} \quad (61)$$

As we mentioned before, the 1-D statement is the approximated model of the 3-D statement. If the boundary conditions (5)-(8) are of the second kind (Neumann's) $(\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0)$, the 1-D problem is an exact statement of the 3-D approach. If the conditions (4), (7) are the second type conditions ($\alpha_1 = \beta_1 = 0$), the formula (22) is valid, but the Green function has the following form if $c \ge 0$ ([27]-[30]):

$$G(x,\xi,t) = \frac{1}{l\sqrt{c}}\sin\left(t\sqrt{c}\right) + \frac{2}{l}\sum_{i=1}^{\infty}\frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2 + c}\right)}{\sqrt{a_\tau^2\lambda_i^2 + c}}, \quad (62)$$

$$\varphi_i(x) = \cos\left(\lambda_i x\right), \lambda_i = \frac{n\pi}{l}.$$

While $c < 0$, it is as follows:

$$G(x,\xi,t) = \frac{1}{l\sqrt{|c|}}\sinh\left(t\sqrt{|c|}\right) + \frac{2}{l}\sum_{i=1}^{m}\frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{\left|a_\tau^2\lambda_i^2 + c\right|}\right)}{\sqrt{\left|a_\tau^2\lambda_i^2 + c\right|}} + \sum_{i=m+1}^{\infty}\frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2 + c}\right)}{\sqrt{a_\tau^2\lambda_i^2 + c}}. \quad (63)$$

We have two inequalities:

$$a_\tau^2\left(\frac{n\pi}{l}\right)^2 + c < 0, i = \overline{1,m},$$

$$a_\tau^2\left(\frac{n\pi}{l}\right)^2 + c \ge 0, i = \overline{m+1,\infty}.$$

All results can be applied to the partial differential equation with the first derivative regarding argument *x*:

$$\frac{\partial^2 w}{\partial t^2} = a_\tau^2\frac{\partial^2 w}{\partial x^2} - \nu\frac{\partial w}{\partial x} - \overline{c}u + \overline{f}\left(x,t\right).$$

We use the following transform in this case:

$$w\left(x,t\right) = \exp\left(\frac{\nu x}{2a_\tau^2}\right)u\left(x,t\right).$$

After this transformation, we have equation (38), where new coefficients are the following:

$$c = \overline{c} - \frac{\nu^2}{4},\ f\left(x,t\right) = \exp\left(-\frac{\nu x}{2a_\tau^2}\right)\overline{f}\left(x,t\right).$$

We would like to finish the section with a comparison remark about the obtained solutions of the time inverse problem and the solution from our paper [6]. The main distinction is between Green functions. In the paper [6], we have used the Green function for the classical (parabolic) heat equation, but here – we have used the Green function for the wave (hyperbolic) equation.

7 Simplifications for Homogeneous Initial Conditions

We would like to finish the one dimensional solution with a simplification for constant initial conditions:

$$u\big|_{t=0} = u_0(x) = u_0 = const, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = w_0(x) = w_0 = const. \quad (64)$$

The solution of the time direct problem is the following (see (44)):

$$u(x,t) = u_0\int_0^l \frac{\partial}{\partial t}G(x,\xi,t)d\xi + v_0\int_0^l G(x,\xi,t)d\xi + H\left(x,t\right). \quad (65)$$

Intensive steel quenching process with initial conditions (64) is very natural [8]-[12]. We have the homogeneous equation (38) and the homogeneous boundary conditions:

$$\frac{\partial^2 u}{\partial \tilde{t}^2} = a_\tau^2\frac{\partial^2 u}{\partial x^2} - cu,$$

$$\left(\frac{\partial u}{\partial x} - \alpha_1 u\right)\Big|_{x=0} = 0,$$

$$\left(\frac{\partial u}{\partial x} + \beta_1 u\right)\Big|_{x=l} = 0.$$

It means that we have:

$$H\left(x,t\right) = 0.$$

The solution (65) can be simplified as follows:

$$u(x,t) = u_0 \int_0^l \frac{\partial}{\partial t} G(x,\xi,t) d\xi + v_0 \int_0^l G(x,\xi,t) d\xi$$
$$= I_1 + I_0.$$

We can integrate both integrals. For the I_0:

$$I_0 = v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin(t\mu_i)}{\|\varphi_i\|^2 \mu_i} \int_0^l \varphi_i(\xi) d\xi$$
$$= v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i \mu_i} B_i, \tag{66}$$
$$B_i = \left[\sin(\lambda_i \xi) - \frac{\alpha_1}{\lambda_i}\cos(\lambda_i \xi) \right] \Bigg|_{\xi=0}^{\xi=l}.$$

Similarly, we can integrate the second integral I_1:

$$I_1 = u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\cos(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i. \tag{67}$$

We can use representations (38), (39) to test our proposed method. For IQP we have positive initial temperature (hot steel sample) and negative heat flux: $u_0 > 0, v_0 < 0$. The temperature field is given by formula:

$$u(x,t) = u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\cos(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i + \\ + v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i \mu_i} B_i. \tag{68}$$

We have the following expression for heat flux:

$$\frac{\partial}{\partial t} u(x,t) = u_0 \int_0^l \frac{\partial^2}{\partial t^2} G(x,\xi,t) d\xi +$$
$$+ v_0 \int_0^l \frac{\partial}{\partial t} G(x,\xi,t) d\xi = I_2 + I_0,$$
$$I_2 = u_0 \int_0^l \frac{\partial^2}{\partial t^2} G(x,\xi,t) d\xi =$$
$$= -u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\mu_i \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i.$$

Finally, we have:

$$\frac{\partial}{\partial t} u(x,t) = -u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\mu_i \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i + \\ + v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i \mu_i} B_i. \tag{69}$$

We can calculate $u(x,T) = u_T(x)$ and $\frac{\partial}{\partial t} u(x,T) = v_T(x)$ by selecting arbitrary $u_0 > 0, v_0 < 0$. We find $\bar{u}_0 > 0, \bar{v}_0 < 0$ by solving time reverse problem (61). Before we use the formulas (44), (45), we can solve two different time reverse problems. The first is the problem with initial conditions as follows:

$$u\big|_{\tilde{t}=0} = \bar{v}_T, \bar{v}_T = \int_0^l v_T(\xi) d\xi$$
$$\frac{\partial u}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -\bar{w}_T, \bar{w}_T = \int_0^l w_T(\xi) d\xi.$$

The second problem – with "small" initial conditions:

$$u(x,T) = v_T(x) - \bar{v}_T,$$
$$\frac{\partial}{\partial t} u(x,T) = -w_T(x) + \bar{w}_T.$$

The difference between u_0, v_0 and $\bar{u}_0$, $\bar{v}_0$ will show the accuracy of our method.

The second approximation way is to use the same constant values as initial conditions for the time inverse problem.

.8 Numerical Results

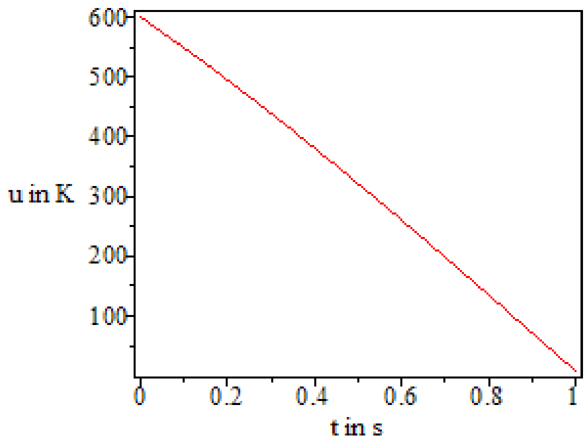
We would like to use physically real parameters, and that is why we choose parameters from the intensive steel quenching process. Let us take typical steel parameters in our model and homogeneous initial conditions:

$$c_m = 477 \frac{J}{kg \cdot K}, \ \rho = 7900 \frac{kg}{m^3}, \ k = 14.9 \frac{W}{m \cdot K},$$
$$l = 1m, \ b = 0.2m, \ w = 0.1m,$$
$$h_i = 100, \ \tau_r = 1.5s,$$
$$u_0 = 600°C, \ v_0 = -500 \frac{K}{s}. \tag{70}$$

We obtain the solution from formulas (68), (69). The first eigenvalues of the transcendental equation are the following:

2.44, 5.00, 7.72, 10.56, 13.49, 16.48, 19.51, 22.57, 25.64, 28.73, 31.83, 34.94, 38.05, 41.16, 44.28, 47.41, 50.53, 53.66, 56.78, 59.91, 63.04, 66.18, 69.31, 72.44, 75.58, 78.71, 81.85, 84.98, 88.12, 91.25, 94.39, 97.53.

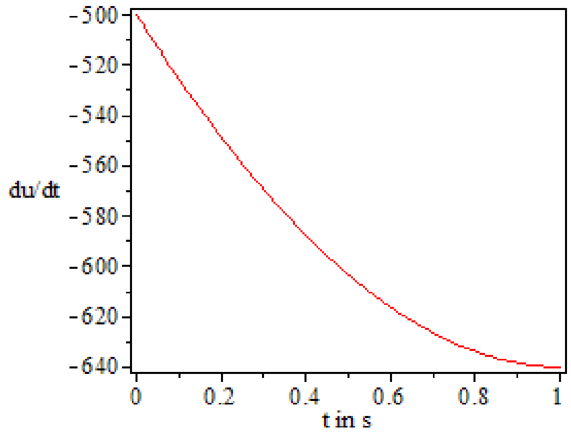
In general, temperature and flux are *x*-dependent, but calculations showed that values are almost constant regarding space dimension. The next figures show temperature and flux distribution in time.



If we take $T=1$ as the final time, values at this moment are the following:

$$u_T = u(x,T) = 8,$$

$$v_T = \frac{\partial u}{\partial t}(x,T) = -640.$$



By using the time reverse problem with these values as initial conditions:

$$u\big|_{\tilde{t}=0} = u_T,$$

$$\left.\frac{\partial u}{\partial \tilde{t}}\right|_{\tilde{t}=0} = -v_T$$

we return to our initial conditions (64) at the moment $\tilde{t}=T$. Calculation errors were negligible in our example ($\Delta u \simeq 0.2$ K).

9 Conclusion

We have constructed some solutions for direct and time inverse problems for hyperbolic heat equation. The solutions for determination of initial heat flux are obtained either in the form of Fredholm integral equation of 1st kind with continuous kernel or in the closed analytical form – in the form of series or ordinary integrals.

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