On a Scale Invariant Model of Statistical Mechanics and Invariant forms of Conservation Equations

SIAVASH H. SOHRAB
Robert McCormick School of Engineering and Applied Science
Department of Mechanical Engineering
Northwestern University, Evanston, Illinois 60208
UNITED STATES OF AMERICA
s-sohrab@northwestern.edu
http://www.mech.northwestern.edu/web/people/faculty/sohrab/

Abstract- A scale-invariant model of statistical mechanics is described and applied to introduce the invariant *Boltzmann* equation and the corresponding invariant *Enskog* equation of change. The invariant modified as well as classical forms of mass, thermal energy, linear momentum, and angular momentum conservation equations are derived. Also, an invariant definition of reaction rate $\Re_{\beta+1} = \sum_{\beta}\Re_{\beta}$ for any scale within the hierarchy of statistical fields is introduced. Following *Cauchy*, the total stress tensor for fluids $\mathbf{P}_{ij\beta} = p_{i\beta}\delta_{ij\beta} - (\mu_{i\beta}/3)\nabla.\mathbf{v}_{i\beta}\delta_{ij\beta}$ is introduced that is consistent with the fact that by definition fluids can only support compressive normal forces. Solutions of modified forms of conservation equations are determination to describe hydro-thermo-diffusive structure of normal shock in pure gas. Also, exact solution of modified form of equation of motion for the problems of laminar and turbulent flow over a flat plate are described and shown to be in close agreement with experimental data in literature. Finally, the solution of the modified *Helmholtz* vorticity equation for the problem of flow within a droplet located at the stagnation point of opposed cylindrically-symmetric gaseous finite jets is presented.

Key Words- Conservation equations, Fluid mechanics, Shock waves, Statistical mechanics, TOE.

1 Introduction

It is well known that the methods of statistical mechanics can be applied to describe physical phenomena over a broad range of scales of space and time from the exceedingly large scale of cosmology to the minute scale of quantum optics as schematically shown in Fig. 1. Although the universality of statistical nature of problems of stochastic quantum fields [1-17] and classical hydrodynamic fields [18-33] is well known, the extent to which exact correspondence exists between the laws of nature amongst the diverse scales of space-time from cosmic to photonic is as yet not recognized. Similarities between the statistical fields shown in Fig. 1 resulted in recent introduction of a scale-invariant model of statistical mechanics [34] and its application to the fields of thermodynamics [35], fluid mechanics [36], statistical mechanics [37] and quantum mechanics [38, 39].

In the present study, following the classical methods [18-22 , 40-44] an invariant model of statistical mechanics is applied to introduce invariant *Boltzmann* equation and the corresponding invariant *Enskog* equation of change. From the equation of change invariant forms of mass, energy, linear momentum, and angular momentum conservation equations are derived. A modified form of the continuity equation is presented that in the absence of convection reduces to diffusion equation thus revealing the internal structure of normal shock wave in pure system. Also, a modified form of equation of motion is introduced

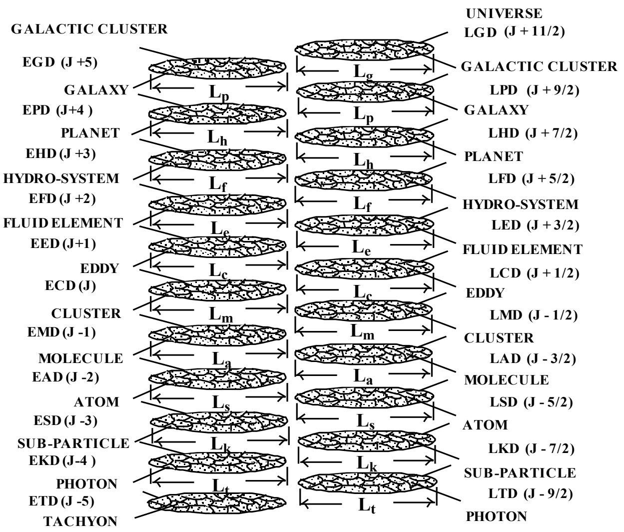


Fig. 1 A scale-invariant model of statistical mechanics. Equilibrium-(β)-Dynamics on the left-hand-side and non-equilibrium Laminar-(β)-Dynamics on the right-hand-side for scales β = g, p, h, f, e, c, m, a, s, k, and t as defined in Section 2. Characteristic lengths of (system, element, "atom") are (L_β, λ_β, ℓ_β) and λ_β is the mean-free-path.

with convective velocity distinguished from local velocity. Finally, a modified form of *Helmholtz* vorticity equation is introduced and its solution for the problem of laminar flow in a liquid droplet located at the stagnation point of gaseous counter flow finite jets is presented. By application of integral methods invariant classical forms of conservation equations are derived and connected to the modified forms in the last Section.

2 A Scale Invariant Model of Statistical Mechanics

The scale-invariant model of statistical mechanics for equilibrium galactic-, planetary-, hydro-system-, fluid-element-, eddy-, cluster-, molecular-, atomic-, subatomic-, kromo-, and tachyon-dynamics corresponding to the scale $\beta = $ g, p, h, f, e, c, m, a, s, k, and t is schematically shown in Fig. 1 [31]. The statistical fields of equilibrium eddy-, cluster-, and molecular-dynamics (EED, ECD, EMD) are shown in Fig. 2 in more details along with the corresponding non-equilibrium laminar flow fields (LED, LCD, LMD). Each statistical field is identified as the "system" and is composed of a spectrum of "elements". Each element is composed of an ensemble of small particles called the "atoms" of the field that are governed by a distribution function $f_{i\beta}(\mathbf{x}_{i\beta}, \mathbf{u}_{i\beta}, t_\beta)$ and are viewed as *point-mass*. The most probable element (system) velocity of the smaller scale *(j)* becomes the velocity of the atom (element) of the larger scale *(j+1)* [38].

Following the classical methods [18-22, 40-44] the invariant definition of density ρ_β, and velocity of *atom* $\mathbf{u}_{i\beta}$, *element* $\mathbf{v}_{i\beta}$, and *system* $\mathbf{w}_\beta$ at the scale β are [38, 39]

$$\rho_{i\beta} = n_{i\beta} m_{i\beta} = m_{i\beta} \int f_{i\beta} du_{i\beta} \quad , \quad \mathbf{u}_{i\beta} = \mathbf{v}_{imp\beta-1} \quad (1)$$

$$\mathbf{v}_{i\beta} = \rho_{i\beta}^{-1} m_{i\beta} \int \mathbf{u}_{i\beta} f_{i\beta} d\mathbf{u}_{i\beta} \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{mp\beta+1} \quad (2)$$

Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$\mathbf{V}'_{i\beta} = \mathbf{u}_{i\beta} - \mathbf{v}_{i\beta} \quad , \quad \mathbf{V}_{i\beta} = \mathbf{v}_{i\beta} - \mathbf{w}_\beta \quad (3)$$

such that

$$\mathbf{V}_{i\beta} = \mathbf{V}'_{i\beta+1} \quad (4)$$

As shown in Fig. 2, the statistical field of ECD at an intermediate scale separates LED from LMD fields. The evidence for the existence of the statistical field of ECD is the phenomena of Brownian motions as discussed in an earlier study [39].

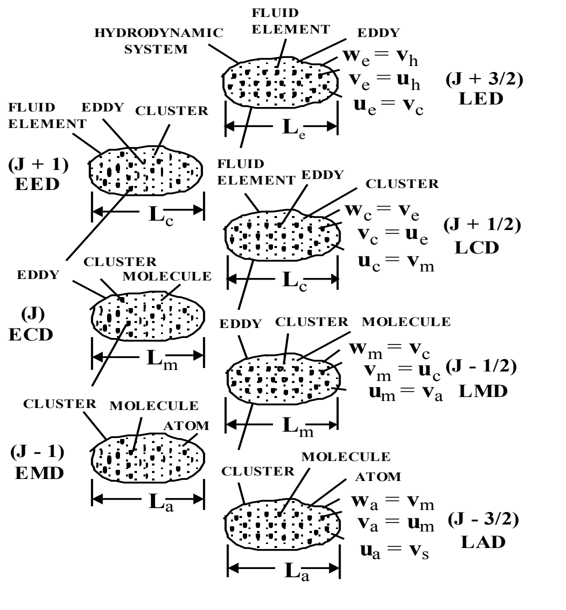


Fig. 2 Hierarchy of statistical fields for equilibrium eddy-, cluster-, and molecular-dynamic scales and the associated laminar flow fields.

For the statistical fields of EED, ECD and EMD, typical characteristic atom, element, and system lengths are

EED $\quad (\ell_e,\ \lambda_e,\ L_e) = (10^{-5},\ 10^{-3},\ 10^{-1})\ \mathrm{m} \qquad (5a)$

ECD $\quad (\ell_c,\ \lambda_c,\ L_c) = (10^{-7},\ 10^{-5},\ 10^{-3})\ \mathrm{m} \qquad (5b)$

EMD $\quad (\ell_m,\ \lambda_m,\ L_m) = (10^{-9},\ 10^{-7},\ 10^{-5})\ \mathrm{m} \qquad (5c)$

The relative system sizes of these statistical fields are schematically shown in Fig. 3.

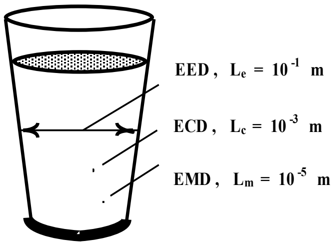


Fig. 3 The system sizes L_β of statistical fields EED, ECD, and EMD relative to a cup of water.

If one applies the same (atom, element, system) = $(\ell_\beta,\ \lambda_\beta, L_\beta)$ relative sizes in (5) to the entire spatial scale of Fig. 1 and considers the relation between scales as $\ell_\beta = \lambda_{\beta-1} = L_{\beta-2}$ then the resulting cascades or hierarchy of overlapping statistical fields will appear as schematically shown in Fig. 4.

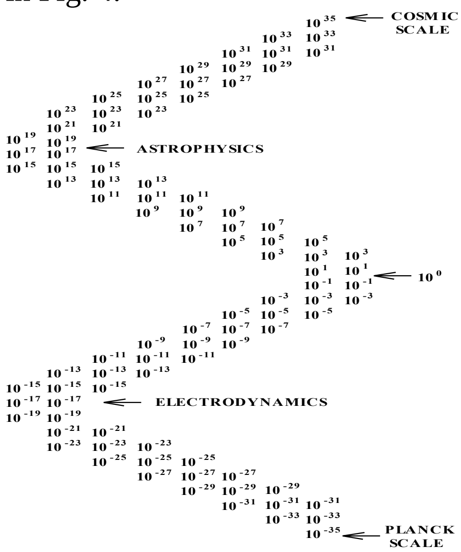


Fig. 4 Hierarchy of statistical fields with $(\ell_\beta,\ \lambda_\beta, L_\beta)$ from cosmic to Planck scales [38].

According to Fig. 4, starting from the hydrodynamic scale $(10^3,\ 10^1,\ 10^{-1},\ 10^{-3})$ after seven generations of statistical fields one reaches the electro-dynamic scale with the element size 10^{-17} and exactly after seven more generations one reaches *Planck* length scale $(\hbar G / c^3)^{1/2} \simeq 10^{-35}\mathrm{m}$, where $\hbar = h / 2\pi$, h is *Planck* constant and G is the gravitational constant. Similarly, seven generations of statistical fields separate the hydrodynamic scale $(10^3, 10^1, 10^{-1}, 10^{-3})$ from the scale of planetary dynamics (astrophysics) 10^{17} and the latter from galactic-dynamics (cosmology) 10^{35} m. Since invariant *Schrödinger* equation was recently derived from invariant *Bernoulli* equation [38], the entire hierarchy of statistical fields shown in Fig. 1 is governed by quantum mechanics. There are no physical or mathematical reasons for the hierarchy shown in Fig. 4 not to continue to larger and smaller scales ad infinitum. Hence, according to Fig. 4 contrary to the often quoted statement by *Einstein* that God does not play dice; the Almighty appears to be playing with infinite hierarchies of embedded dice.

The left hand side of Figs. 1 and 2 correspond to equilibrium statistical fields when the velocities of elements of the field are random since at thermodynamic equilibrium particles i.e. oscillators of such statistical fields will have normal or *Gaussian* velocity distribution. For example, for stationary homogeneous isotropic turbulence at EED scale, the experimental data of *Townsend* [45] confirms the *Gaussian* velocity distribution of eddies as shown in Fig. 5.

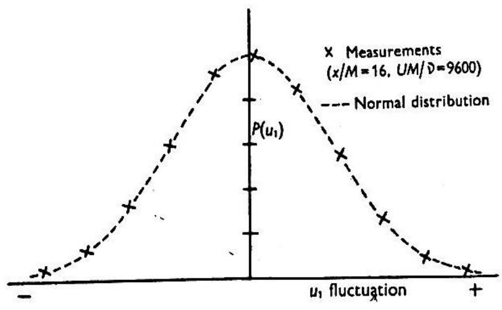


Fig. 5 Measured velocity distribution in isotropic turbulent flow by Townsend [45].

The invariant model of statistical mechanics (1)-(4) suggests that all statistical fields shown in Fig. 1 are turbulent fields [37, 38]. First, let us start with the field of laminar molecular dynamics LMD when molecules, clusters of molecules (cluster), and cluster of clusters of molecules (eddy) form the “atom”, the “element”, and the “system” with the velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m)$. Similarly, from (1)-(2) the fields of laminar cluster-dynamics LCD and eddy-dynamics LED will have the velocities

$$\text{LED} \qquad (\mathbf{u}_e, \mathbf{v}_e, \mathbf{w}_e) \qquad (6a)$$

$$\text{LCD} \qquad (\mathbf{u}_c, \mathbf{v}_c, \mathbf{w}_c) \qquad (6b)$$

$$\text{LMD} \qquad (\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m) \qquad (6c)$$

With *Gaussian* velocity distribution as in Fig. 5, the same chain of reasoning as employed in the classical kinetic theory of *Maxwell* and *Boltzmann* [18-20] requires that the distribution of the speeds of oscillators (eddies) in stationary isotropic turbulence be given by the invariant *Maxwell-Boltzmann* distribution function [38-39]

$$\mathbf{u}_c = \mathbf{v}_{m,mp} \quad \frac{dN_{u\beta}}{N} = 4\pi \left(\frac{m_\beta}{2\pi k T_\beta}\right)^{3/2} u_\beta^2 \, e^{-m_\beta u_\beta^2/2kT_\beta} du_\beta \qquad (7)$$

By (7), one arrives at a hierarchy of embedded *Maxwell-Boltzmann* distribution functions for ECD, EMD, and EAD scales shown in Fig. 6.

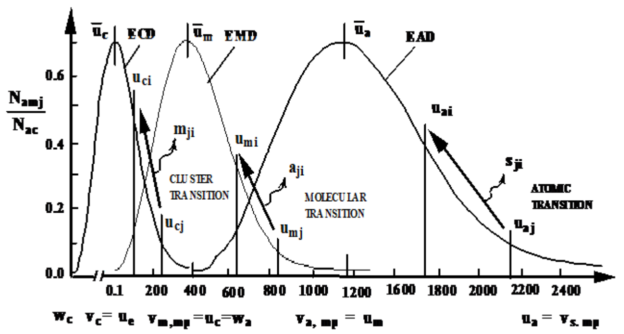


Fig. 6 Maxwell-Boltzmann speed distribution viewed as stationary spectra of cluster sizes for ECD, EMD, and EAD scales at 300 K.

According to (1)-(2) and as shown in Fig. 6, the “atomic” velocity of ECD field will be the most probable speed of the adjacent lower scale of EMD $\mathbf{u}_c = \mathbf{v}_{m,mp}$. Similarly, the “system” speed of EMD scale will be the most probable speed of ECD and “atomic” speed of EED fields such that $\mathbf{w}_m = \mathbf{v}_{c,mp} = \mathbf{u}_e$.

Because at thermodynamic equilibrium particles’ velocity field is governed by a *Gaussian* profile (Fig. 5) namely *Boltzmann* distribution function and the particle speeds must follow *Maxwell-Boltzmann* distribution function, it was recently shown that the energy spectrum of particles will follow *Planck* spectrum of equilibrium radiation [37, 38, 46, 47]. This correspondence between monatomic gas and photon gas becomes possible because of the recent closure of the gap between ideal gas theory on the one hand and *Planck* equilibrium radiation theory on the other hand [39]. For example, the field of isotropic homogeneous turbulence is identified as equilibrium eddy dynamics EED, Figs.1 and 2, with turbulent eddies defined as clusters of molecular clusters or super-clusters constituting the elements of the field. At thermodynamic equilibrium the energy spectrum of eddies in such isotropic stationary turbulent field will be governed by invariant *Planck* energy distribution law [37, 46-47]

$$\frac{\varepsilon_\beta dN_\beta}{V} = \frac{8\pi h}{u_\beta^3} \frac{\nu_\beta^3}{e^{h\nu_\beta/kT} - 1} d\nu_\beta \qquad (8)$$

shown in Fig. 7.

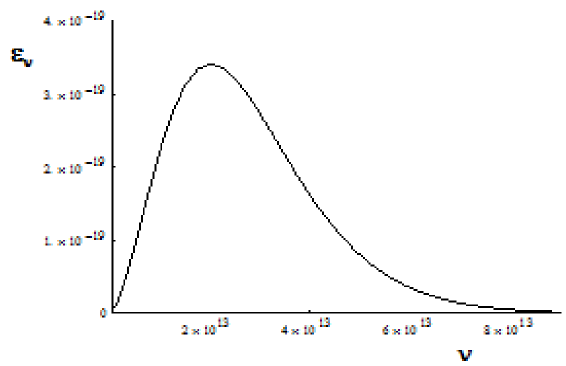


Fig. 7 Planck energy distribution law governing the energy spectrum of eddies at the temperature T = 300 K.

The three-dimensional energy spectrum E(k) for isotropic turbulence measured by *Van Atta* and *Chen* [48, 49] and shown in Fig. 8 is in qualitative agreement with *Planck* energy spectrum shown in Fig. 7.

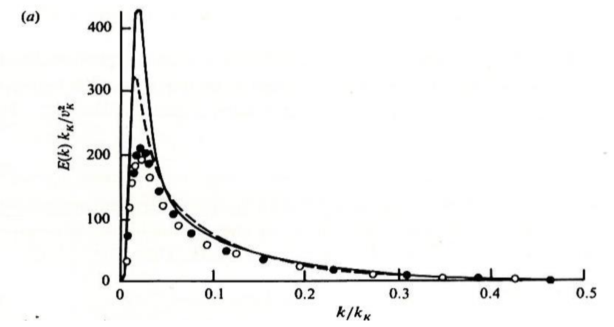


Fig. 8 Normalized three-dimensional energy spectra for isotropic turbulence [48].

In a more recent experimental investigation the energy spectrum of turbulent flow within the boundary layer in close vicinity of rigid wall was measured by *Marusic et al.* [50] and the reported energy spectrum have profiles quite similar to *Planck* distribution law.

To maintain stationary isotropic turbulence it is expected that both energy supply as well as energy dissipation spectrum should follow *Planck* law in (8). The experimental data obtained for one dimensional dissipation spectrum [51] along with *Planck* energy distribution (8) as well as this same distribution shifted by a constant amount of energy are shown in Fig. 9.

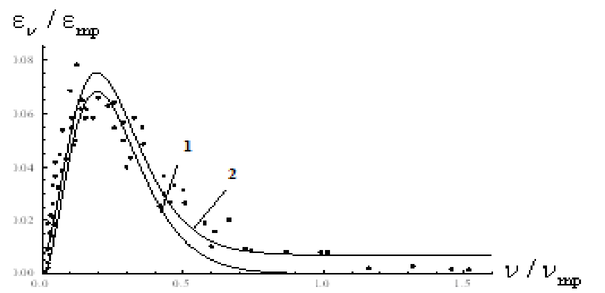


Fig. 9 One-dimensional dissipation spectrum [51] compared with (1) Planck energy distribution (2) Planck energy distribution with constant displacement.

Similar comparison with *Planck* energy distribution as shown in Fig. 9 is obtained with the experimental data of *Saddoughi and Veeravalli* [52] for one-dimensional dissipation spectrum of isotropic turbulence. Also, the normalized three-dimensional energy spectrum for homogeneous isotropic turbulence obtained from transformation of one-dimensional energy spectrum of *Lin* [53] by *Ling and Huang* [54]

$$E^* = \frac{\alpha^{*2}}{3}(K^* + \alpha^* K^{*2}) \exp(-\alpha^* K^*) \qquad (9)$$

is in close agreement with *Planck* law (8).

An important aspect of *Planck* law (8) is that at a given fixed temperature the energy spectrum of equilibrium field is time invariant. Since one may view *Planck* distribution as energy spectrum of eddy cluster sizes [38] this means that cluster sizes are stationary. Therefore, even though the number of eddies N_{jf} and their energy ε_{jf} in different fluid elements (energy levels) are different their product that is the total energy of all energy levels is the same [38]

$$U_j = \sum_j \varepsilon_j = N_j \varepsilon_j = U_{j+1} = ... = U_{mp} = \overline{U} \qquad (10a)$$

Thus *Boltzmann's* equipartition principle is satisfied in order to maintain time independent spectrum (Fig. 7) and avoid *Maxwell's* demon paradox [37]. Therefore, in stationary isotropic turbulence, energy flux occurs between fluid elements by transition of eddies of diverse sizes while leaving the fluid elements stochastically stationary in time. A schematic diagram of energy flux across hierarchies of eddies from large to small size is shown in Fig. 10 from the study by *Lumley et al.* [55].

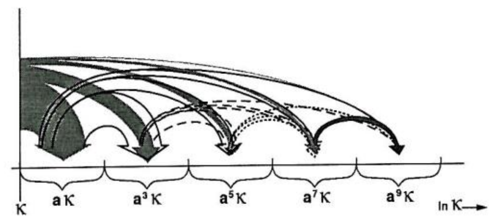


Fig. 10 A realistic view of spectral energy flux [55].

According to Fig. 6, *Maxwell-Boltzmann* distribution could be viewed as a spectrum of particle-cluster sizes that are stochastically stationary [38, 39]. Hence one arrives at a new paradigm of the physical foundation of quantum mechanics according to which *Bohr* stationary states will correspond to the stochastically stationary sizes of particle clusters, *de Broglie* wave packets, which will be governed by *Maxwell-Boltzmann* distribution function as shown in Fig. 6. Different energy levels of quantum mechanics could be identified as different size particle clusters (elements). Transfer of particle from a small rapidly- oscillating cluster j to a large slowly-oscillating cluster i constitutes transition from the high energy level j to the low energy level i (see Fig. 6). A scale invariant description of such transitions between energy levels at arbitrary scale β is schematically shown in Fig. 11.

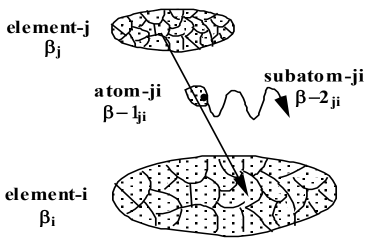


Fig. 11 Transition of "atom" a_{ij} from element-j to element-i leading to emission of sub-atomic particle s_{ij}.

Particle transitions will be accompanied with emission of a "sub-particle" that will carry away the excess energy

$$\Delta\varepsilon_{ji\beta} = \varepsilon_{j\beta} - \varepsilon_{i\beta} = h(\nu_{j\beta} - \nu_{i\beta}) \qquad (10b)$$

in harmony with *Bohr* theory of *atomic* spectra [38, 39].

3 Scale Invariant forms of Conservation Equations for Chemically Reactive Fields

Following the classical methods [18-22, 40-44], the scale-invariant forms of mass, thermal energy, linear and angular momentum conservation equations [38, 39] at scale β are given as

$$\frac{\partial \rho_{i\beta}}{\partial t_\beta} + \nabla.\left(\rho_{i\beta}\mathbf{v}_{i\beta}\right) = \Re_{i\beta} \qquad (11)$$

$$\frac{\partial \varepsilon_{i\beta}}{\partial t_\beta} + \nabla.\left(\varepsilon_{i\beta}\mathbf{v}_{i\beta}\right) = 0 \qquad (12)$$

$$\frac{\partial \mathbf{p}_{i\beta}}{\partial t_\beta} + \nabla.\left(\mathbf{p}_{i\beta}\mathbf{v}_{i\beta}\right) = -\nabla.\mathbf{P}_{ij\beta} \qquad (13)$$

$$\frac{\partial \boldsymbol{\pi}_{i\beta}}{\partial t_\beta} + \nabla.\left(\boldsymbol{\pi}_{i\beta}\mathbf{v}_{i\beta}\right) = \rho_{i\beta}\boldsymbol{\omega}_\beta.\nabla\mathbf{v}_{i\beta} \qquad (14)$$

involving the *volumetric density* of thermal energy $\varepsilon_{i\beta} = \rho_{i\beta}\tilde{h}_{i\beta}$, linear momentum $\mathbf{p}_{i\beta} = \rho_{i\beta}\mathbf{v}_{i\beta}$, and angular momentum $\boldsymbol{\pi}_{i\beta} = \rho_{i\beta}\boldsymbol{\omega}_{i\beta}$. Also, $\Re_{i\beta}$ is the chemical reaction rate, $\tilde{h}_\beta$ is the absolute enthalpy [34]

$$\tilde{h}_{i\beta} = \int_0^T c_{pi\beta} dT_\beta \qquad (15)$$

In the energy conservation equation (12) instead of the classical practice of considering the internal energy the total thermal energy $\varepsilon_{i\beta} = \rho_{i\beta}\tilde{h}_{i\beta}$ namely absolute enthalpy (15) is considered. Therefore, the "potential" energy p/ρ of the moving fluid often referred to as "flow work" is also taken into account. Also, an important correction to invariant *Helmholtz* vorticity equation (14) is made herein by inclusion of *Coriolis* force that was neglected in earlier studies [38, 39] as discussed in Section 8. Finally, the scale dependence and quantum nature of physical "time" discussed in [38] is not addressed in the present study by taking $t_\beta = t$.

The partial stress tensor $\mathbf{P}_{ij\beta}$ is [40]

$$\mathbf{P}_{ij\beta} = m_\beta \int (\mathbf{u}_{i\beta} - \mathbf{v}_{i\beta})(\mathbf{u}_{j\beta} - \mathbf{v}_{j\beta}) f_\beta du_\beta \qquad (16)$$

The derivation of (13) involves the definition of the peculiar velocity (3) along with the identity

$$\overline{\mathbf{V}'_{i\beta}\mathbf{V}'_{j\beta}} = \overline{(\mathbf{u}_{i\beta} - \mathbf{v}_{i\beta})(\mathbf{u}_{j\beta} - \mathbf{v}_{j\beta})} = \overline{\mathbf{u}_{i\beta}\mathbf{u}_{j\beta}} - \mathbf{v}_{i\beta}\mathbf{v}_{j\beta} \qquad (17)$$

Also, summation of (13) over all the species results in *Cauchy* equation of motion at the next larger scale β+1

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla.(\rho\mathbf{v}\mathbf{v}) = -\nabla.\mathbf{P} \qquad (18)$$

where $\rho = \rho_{\beta+1}$, $\mathbf{v} = \mathbf{v}_{\beta+1}$ and the total or mixture stress tensor is [40, 42]

$$\mathbf{P} = \sum_i \mathbf{P}_{i\beta} = \sum_i m_{i\beta} \int (\mathbf{u}_{i\beta} - \mathbf{v}_{i\beta})(\mathbf{u}_{j\beta} - \mathbf{v}_{j\beta}) f_{i\beta} du_{i\beta} \qquad (19)$$

The subscript "i" in (11)-(14) that conventionally refers to chemical "specie i" is being also employed to identify the scale of "element" of the field. For example, in case of velocity with (atomic, element, and system) velocities denoted as $(\mathbf{u}_{i\beta}, \mathbf{v}_{i\beta}, \mathbf{w}_{\beta})$ one associates the average of a local group of specie i particles each with velocity $\mathbf{u}_{i\beta}$ as "element" or mean local velocity $\mathbf{v}_{i\beta}$. In moving to the next larger scale of β+1, the system velocity $\mathbf{w}_{\beta} = <\mathbf{v}_{i\beta}>$ is then identified as the "mixture" velocity. When moving to the yet higher scale, one identifies $(\text{system})_{\beta} \Rightarrow (\text{element})_{\beta+1}$ such that $\mathbf{w}_{\beta} \Rightarrow \mathbf{v}_{j\beta+1}$ as the new "specie j" or "element" velocity and repeats the same procedure as before. According to this convention, the ordinary mass-average velocity of fluid mechanics $\mathbf{v}_{o}$ will correspond to the system or "mixture" velocity $\mathbf{w}_{m} = \mathbf{v}_{c}$ of molecular-dynamic scale. This is because the mean thermal speed of molecules is the speed of sound $\mathbf{v}_{m,mp}$ that is a stationary random variable for a given temperature and constitutes the "atomic" velocity $\mathbf{u}_{c} = \mathbf{v}_{m,mp}$ of the next higher scale of ECD field by (1).

Considering (14), the classical definition of vorticity involves the curl of linear velocity $\nabla \times \mathbf{v}_{\beta} = \boldsymbol{\omega}_{\beta}$ thus giving rotational velocity of particle a secondary status in that it depends on its translational velocity $\mathbf{v}_{\beta}$. However, it is known that particle's rotation about its center of mass is independent of the translational motion of its center of mass. In other words, translational, rotational, and vibrational (pulsational) motions of particle are independent degrees of freedom that should not be necessarily coupled.

To resolve this paradox, iso-spin of the particle at scale β is defined as the curl of the velocity at the next lower scale of β−1 [56]

$$\boldsymbol{\varpi}_{i\beta} = \nabla \times \mathbf{v}_{i\beta-1} = \nabla \times \mathbf{u}_{i\beta} \tag{20}$$

With such a definition, the rotational velocity while having a connection to some type of translational motion at internal scale of β−1, retains its independent degree of freedom at the external scale β as required. A schematic description of iso-spin and vorticity fields is shown in Fig. 12.

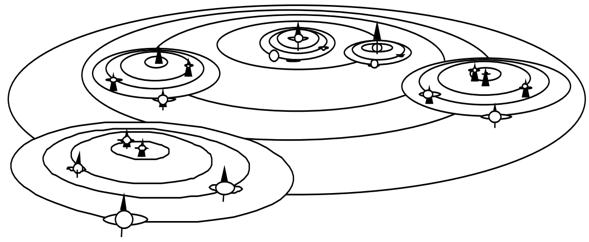


Fig. 12 Description of internal (iso-spin) versus external vorticity fields in cosmology.

Thus, what appears as rotational motion at scale β, when viewed at the lower scale β−1 is identified as orbital translational motion and a new local rotational motion is identified at this smaller scale. The nature of galactic vortices in cosmology and the associated dissipation have been discussed [29, 57].

The local velocity $\mathbf{v}_{\beta}$ in (11)-(14) is expressed in terms of the convective $\mathbf{w}_{\beta}$ and the diffusive $\mathbf{V}_{\beta}$ velocities [34, 58]

$$\mathbf{w}_{\beta} = \mathbf{v}_{\beta} - \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_{i\beta}\nabla \ln(\rho_{i\beta}) \tag{21a}$$

$$\mathbf{w}_{\beta} = \mathbf{v}_{\beta} - \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_{i\beta}\nabla \ln(\varepsilon_{i\beta}) \tag{21b}$$

$$\mathbf{w}_{\beta} = \mathbf{v}_{\beta} - \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_{i\beta}\nabla \ln(\mathbf{p}_{i\beta}) \tag{21c}$$

$$\mathbf{w}_{\beta} = \mathbf{v}_{\beta} - \mathbf{V}_{\beta rhg} \quad , \quad \mathbf{V}_{\beta rhg} = -\nu_{i\beta}\nabla \ln(\boldsymbol{\pi}_{i\beta}) \tag{21d}$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$, $\mathbf{V}_{\beta rhg}$) are respectively the diffusive, the thermo-diffusive, the linear and the angular hydro-diffusive velocities and $\nu_{i\beta} = \mu_{i\beta} / \rho_{i\beta}$. Since for an ideal gas $\tilde{h}_{\beta} = c_{p\beta}T_{\beta}$, when $c_{p\beta}$ is constant and $T = T_{\beta}$, (21b) reduces to the *Fourier* law of heat conduction

$$\mathbf{q}_{i\beta} = \rho_{i\beta}\tilde{h}_{i\beta}\mathbf{V}_{\beta t} = -\kappa_{i\beta}\nabla T \tag{22}$$

where $\kappa_{i\beta}$ and $\alpha_{i\beta} = \kappa_{i\beta} / (\rho_{i\beta}c_{pi\beta})$ are the thermal conductivity and diffusivity. Similarly, (21c) may be identified as the shear stress associated with diffusional flux of linear momentum and expressed by the generalized *Newton* law of viscosity [34, 58]

$$\boldsymbol{\tau}_{ij\beta} = \rho_{i\beta}\mathbf{v}_{j\beta}\mathbf{V}_{ij\beta h} = -\mu_{i\beta}\partial \mathbf{v}_{j\beta} / \partial \mathbf{x}_{i} \tag{23}$$

Finally, (21d) may be identified as the torsional stress induced by diffusional flux of angular momentum and expressed as [36]

$$\boldsymbol{\tau}_{\mathrm{ijr\beta}} = \rho_\beta \boldsymbol{\omega}_{\mathrm{j\beta}} \mathbf{V}_{\mathrm{ij\beta rh}} = -\mu_\beta \partial \boldsymbol{\omega}_{\mathrm{j\beta}} / \partial \mathbf{x}_{\mathrm{i}} \quad (24)$$

Substituting from (21a)-(21d) into (11)-(14), neglecting the cross-diffusion terms and assuming constant transport coefficients with unity *Prandtl* and *Schmidt* numbers $\mathrm{Sc}_\beta = \mathrm{Pr}_\beta = 1$ result in

$$\frac{\partial \rho_{\mathrm{i\beta}}}{\partial t} + \mathbf{w}_\beta . \nabla \rho_{\mathrm{i\beta}} = D_{\mathrm{i\beta}} \nabla^2 \rho_{\mathrm{i\beta}} + \Re_{\mathrm{i\beta}} \quad (25)$$

$$\frac{\partial \mathrm{T}_{\mathrm{i\beta}}}{\partial t} + \mathbf{w}_\beta . \nabla \mathrm{T}_{\mathrm{i\beta}} = \alpha_\beta \nabla^2 \mathrm{T}_{\mathrm{i\beta}} - \tilde{h}_{\mathrm{i\beta}} \Re_{\mathrm{i\beta}} / (\rho_{\mathrm{i\beta}} c_{\mathrm{pi\beta}}) \quad (26)$$

$$\frac{\partial \mathbf{v}_{\mathrm{i\beta}}}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_{\mathrm{i\beta}} = \nu_{\mathrm{i\beta}} \nabla^2 \mathbf{v}_{\mathrm{i\beta}} - \frac{\nabla . \mathbf{P}_{\mathrm{ij\beta}}}{\rho_{\mathrm{i\beta}}} - \frac{\mathbf{v}_{\mathrm{i\beta}} \Re_{\mathrm{i\beta}}}{\rho_{\mathrm{i\beta}}} \quad (27)$$

$$\frac{\partial \boldsymbol{\omega}_{\mathrm{i\beta}}}{\partial t} + \mathbf{w}_\beta . \nabla \boldsymbol{\omega}_{\mathrm{i\beta}} = \nu_{\mathrm{i\beta}} \nabla^2 \boldsymbol{\omega}_{\mathrm{i\beta}} + \boldsymbol{\omega}_\beta . \nabla \mathbf{v}_{\mathrm{i\beta}} - \frac{\boldsymbol{\omega}_{\mathrm{i\beta}} \Re_{\mathrm{i\beta}}}{\rho_{\mathrm{i\beta}}} \quad (28)$$

The main new feature of the modified form of the equation of motion (27) is its linearity due to the difference between convective $\mathbf{w}_\beta$ versus local $\mathbf{v}_\beta$ velocities as compared with the non-linear classical *Navier-Stokes* equation of motion. The linearity of (27) in harmony with *Carrier* [59] equation resolves the classical paradox of drag reciprocity [60, 61].

4 The Invariant Boltzmann Equation and the Associated Invariant Enskog Equation of Change

Following the classical methods [19, 22, 40-44], the scale-invariant form of *Boltzmann* equation in the absence of body forces is expressed as

$$\frac{\partial f_{\mathrm{i\beta}}}{\partial t} + \mathbf{u}_{\mathrm{i\beta}} . \nabla f_{\mathrm{i\beta}} = \frac{\delta f_{\mathrm{i\beta}}}{\delta t} \quad (29)$$

where $f_{\mathrm{i\beta}}(\mathbf{x}_\beta, \mathbf{u}_\beta, t_\beta)$ is the invariant distribution function at the scale β. Multiplication of (29) with an arbitrary invariant function of velocity $\psi_{\mathrm{i\beta}}$ for specie i and integration over all velocity space under appropriate assumptions [22, 40-43] results in the *Enskog* equation of change

$$\frac{\partial}{\partial t}(\mathrm{n}_{\mathrm{i\beta}} \overline{\psi_{\mathrm{i\beta}}}) + \nabla . (\mathrm{n}_{\mathrm{i\beta}} \overline{\psi_{\mathrm{i\beta}} \mathbf{u}_{\mathrm{i\beta}}}) = \int \psi_{\mathrm{i\beta}} \frac{\delta f_{\mathrm{i\beta}}}{\delta t} \mathrm{du}_{\mathrm{i\beta}} \quad (30)$$

In the generalized *Boltzmann* equation [42] we consider that only collisions that lead to "chemical reactions" contribute to rate of change of volumetric number density dn_i/dt of particles of species i. Therefore, the chemical reaction source term on the RHS of (30) will be identically zero except for mass conservation equation.

Following the classical methods [40-43], the summational invariants $\Psi_{\mathrm{i\beta}}$ for mass, thermal energy, linear, and angular momentum are defined as

$\psi_{\mathrm{i\beta}} = \mathrm{m}_{\mathrm{i\beta}}$	Mass	(31a)
$\psi_{\mathrm{i\beta}} = \mathrm{m}_{\mathrm{i\beta}} \tilde{h}_{\mathrm{i\beta}}$	Thermal energy	(31b)
$\psi_{\mathrm{i\beta}} = \mathrm{m}_{\mathrm{i\beta}} \mathbf{u}_{\mathrm{i\beta}}$	Linear momentum	(31c)
$\psi_{\mathrm{i\beta}} = \mathrm{m}_{\mathrm{i\beta}} \boldsymbol{\varpi}_{\mathrm{i\beta}}$	Angular momentum	(31d)

The new summational invariant (31d) involves the fluctuating iso-spin $\boldsymbol{\varpi}_{\mathrm{i\beta}}$ of particle [56] of mass $\mathrm{m}_{\mathrm{i\beta}}$ defined in (20).

5 The Invariant forms of Continuity Equation

Following the classical methods [40-44], substitution of $\psi_{\mathrm{i\beta}} = \mathrm{m}_{\mathrm{i\beta}}$ into (30) gives invariant continuity equation

$$\frac{\partial \rho_{\mathrm{i\beta}}}{\partial t} + \nabla . \left(\rho_{\mathrm{i\beta}} \mathbf{v}_{\mathrm{i\beta}} \right) = \Re_{\mathrm{i\beta}} \quad (32)$$

with the invariant reaction rate defined as

$$\Re_{\mathrm{i\beta}} = \mathrm{m}_{\mathrm{i\beta}} \int (\delta f_{\mathrm{i\beta}} / \delta t) \mathrm{du}_{\mathrm{i\beta}} \quad (33)$$

By summing (32) over all species one obtains the continuity equation for the next higher scale [36]

$$\frac{\partial \rho_{\beta+1}}{\partial t} + \nabla . \left(\rho_{\beta+1} \mathbf{v}_{\beta+1} \right) = \Re_{\beta+1} \quad (34)$$

because

$$\sum_i{}' \Re_{i\beta} = \sum_i{}' m_{i\beta} \int (\delta f_{i\beta} / \delta t) du_{i\beta} =$$
$$= \frac{\delta}{\delta t} \sum_i m_{i\beta} \int f_{i\beta} du_{i\beta} = \frac{\delta}{\delta t} \sum_i m_{i\beta} n_{i\beta} =$$
$$= \frac{\delta}{\delta t} \sum_i \rho_{i\beta} = \frac{\delta}{\delta t} \rho_{\beta+1} = \frac{\delta}{\delta t} (m_{\beta+1} n_{\beta+1}) =$$
$$= m_{\beta+1} \int (\delta f_{\beta+1} / \delta t) du_{\beta+1} = \Re_{\beta+1} \quad (35)$$

The result (35) is important since it represents a generalized expression for the reaction rate valid for all scales within the hierarchy shown in Fig. 1. The gravitational mass at any scale is therefore convertible into energy through "chemical reactions" at the rate defined by (35).

When the local velocity in (32) is expressed in terms of the convective and diffusive velocities from (21a) one obtains (25) that for a pure system $\rho_{i\beta} = \rho_\beta$ in the absence of reactions $\Re_{i\beta} = 0$ results in the modified form of continuity equation [36]

$$\frac{\partial \rho}{\partial t} + \mathbf{w}.\nabla\rho - D\nabla^2\rho = 0 \quad (36)$$

where *D* is the coefficient of self-diffusion.

The importance of the modified form of the continuity equation (36) is that in the absence of convective velocity it reduces to diffusion equation. The second order derivative of diffusion equation allows for determination of internal structure of normal shock waves [62]. For one-dimensional problem of normal shock with an imposed convective velocity w' the conservation equations in a moving coordinate $z' = x' - w'_s t'$ reduce to

$$w \frac{df}{dz} = \frac{d^2 f}{dz^2} \qquad f = v_x, \theta, Y \quad (37)$$

where the dimensionless coordinates are

$$z = z' / \ell_H \quad , \quad x = x' / \ell_H \quad , \quad \ell_H = \nu / w'_s$$

, w'_s is the shock velocity, and $w = w' / w'_s$. Also, the dimensionless velocity, temperature, and density are defined as

$$v = (v' - v'_\infty) / (v'_{-\infty} - v'_\infty) \quad (38)$$
$$\theta = (T - T_\infty) / (T_{-\infty} - T_\infty) \quad (39)$$
$$Y = (\rho - \rho_\infty) / (\rho_{-\infty} - \rho_\infty) \quad (40)$$

When viewed at the outer laboratory coordinates (x', z') at LCD scale the shock appears as a mathematical surface of discontinuity. However, in terms of the stretched hydro-thermo-diffusive coordinate (x, z) the internal structure of the shock is revealed at the scale of LMD (Fig. 2). At this smaller scale the constant convective velocity will manifest its coordinate dependence and is expressed as $w = (1 - z)/2$ such that in terms of the stretched coordinate $\eta = (z - 1)/2$ Eq. (37) becomes

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \qquad f = v_x, \theta, Y \quad (41)$$

$$v = \theta = Y = 0 \qquad \eta = \infty \quad (42a)$$
$$v = \theta = Y = 1 \qquad \eta = -\infty \quad (42b)$$

with the solution

$$v_x = \theta = Y = \frac{1}{2} \mathrm{erfc}(\eta) \quad (43)$$

The error-function shape of the solution in Eq. (43) is in agreement with both experimental data [63, 64] as well as numerical calculations [65-68] of normal shock. However, the experimental data of Sherman [63] shows a shift of about 0.2 in the origin of normalized temperature profile as seen in Fig. 13a.

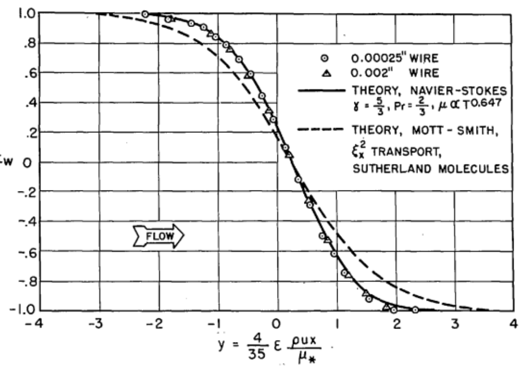


Fig. 13a Comparison between experiment and theory from Sherman [63] for normal shock in helium at *Ma* = 1.82.

It is possible to account for this shift of coordinate origin in terms of the location of the shock relative to the center of the hydrodynamic shock structure. This is because the position of shock is at $z = -z^*$ where w = 1 while the center of the shock hydrodynamic structure is at z = 0. From the solution in Eq. (43) the thickness of the shock to the accuracy of 0.99954 and in units of characteristic viscous length ℓ_H is $z^* = \delta_s / 2\ell_H = 2.5$ hence

$$\delta_s = 5\ell_H \tag{44}$$

Therefore, with normalized coordinate $\xi = \eta/(\delta_s/2\ell_H)$ and $y = z/(\delta_s/2\ell_H)$ based on Eq. (44) one can express the *normalized* temperature profile as [62].

$$\Theta = -\mathrm{erf}(\xi) = \mathrm{erf}(0.2 - y) \tag{45}$$

The predicted temperature profile (45) is in close agreement with the experimental data of normalized wire temperature Θ across normal shock in helium at Mach number Ma = 1.82 reported by *Sherman* [63] as shown in Fig. 13b

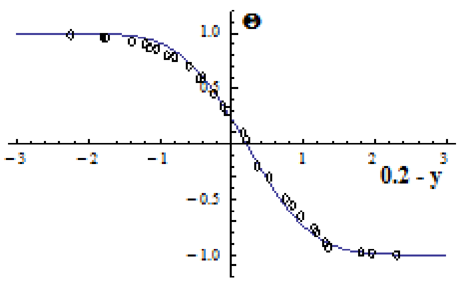


Fig. 13b Comparison between measured normalized wire temperature Θ versus position (0.2 – y) in normal shock [63] and theory (45).

It is interesting to examine the implication of the present theory to the results of a recent study of normal shocks in a rarefied polyatomic gas by Taniguchi et al., [68]. In this study, besides the symmetric type A shock structure like Fig. 13b, the authors identified non-symmetric type B and type C shock structures composed of thin and thick layers. An example of type C structure is shown in Fig. 14 to be compared with FIG. 6 of [68]. According to Taniguchi et al., [68] the thickness of the thick layer could be as long as several centimeters.

It is suggested that the thick layer identified by Taniguchi et al., [68] should correspond to conventional gas-dynamics of viscous flow at LCD scale. To describe the type C shock structure, one notes that for hypersonic flows no signals could propagate to the left of the shock location at point A of Fig. 14 because $v_1 > a_1$ and $Ma_1 > 1$. Also, the velocity at point C at the exit of the shock surface of discontinuity (Fig. 14) will be equal to the local speed of sound $v_2 = v^* = a_2$ such that $Ma_2 = 1$. Therefore, beyond point C the flow will be subsonic and the velocity profile will be governed by the viscous equation of motion (41) but at the larger scale of LCD with the solution

$$v = v'/a = \mathrm{erfc}(z) \tag{46}$$

as shown in Fig. 14.

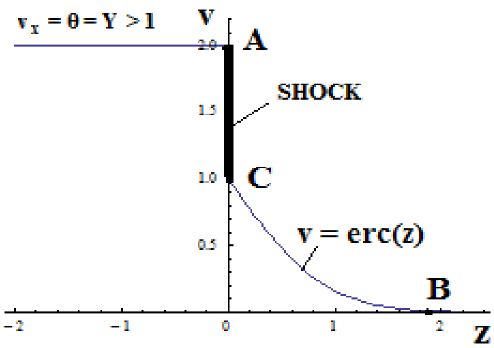


Fig. 14 Structure of shock in polyatomic gas with thin and thick layers corresponding to type C shock structure of Taniguchi et al., [68].

To summarize, a hypersonic flow at LCD scale β = c corresponding to conventional gas dynamics (Fig. 2) will retain its constant velocity until it arrives at the shock location A in Fig. 14. From this point one must move to the lower scale of LMD β = m and the shock structure will be given by the solutions (43) and (45) with structure shown in Fig. 13b. However, at LCD scale the shock will appear as a mathematical surface of discontinuity as shown in Fig. 14. Finally, from point C to the far downstream point B in stationary gas the viscous flow at LCD scale will be governed by the solution in Eq. (46).

As a second example that reveals the important difference between (32) versus (36), let us consider the one-dimensional unsteady problem of diffusion of an infinite yz- plane source of mass placed at the origin x = 0 at time t = 0. Because of the symmetry of the problem the center of mass will remain stationary $\mathbf{w}_\beta = 0$ and (36) reduces to the diffusion equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \tag{47}$$

that governs the time evolution of density discontinuity in a background field composed of the same fluid. With appropriate boundary and initial conditions the solution of (47) is

$$\rho = \frac{M}{\sqrt{4\pi D t}}\exp\left(-\frac{x^2}{4Dt}\right) \tag{48}$$

where

$$M = \int_{-\infty}^{+\infty} \rho dx \tag{49}$$

6 The Invariant form of Energy Conservation Equation

To obtain the scale-invariant form of the energy equation the exact nature of the summational invariant $\psi_{i\beta}$ to be substituted in (30) must first be identified. First the energy for translational oscillations of particles in two directions (x^+, x^-) is considered. According to (3), particle translational velocity is the sum of the mean or cluster velocity and the random peculiar velocity

$$\mathbf{u}_{mj} = \mathbf{v}_{mj} + \mathbf{V}'_{mj} = \mathbf{u}_{cj} + \mathbf{V}'_{mj} \tag{50}$$

The above definition is in the same spirit as in cosmology where the peculiar velocity of a galaxy is defined as the difference between its velocity and the mean velocity of the cluster of galaxies to which it belongs. The thermodynamic system being considered herein is composed of a spectrum of molecular clusters under stochastically stationary state. In a recent study [38], it was shown that three different flow regimes based on the nature of cluster velocity $\mathbf{v}_m$ in (3) could be identified. For a system of ideal gas at thermodynamic equilibrium all three velocities in (50) will be random such that (50) when squared, averaged, and multiplied by particle mass leads to the kinetic energy

$$m\overline{u^2_{mxj}} = m\overline{v^2_{mxj}} + m\overline{V'^2_{mxj}} = m\overline{u^2_{cxj}} + m\overline{V'^2_{mxj}} \tag{51}$$

since $\overline{\mathbf{u}_{cxj}\mathbf{V}'_{mxj}} = 0$. The internal energy of particle due to translational motion of particle in two directions (x^+, x^-) is expressed as

$$\hat{u}_t = \frac{1}{2}m\overline{v^2_{mx+}} + \frac{1}{2}m\overline{v^2_{mx-}} =$$

$$= \frac{1}{2}m\overline{u^2_{cx+}} + \frac{1}{2}m\overline{u^2_{cx-}} = m\overline{v^2_{mx+}} \tag{52}$$

Next, the potential energy due to "stress" is related to the kinetic energy of peculiar velocity as

$$\hat{\varepsilon}_p = \frac{1}{2}m\overline{V'^2_{mx+}} + \frac{1}{2}m\overline{V'^2_{mx-}} =$$

$$= m\overline{V'^2_{mx+}} = \frac{1}{3}m\overline{V'^2_{m+}} = p\hat{v} \tag{53}$$

where pressure is defined as $p = nm\overline{V'^2_{m+}}/3$. Hence, the total energy associated with harmonic translational motion of particle will become [69]

$$\hat{\varepsilon}_t = \hat{u}_t + \hat{\varepsilon}_p = \hat{u}_t + p\hat{v} \tag{54}$$

Since particles are neither point masses without any physical extent nor absolutely rigid, their rotational and vibrational energies cannot be properly neglected as was emphasized by *Clausius* in his pioneering investigation of the mechanical theory of heat [70]

> **"In liquids, therefore, an oscillatory, a rotatory, and a translator motion of the molecules take place, but in such a manner that these molecules are not themselves separated from each other, but even in the absence of external forces, remain within a certain volume"**

Therefore, following *Clausius* the internal kinetic energy of rotational and vibrational motion of particles in two directions (θ^+, θ^-), (r^+, r^-) are written as [35, 69]

$$\hat{u}_r = \varepsilon_r = \frac{1}{2}\mathrm{I}\overline{\omega^2_{m\theta+}} + \frac{1}{2}\mathrm{I}\overline{\omega^2_{m\theta-}} = \mathrm{I}\overline{\omega^2_{m\theta+}} \tag{55}$$

$$\hat{u}_v = \varepsilon_v = \frac{1}{2}\chi\overline{r^2_{m+}} + \frac{1}{2}\chi\overline{r^2_{m-}} = \chi\overline{r^2_{m+}} \tag{56}$$

where (I, χ) are respectively the moment of inertia and the spring constant. Hence, the internal "atomic" energy of particle is defined as the sum of its translational, rotational, and vibrational kinetic energies from equations (52), (55) and (56)

$$\hat{u} = \hat{u}_t + \hat{u}_r + \hat{u}_v = \mathrm{m}\overline{v^2_{x+}} + \mathrm{I}\overline{\omega^2_{\theta+}} + \chi\overline{r^2_+} \tag{57}$$

Finally, by (53) and (57) the total atomic energy or the atomic enthalpy is defined as the sum of atomic internal kinetic energy and atomic external potential energy written as [69]

$$\hat{h} = \hat{u}_t + \hat{u}_r + \hat{u}_v + \mathrm{p}\hat{v} = \hat{u} + \mathrm{p}\hat{v} \tag{58}$$

such that the total enthalpy becomes

$$H = U + \mathrm{p}V \tag{59}$$

where $(H,U,V) = \mathrm{N}(\hat{h},\hat{u},\hat{v})$. According to equation (58) the system has four degrees of freedom and at equilibrium *Boltzmann* principle of equipartition of energy requires

$$\hat{u}_t = \hat{u}_r = \hat{u}_v = \mathrm{p}\hat{v} \tag{60}$$

In view of (58) and (60) the atomic enthalpy per unit mass $\tilde{h}_m$ is defined as

$$\tilde{h}_{i\beta} = \hat{h}_{i\beta} / \mathrm{m}_{i\beta} \tag{61}$$

By (61) the expression for summational invariant for thermal energy in (31b) becomes

$$\psi_{i\beta} = \mathrm{m}_{i\beta}\tilde{h}_{i\beta} = \hat{h}_{i\beta} \tag{62}$$

Substituting (62) in *Enskog* equation of change (30) without the source term on the RHS results in the invariant form of energy conservation equation (12)

$$\frac{\partial \varepsilon_{i\beta}}{\partial t} + \nabla.\left(\varepsilon_{i\beta}\mathbf{v}_{i\beta}\right) = 0 \tag{63}$$

with the volumetric density of total thermal energy defined as $\varepsilon_{i\beta} = \rho_{i\beta}\tilde{h}_{i\beta}$. Substituting from (21b) in (63), neglecting the cross diffusion terms, assuming constant transport coefficients such that the absolute enthalpy of ideal gas could be expressed as $\tilde{h}_{i\beta} = c_{pi\beta}T_{i\beta}$ and setting $T_{i\beta} = T_\beta$ results in scale invariant form of energy conservation equation [36]

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta.\nabla T_\beta - \alpha_{i\beta}\nabla^2 T_\beta = -\tilde{h}_{i\beta}\Re_{i\beta} / (\rho_\beta c_{pi\beta}) \tag{64}$$

As an example of solution of (64) the simple problem of steady plane parallel laminar flow over a flat plate schematically shown in Fig. 15 is considered.

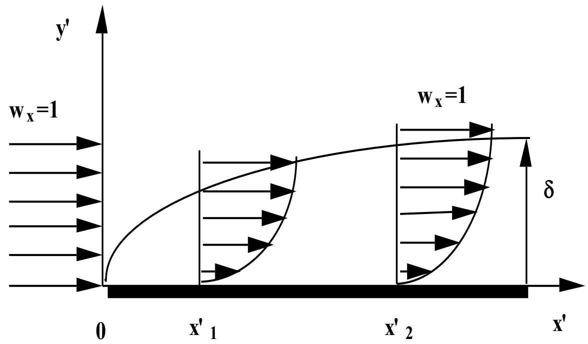


Fig. 15 Laminar boundary layer over a flat plate.

In the absence of chemical reactions and with negligible pressure gradient the equations governing partial density (37), temperature (64) , and velocity become identical for unity *Prandtl* and *Schmidt* numbers $Sc_\beta = Pr_\beta = 1$ [71]

$$\frac{d^2 g}{d\xi^2} + 2\xi\frac{dg}{d\xi} = 0 \qquad g = \theta, v_x, Y_i \tag{65}$$

where similarity variable ξ is defined as

$$\xi = \frac{y}{2\sqrt{2x}} = \frac{\eta}{2\sqrt{2}} \tag{66}$$

Furthermore, defining the dimensionless temperature, velocity, partial density, and coordinates as

$$\theta = \frac{T - T_w}{T_\infty - T_w} \ , \ v_x = \frac{v'_x}{w'_{xo}} \ , \ Y_i = \frac{\rho_i - \rho_{iw}}{\rho_{i\infty} - \rho_{iw}} \tag{67}$$

$$x = x'/\ell_H \ , \ y = y'/\ell_H \ , \ \ell_H = \nu / w'_{xo} \tag{68}$$

leads to identical boundary conditions

$$\theta = v = Y_i = 0 \qquad \xi = 0 \tag{69a}$$

$$\theta = v = Y_i = 1 \qquad \xi = \infty \tag{69b}$$

such that the solution of (65) and (69) for temperature, velocity, and concentration profiles become identical [71]

$$\theta = v_x = Y_i = \mathrm{erf}(\eta / 2\sqrt{2}) \tag{70}$$

The solution (70) leads to the Nusselt number Nu

$$\mathrm{Nu} = hL/\kappa = (1/\sqrt{2\pi})\,\mathrm{Re}_x^{1/2} = 0.399\,\mathrm{Re}_x^{1/2} \tag{71}$$

to be compared with the classical result [41]

$$\mathrm{Nu} = 0.332\,\mathrm{Re}_x^{1/2} \tag{72}$$

Following steps parallel to (47)-(49) for mass diffusion, the diffusion of a plane source of heat placed at the origin x = 0 at time t = 0 in a none-reactive $\Re_{i\beta} = 0$ field leaves the center of mass stationary hence $\mathbf{w}_\beta$ = 0 and (64) reduces to diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{73}$$

that with the appropriate initial and boundary conditions has the solution

$$T = \frac{Q_\beta}{\rho c_p \sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right) \tag{74}$$

where

$$Q_\beta = \int_{-\infty}^{+\infty} \rho \tilde{h} dx = \int_{-\infty}^{+\infty} \rho c_p T dx \tag{75}$$

7 The Invariant form of Equation of Motion

The invariant form of equation of motion is obtained by the classical methods [40-44] of substituting $\psi_{i\beta} = m_{i\beta}\mathbf{u}_{i\beta}$ into (30) without the source term on its' RHS to obtain invariant *Cauchy* equation of motion

$$\frac{\partial(\rho_{i\beta}\mathbf{v}_{i\beta})}{\partial t} + \nabla.\left(\rho_{i\beta}\mathbf{v}_{i\beta}\mathbf{v}_{j\beta}\right) = -\nabla.\mathbf{P}_{ij\beta} \tag{76}$$

where the stress tensor on the right hand-side has been defined in (16).

The history of derivation of *Navier–Stokes* equation from (76) has been described in an excellent review by *Darrigol* [72]. The derivation begins with the introduction of *Cauchy* total stress tensor in the form [40-44, 72-74]

$$\sigma_{ij} = -p\delta_{ij} + \lambda' \, \nabla.\mathbf{v}\delta_{ij} + 2\mu e_{ij} \tag{77}$$

where the rate of strain is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{78}$$

For fluids the two *Lame* constants (μ, λ') are called the first and the second viscosity coefficients [72-74]. The classical expression for the stress tensor of fluids (77) was introduced in close analogy with the stress tensor for deformation of elastic solids developed by the founders *Cauchy* and *Poisson* [72, 73]. Thus the role played by the strain tensor ε_{ij} in solids was related to the rate of strain tensor e_{ij} in liquids.

In view of the negative sign in (76) the total stress tensor for fluids is expressed as

$$\sigma_{ij\beta} = p_{i\beta}\delta_{ij\beta} + \lambda'_{i\beta}\nabla.\mathbf{v}_{i\beta}\delta_{ij\beta} + 2\mu_{i\beta}e_{ij\beta} \tag{79}$$

By classical methods [72-74] the mean pressure of fluid in motion is defined as

$$\mathbf{P}_{ij\beta} = \frac{1}{3}\sigma_{ii\beta}\delta_{ij\beta} = p_{i\beta}\delta_{ij\beta} + (\lambda'_{i\beta} + \frac{2}{3}\mu_{i\beta})\nabla.\mathbf{v}_{i\beta}\delta_{ij\beta} \quad (80)$$

According to the conventional practice one makes the *Stokes* assumption of zero bulk viscosity $b_{s\beta} = 0$ such that the two *Lame* constants $(\lambda'_\beta, \mu_\beta)$ will be related by [74]

$$b_{s\beta} = \lambda'_\beta + \frac{2}{3}\mu_\beta = 0 \quad (81)$$

However, according to both *Cauchy* and *Poisson* as the intermolecular spacing vanish $R \to 0$ the two *Lame* constants must satisfy the limiting expression [73]

$$\lambda'_\beta + \mu_\beta = \underset{R\to 0}{Lim} R^4 f(R) = 0 \quad (82)$$

that by (81) leads to a finite coefficient of bulk viscosity [39]

$$b_\beta = \lambda'_\beta + \frac{2}{3}\mu_\beta = -\frac{\mu_\beta}{3} \quad (83)$$

Therefore, in *Cauchy-Poisson* limit (82) all tangential stresses will vanish as was emphasized by *Darrigol* [73]

"Poisson and Cauchy both assumed the limit to be zero. Then the medium loses its rigidity since the transverse pressures disappear."

leaving only normal stresses. By (80) and (83) the total normal stress tensor for fluids becomes [39]

$$\mathbf{P}_{ij\beta} = p_{i\beta}\delta_{ij\beta} - \frac{1}{3}\mu_{i\beta}\nabla.\mathbf{v}_{i\beta}\delta_{ij\beta} = (p_{ti\beta} + p_{hi\beta})\delta_{ij\beta} \quad (84)$$

when the hydrodynamic pressure is defined as

$$p_{hi\beta} = -\frac{\mu_{i\beta}}{3}\nabla.\mathbf{v}_{i\beta} \quad (85)$$

According to (84) a moving fluid besides the thermodynamic pressure $p_{t\beta}$ experiences the hydrodynamic pressures $p_{h\beta}$ due to its motion.

The expression for hydrodynamic pressure in (84) could also be arrived at directly by first noting that classically hydrodynamic pressure is defined as the mean normal stress

$$p_{hi\beta} = (\tau_{xxi\beta} + \tau_{yyi\beta} + \tau_{zzi\beta})/3 \quad (86)$$

because shear stresses in fluids must vanish by definition. Next, normal stresses are expressed as diffusional flux of the corresponding momenta by (21c) as

$$\tau_{ii\beta} = \rho_{i\beta}\mathbf{v}_{i\beta}\mathbf{V}_{ii\beta} = -\mu_{i\beta}\nabla.\mathbf{v}_{i\beta} \quad (87)$$

Substituting from (87) into (86) results in

$$p_{hi\beta} = \frac{1}{3}(\tau_{xxi\beta} + \tau_{yyi\beta} + \tau_{zzi\beta}) = -\frac{\mu_{i\beta}}{3}\nabla.\mathbf{v}_{i\beta} \quad (88)$$

that is in accordance with (85).

Because by definition fluids satisfy *Cauchy-Poisson* limit (82) and hence are incapable of supporting tangential forces, the normal stress tensor (84) is the total stress tensor for fluids [39] and only involves a single *Lame* coefficient as anticipated by *Navier* [72]. Substituting from (84) into (76) results in the modified invariant equation of motion for compressible fluids [39]

$$\frac{\partial(\rho_{i\beta}\mathbf{v}_{i\beta})}{\partial t} + \nabla.\left(\rho_{i\beta}\mathbf{v}_{i\beta}\mathbf{v}_{j\beta}\right) = -\nabla p_{i\beta} + \frac{\mu_{i\beta}}{3}\nabla(\nabla.\mathbf{v}_{i\beta}) \quad (89)$$

It is noted that unlike the classical result the diffusion term does not occur in (89). This is because by definition fluids are substances that cannot support any shear stress and hence shear force. As a result, in the application of *Newton* law of motion to fluids the only forces on the right-hand-side of *Cauchy* equation of motion (76) must be normal forces. Tangential or shear forces must therefore arise from diffusional flux of momentum and originate from the left-hand-side of (76) to be further described in the following.

A most significant aspect of the modified equation of motion (89) is that for an incompressible fluid $\nabla.\mathbf{v}_\beta = 0$ by continuity equation (11) it reduces to *Euler* equation

$$\frac{\partial\mathbf{v}_{i\beta}}{\partial t} + \mathbf{v}_\beta.\nabla\mathbf{v}_{i\beta} = -\frac{\nabla p_{i\beta}}{\rho_{i\beta}} \quad (90)$$

Therefore, for incompressible flows *Euler* equation (90) that has been conventionally only associated with potential hence non-viscous flows is now identified as exact equation of motion even for viscous flows. This is because the only viscous effect appears in the last term of (89) and vanishes due to solenoidal velocity field. In other words, in fluids shear stresses can only arise from diffusional flux of momenta that according to (21c) arise from difference between convective versus local velocities. However, such distinction between convective and local velocities does not occur in *Euler* equation (90).

If the local velocity that is coupled with the divergence operator $\mathbf{v}_{j\beta}$ in the second term of (89) is expressed in terms of the sum of convective and diffusive velocities from (21c), and following conventional practice the cross diffusion effects are neglected and transport coefficients are assumed to be constant, one obtains the modified invariant equation of motion for compressible fluids as

$$\frac{\partial \mathbf{v}_{i\beta}}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_{i\beta} - \nu_{i\beta}\nabla^2 \mathbf{v}_{i\beta} = -\frac{\nabla p_{i\beta}}{\rho_{i\beta}} + \frac{\nu_{i\beta}}{3}\nabla(\nabla . \mathbf{v}_{i\beta}) - \frac{\mathbf{v}_{i\beta}\Re_{i\beta}}{\rho_{i\beta}} \tag{91}$$

The result (91) is to be compared with the classical Navier-Stokes equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} - \nu\nabla^2 \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{\nu}{3}\nabla(\nabla . \mathbf{v}) \tag{92}$$

An important feature of the modified equation of motion (91) as compared to (92) is that it is linear since it involves a convective velocity $\mathbf{w}_\beta$ that is different from the local fluid velocity $\mathbf{v}_\beta$. Also, the diffusion term in (91) arises from the LHS of (76) and relates to diffusional flux of momentum [34, 58] by (21c). In (92) on the other hand, the diffusion term arises from the RHS of (76) as a force that relates to the classical form of the stress tensor (77). Finally, in the absence of convective velocity $\mathbf{w}_\beta$ = 0, the temporal and diffusion terms in (91) remain finite. With the *Navier-Stokes* equation (92) on the other hand, the absence of velocity $\mathbf{v}_\beta$ = 0 results in the vanishing of the entire equation of motion.

Parallel to the steps (44)-(46) for diffusion of mass. let a yz-plane source of momentum be placed at the origin x = 0 at time t = 0 in an infinite and otherwise stationary fluid. Because of symmetry, momentum conservation leads to stationary center of mass $\mathbf{w}_\beta$ = 0. Neglecting the pressure gradient, i.e. assuming that the total initial imparted momentum is small and for an incompressible fluid in the absence of chemical reactions $\Re_{i\beta} = 0$, (91) reduces to

$$\frac{\partial \mathbf{v}_\beta}{\partial t} = \nu_\beta \nabla^2 \mathbf{v}_\beta \tag{93}$$

With the appropriate initial and boundary conditions the solution of (93) is

$$\mathbf{v}_\beta = \frac{\mathbf{\Lambda}_\beta}{\rho_\beta\sqrt{4\pi\nu t}}\exp(-\frac{x^2}{4\nu t}) \tag{93a}$$

where

$$\mathbf{\Lambda}_\beta = \int_{-\infty}^{+\infty} \rho \mathbf{v}_\beta dx \tag{93b}$$

is linear momentum per unit area of the plane source. Derivation of results (93) from the nonlinear classical form of the equation of motion (92) will be more complex.

As another example of solution of (91) the problem of laminar flow over a flat plate described in the previous Section with $Pr_\beta = \nu_\beta / \alpha_\beta = 1$ is considered. The predicted velocity profile [71] given in (70) is in good agreement with experimental data of Dhawan [75] shown in Fig. 16.

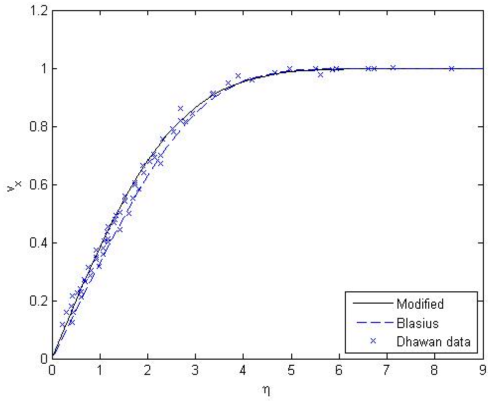


Figure 16. Predicted velocity profile (70) compared with experimental data of Dhawan [75].

The close agreement with experimental data of Dhawan [75] and the small deviation from classical boundary layer solution by Blasius [76, 41] shown in Fig. 16 were found to persist in other independent studies by Burgers and Zijnen [77], Zijnen [78], Hansen [79], and Büttner and Czarske [80] as described in [81]. For example, the comparison between predictions of modified and classical theories with the experimental data of Büttner and Czarske [80] from previous study [81] is shown in Fig. 17.

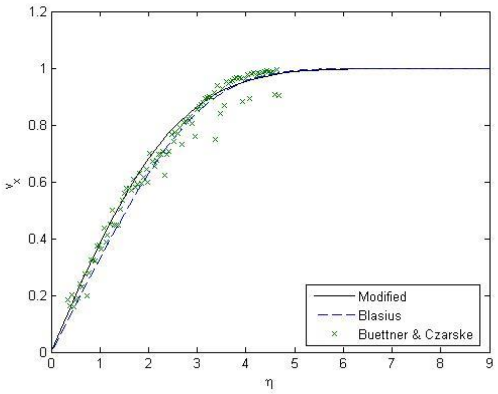


Figure 17. Comparisons between the experimental data of Büttner and Czarske [80] and the predictions of Blasius [76] and modified (70) theories.

The only data showing very close agreement with Blasius [76] solution of boundary layer equation is that of Nikuradse [82] that are likely to be defective because of his "data selection" to correct for entry-length effects described by Schlichting [41].

It is important to note that the exact similarity between hydro-thermo-diffusive (**v**, θ, Y_i) fields for 1-dimensional laminar boundary layer flow governed by equations (62) and (65) will no longer be valid if instead of (91) the non-linear *Navier-Stokes* equation of motion (92) is considered to obtain the velocity field.

The velocity profile (70) leads to friction coefficient

$$c_f = \tau / (\rho w'^2_{xo} / 2) = \sqrt{2/\pi}\, Re_x^{-1/2} = 0.798\, Re_x^{-1/2} \tag{94a}$$

to be compared with the classical result [41].

$$c_f = 0.664\, Re_x^{-1/2} \tag{94b}$$

With the longitudinal velocity (70) the continuity equation leads to the transverse velocity

$$v_y = \sqrt{2/x}\left[\xi \operatorname{erf}(\xi) - \int_0^{\xi} \operatorname{erf}(z)dz\right] \tag{95}$$

The stream function and vorticity associated with the velocity field (70) and (95) are

$$\Psi = 2\sqrt{2x}\int_0^{\xi} \operatorname{erf}(\xi)d\xi \tag{96}$$

and

$$\omega_z = 1/(x\sqrt{2x})\left[\int_0^{\xi} \operatorname{erf}(z)dz - \xi \operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}}\xi^2 e^{-\xi^2}\right] - (1/\sqrt{2\pi x})e^{-\xi^2} \tag{97}$$

The scale invariant model of statistical mechanics naturally leads to a statistical theory of turbulence that is in accordance with the perceptions of *Heisenberg* and *Chandrasekhar* [29, 30, 38]. Hence, following *Heisenberg* the problem of turbulence is considered to be similar to that of *Maxwell* and *Boltzmann* kinetic theory of gas namely the problem of distribution of a given amount of energy amongst large numbers of degrees of freedom as described by *Heisenberg* [30]

"Turbulence is an essentially statistical problem of the same type as one meets in statistical mechanics, since it is the problem of distribution of energy among a very large number of degrees of freedom. Just as in Maxwell theory this problem can be solved without going into details of the mechanical motions, so it can be solved here by simple considerations of similarity."

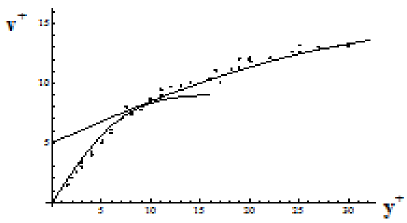
The model shown in Figs. 1 and 2 suggests a hierarchy of flows that appear laminar at scale β but are actually bulk advection of turbulent flows at the smaller scale of $\beta-1$. Such hierarchies of embedded turbulent flows are most clearly seen in steady turbulent boundary layer over a flat plate when the solutions of (91) at scales β and $\beta+1$ were respectively found to be [83]

$$v^+_{\beta+1} = 5 + 8(2/\sqrt{\pi})^2 \operatorname{erf}(y^+_{\beta}/32) \tag{98a}$$

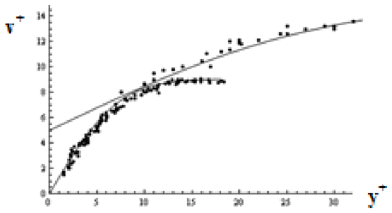
and

$$v_\beta^+ = 8\,(2/\sqrt{\pi})\ \mathrm{erf}\,(y_{\beta-1}^+ / 8) \tag{98b}$$

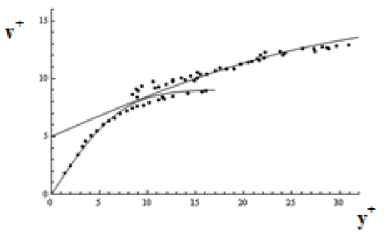
For example, the solutions in (98a) and (98b) at β+1 = e and β = c correspond to LED and LCD and their comparisons with experimental data [41, 49, 84-86] are shown in Fig.18a.



(a)



(b)



(c)

Fig. 18 Comparison between the predicted velocity profiles (a) LED-LCD, (b) LCD-LMD, (c) LMD-LAD with experimental data in the literature over spatial range of 10^8 [83].

Similarly, the solutions in (98a) and (98b) at β+1 = c and β = m correspond to LCD and LMD and at β+1 = m and β = a corresponding to LMD and LAD and their comparisons with experimental data of *Lancien et al.* [87] and *Meinhart et al.* [88] are shown in Figs. 18b and 18c, respectively.

8 The Invariant form of Angular Momentum Conservation Equation

To obtain the invariant form of the conservation equation of angular momentum, one introduces the new summational invariant $\psi_{i\beta} = m_{i\beta}\boldsymbol{\varpi}_{i\beta}$ from (31d) where $\boldsymbol{\varpi}_{i\beta}$ is the iso-spin of particle [56] defined in (20). Next, parallel to the distribution function for translational velocity $f_{i\beta}(\mathbf{x}_{i\beta}, \mathbf{u}_{i\beta}, t_\beta)$ in *Boltzmann* equation (29) one introduces the distribution function for iso-spin or rotational velocity $f_{ri\beta}(\mathbf{x}_{i\beta}, \boldsymbol{\varpi}_{i\beta}, t_\beta)$ at the scale β that gives the total number of rotators per unit volume as

$$n_{i\beta} = \int f_{ri\beta}\, d\boldsymbol{\omega}_{i\beta} \tag{99}$$

The introduction of particle iso-spin $\boldsymbol{\varpi}_{i\beta}$ in (99) gives the mean iso-spin i.e. vorticity as

$$\boldsymbol{\omega}_{i\beta}(x_{i\beta}, t) = \int \boldsymbol{\varpi}_{i\beta}\, f_{ri\beta}\, d\boldsymbol{\omega}_{i\beta} / n_{i\beta} \tag{100}$$

Substitutions form (31d) and (100) into equivalent of (30) without source term on RHS and with $f_{ri\beta}$ for rotational motion results in the modified invariant vorticity equation

$$\frac{\partial \boldsymbol{\pi}_{i\beta}}{\partial t} + \nabla.\left(\boldsymbol{\pi}_{i\beta}\mathbf{v}_{i\beta}\right) = \rho_\beta \boldsymbol{\Omega}_\beta\ .\nabla \mathbf{v}_{i\beta} \tag{101}$$

In (101) diffusion vorticity $\boldsymbol{\Omega}_{i\beta}$ is the mean of peculiar vorticity $\boldsymbol{\Omega}'_{i\beta}$ defined by curl of (3). The last term of (101) corresponds to vortex stretching and arises from *Coriolis* force [89] as

$$-\nabla.(\rho_{i\beta}\boldsymbol{\Omega}_{i\beta} \times \mathbf{v}_{j\beta}) = -\nabla.(\rho_{i\beta}\varepsilon_{ijk}\boldsymbol{\Omega}_{i\beta}\mathbf{v}_{j\beta})$$

$$= \nabla.(\rho_{i\beta}\varepsilon_{ijk}\boldsymbol{\Omega}_{j\beta}\mathbf{v}_{i\beta}) = \rho_{i\beta}\boldsymbol{\Omega}_\beta.\nabla\mathbf{v}_{i\beta} \tag{102}$$

The issue of sign discrepancy of the vortex stretching term in *Helmholtz* vorticity equation has been a source of difficulty in past studies [38, 39]. The error is now identified to originate from the neglect of *Coriolis* force in the last term on the RHS of (101) to be further discussed in the following.

As stated before, vorticity diffusion $\boldsymbol{\Omega}_\beta$ arises from the difference between the bulk or mean vorticity $\bar{\boldsymbol{\varpi}}_\beta$ and the local vorticity $\boldsymbol{\omega}_\beta$ given by the curl of equation (3) as

$$\boldsymbol{\omega}_{i\beta} - \bar{\boldsymbol{\omega}}_{\beta} = \boldsymbol{\Omega}_{i\beta} \tag{103}$$

When the mean or bulk vorticity is absent $\bar{\boldsymbol{\omega}}_{i\beta} = \nabla \times \mathbf{w} = 2\mathbf{w}_{\theta} = 0$, i.e. the absence of what *Kundu* [89] calls *planetary vorticity*, by (103) the local vorticity and diffusion vorticity relate by

$$\boldsymbol{\omega}_{i\beta} = \boldsymbol{\Omega}_{i\beta} \tag{104}$$

Substitution from (104) into (101) results in scale-invariant *Helmholtz* vorticity equation

$$\frac{\partial \boldsymbol{\pi}_{i\beta}}{\partial t} + \nabla.\left(\boldsymbol{\pi}_{i\beta}\mathbf{v}_{i\beta}\right) = \rho_{i\beta}\boldsymbol{\omega}_{\beta}.\nabla\mathbf{v}_{i\beta} \tag{105}$$

By expressing the local velocity in (105) in terms of convective and diffusive velocities from (21d) and neglecting cross diffusion terms and assuming constant transport coefficients one obtains the invariant modified *Helmholtz* vorticity equation

$$\frac{\partial \boldsymbol{\omega}_{i\beta}}{\partial t} + \mathbf{w}_{\beta}.\nabla\boldsymbol{\omega}_{i\beta} = \boldsymbol{\omega}_{\beta}.\nabla\mathbf{v}_{i\beta} + \nu_{i\beta}\nabla^2\boldsymbol{\omega}_{i\beta} - \frac{\boldsymbol{\omega}_{i\beta}\Re_{i\beta}}{\rho_{i\beta}} \tag{106}$$

Equation (106) is to be compared with the classical form of *Helmholtz* vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v}.\nabla\boldsymbol{\omega} = \boldsymbol{\omega}.\nabla\mathbf{v} + \nu\nabla^2\boldsymbol{\omega} \tag{107}$$

An important difference between the modified (106) and the classical (107) forms of *Helmholtz* vorticity equation is the occurrence of convective velocity $\mathbf{w}_{\beta}$ as opposed to local velocity $\mathbf{v}_{\beta}$ in the second term. Because local vorticity $\boldsymbol{\omega}_{\beta}$ in (107) is itself related to the curl of local velocity it cannot be convected by this same velocity. On the other hand, the advection of local vorticity by convective velocity $\mathbf{w}_{\beta}$ in (106) is possible. Moreover, in absence of convection (106) reduces to the diffusion equation similar to that in (47), (73), and (93) for mass, heat, and momentum transport. However, the absence of local velocity in (107) will lead to the vanishing of the entire equation.

Parallel to diffusion of mass (47), heat (73), and momentum (93), for a plane source of vorticity placed at x = 0, at time t = 0, by symmetry the convection vanishes $\mathbf{w}_{\beta} = 0$ and in the absence of reaction $\Re_{i\beta} = 0$ (106) reduces to diffusion equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu\frac{\partial^2\boldsymbol{\omega}}{\partial x^2} \tag{108}$$

with the solution

$$\boldsymbol{\omega} = \frac{\mathbf{S}}{\sqrt{4\pi\nu t}}\exp\left(-\frac{x^2}{4\nu t}\right) \tag{109}$$

where

$$\mathbf{S} = \int_{-\infty}^{+\infty}\boldsymbol{\omega}dx \tag{110}$$

is the vorticity per unit area of the plane source.

To show that the modified *Helmholtz* vorticity equation (106) does lead to consistent predictions, we consider the problem of spherical flow within a droplet located at the stagnation point of two cylindrically-symmetric opposed gaseous finite jets. The convective velocity field is known and given as [41, 90]

$$\mathbf{w}_r = \boldsymbol{\xi} \quad , \quad \mathbf{w}_z = -2\boldsymbol{\zeta} \tag{111}$$

and the dimensionless velocity and coordinates are defined as

$$(\mathbf{w}_r, \mathbf{w}_z) = (\mathbf{w}'_r, \mathbf{w}'_z)/\sqrt{\nu\Gamma} \quad , \quad \boldsymbol{\xi} = \mathbf{r}/\sqrt{\nu/\Gamma}$$
$$\boldsymbol{\zeta} = \mathbf{z}/\sqrt{\nu/\Gamma} \tag{112}$$

where Γ is the stagnation flow velocity gradient. The modified steady *Helmholtz* vorticity equation (106) for axi-symmetric cylindrical coordinates in absence of chemical reactions $\Re_{i\beta} = 0$ reduces to

$$\mathrm{w}_r\frac{\partial \boldsymbol{\omega}_{\theta}}{\partial \xi} + \mathrm{w}_z\frac{\partial \boldsymbol{\omega}_{\theta}}{\partial \zeta} = \frac{\boldsymbol{\omega}_{\theta}\mathrm{v}_r}{\xi} + \left[\frac{\partial^2\boldsymbol{\omega}_{\theta}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \boldsymbol{\omega}_{\theta}}{\partial \xi} - \frac{\boldsymbol{\omega}_{\theta}}{\xi^2} + \frac{\partial^2\boldsymbol{\omega}_{\theta}}{\partial \zeta^2}\right] \tag{113}$$

The solution of (111)-(113) after substitution for local radial velocity $\mathbf{v}_{r\beta} = \mathbf{w}_{r\beta-1} = -\boldsymbol{\xi}$ that is in opposite direction as compared to (111) in accordance with (117) and under appropriate boundary conditions is given as [91, 92]

$$\boldsymbol{\omega}_\theta = 14\xi\zeta / R^2 \quad (114)$$

and the corresponding flow within the droplet is described by the stream function

$$\Psi = \zeta\xi^2[1-(\xi/R)^2-(\zeta/R)^2] \quad (115)$$

The dimensionless vorticity, stream function, and droplet radius R' are defined as $\boldsymbol{\omega}_\theta = \boldsymbol{\omega}'_\theta/\Gamma$, $\Psi = \Psi'/(\nu^3/\Gamma)^{1/2}$, and $R = R'/\sqrt{\nu/\Gamma}$. Some of the streamlines calculated from (115) are shown in Fig. 19.

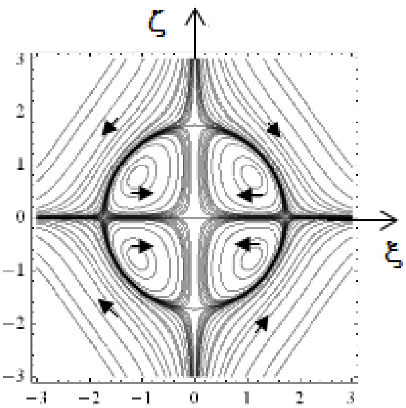


Fig. 19 Flow field within liquid droplet at the stagnation point of viscous counter flow [92].

The local radial and axial velocities corresponding to the stream function (115) are

$$v_r = -\xi[1-(\xi/R)^2-3(\zeta/R)^2]$$

$$v_z = 2\zeta[1-2(\xi/R)^2-(\zeta/R)^2] \quad (116)$$

One notes that with the local velocity field given in (116) the solution (114) does not satisfy the classical form of *Helmholtz* vorticity equation (107). However, in close vicinity of the stagnation point ($r \approx 0$, $z \approx 0$) the local velocity field (116) reduces to

$$\mathbf{v}_{r\beta} = \mathbf{w}_{r\beta-1} = -\boldsymbol{\xi} \quad , \quad \mathbf{v}_{z\beta} = \mathbf{w}_{z\beta-1} = 2\boldsymbol{\zeta} \quad (117)$$

that is identical in form but opposite in direction to the outer convective velocity field in (111). Therefore, as was noted earlier [92], because of the scale-invariant nature of the conservation equations, one expects a cascade of embedded concentric spherical flows at ever smaller scales to form around the stagnation point.

It is interesting to note that even if there were no liquid droplet at the stagnation point, it is expected that a small spherical region of gaseous recirculation flow like that shown in Fig. 19 will form around the stagnation point of viscous counter flow. For fluids with finite viscosity the critical singularity located at the stagnation point will be avoided by the formation of rigid-body core within the region of secondary flow recirculation. The radius of such rigid-body core region is given by $R' = \sqrt{\nu/\Gamma}$ and hence depends on the viscosity and the rate of strain [91, 92].

Because of linearity of (113) the superposition principle applies such that the sum or products of solutions of (113) will also satisfy this equation. For example by (115) it can be shown that the flow within three concentric immiscible liquid droplets of radii R_1 = 2, R_2 = 5, and R_3 = 10 located at the stagnation point of gaseous counter flow finite jets is described by the product solution

$$\Psi = \xi^6\zeta^3(2-\xi^2/R^2-\zeta^2/R^2)(5-\xi^2/R^2-\zeta^2/R^2)$$
$$(10-\xi^2/R^2-\zeta^2/R^2) \quad (118)$$

Some of the streamlines for flow within three concentric droplets calculated from (118) are shown in Fig. 20.

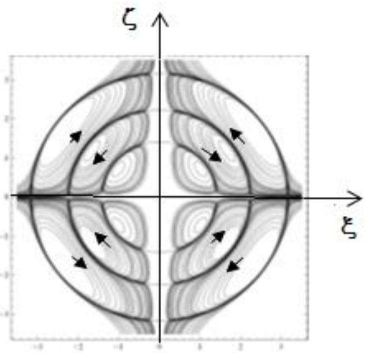


Fig. 20 Streamlines of concentric embedded spherical flows calculated from (118).

Looking at Figs. 19 and 20 it is seen that as one moves away from the central inner droplet the outer convective flow (111) is identified as the local feature of a larger spherical flow within the second droplet and hence $\mathbf{w}'_\beta = \mathbf{v}'_{\beta+1}$ gives

$$w_{r\beta} = v_{r\beta+1} = \xi[1-(\xi/R)^2 - 3(\zeta/R)^2] \quad ,$$

$$w_{z\beta} = v_{z\beta+1} = -2\zeta[1-2(\xi/R)^2 - (\zeta/R)^2] \qquad (119)$$

leading to the vorticity

$$\boldsymbol{\omega}_{\theta\beta+1} = -14\xi\zeta/R^2 \qquad (120)$$

with opposite sign of $\boldsymbol{\omega}_{\theta\beta}$ in (114). The second droplet experiences yet a new "convective" velocity $\mathbf{w}_{\beta+1}$ at the next larger scale given by

$$\mathbf{w}_{r\beta+1} = -\boldsymbol{\xi} \quad , \quad \mathbf{w}_{z\beta+1} = 2\boldsymbol{\zeta} \qquad (121)$$

that is similar to (111) but has opposite direction. The above hierarchy of embedded spherical flows with alternating sense of rotation is recognized as one aspect of the important problem of cascades of vortices in turbulent flows and the well-known rhyme attributed to *Richardson* about big and little eddies [93].

Big whirls have little whirls,
That feed on their velocity;
And little whirls have lesser whirls,
And so on to viscosity,

Generation of cascades of embedded spherical vortices within locally strained flows (Fig. 20) could be identified as one possible mechanism of turbulent dissipation.

The result in Fig. 20 is also in harmony with perceptions of *Weizäcker* in cosmology as was emphasized by *Chandrasekhar* [93].

"Prominent role that Weizäcker ascribed to interplay between turbulence and rotation"

Another significant astrophysical aspect of the spherical flows shown in Figs. 19 and 20 is their relations to the well-known *Hill* spherical vortex [94, 95] formed within a liquid droplet in uniform gaseous flow. Hence, Fig. 19 could be viewed as two semi-spherical *Hill* vortices each vortex occupying a semi-spherical volume. Clearly the cascade of spherical flows in Fig. 20 is a good model of convective flows within stars. As an example it is known that the direction of magnetic polarization of volcanic rocks alternate every few million years with no known mechanism to account for such behavior. Clearly, reversal of polarization could not be due to change of the direction of the magnetic field of the entire planet earth that would be catastrophic. Examination of Fig. 20 suggests that successive generation and evolution of embedded spherical flows with alternating sense of rotation within the ionic plasma of a dynamo such as the planet earth discussed by *Elsasser* [96] could possibly account for such periodic changes in the direction of polarization.

9 Derivation of Invariant Classical forms of Conservation Equations

The invariant forms of conservation equations could be derived following classical integral methods by considering a volume element Γ shown in Fig. 21

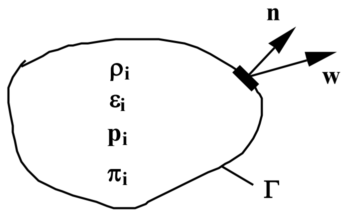


Fig. 21 Conservation of $(\rho_i, \varepsilon_i, \mathbf{p}_i, \boldsymbol{\pi}_i)$ in an arbitrary domain Γ with unit outward normal n.

with the unit outward normal **n** and expressing mass, thermal energy, linear and angular momentum conservations as

$$\frac{\partial}{\partial t}\int \rho_{i\beta} dV_\beta = -\int \rho_{i\beta}\mathbf{w}_\beta.\mathbf{n} dA_\beta + \int \Re_{i\beta} dV_\beta \qquad (122)$$

$$\frac{\partial}{\partial t}\int \varepsilon_{i\beta} dV_\beta = -\int \varepsilon_{i\beta}\mathbf{w}_\beta.\mathbf{n} dA_\beta \qquad (123)$$

$$\frac{\partial}{\partial t}\int \mathbf{p}_{i\beta} dV_\beta = -\int \mathbf{p}_{i\beta}\mathbf{w}_\beta.\mathbf{n} dA_\beta - \int \mathbf{P}_{ij\beta}.\mathbf{n} dA_\beta \qquad (124)$$

$$\frac{\partial}{\partial t}\int \boldsymbol{\pi}_{i\beta} dV_\beta = -\int \boldsymbol{\pi}_{i\beta}\mathbf{w}_\beta.\mathbf{n} dA_\beta + \int \rho_{i\beta}\varepsilon_{ijk}\boldsymbol{\omega}_{j\beta}\mathbf{v}_i.\mathbf{n} dA_\beta \qquad (125)$$

For *Coriolis* force [89] in the last term of (125) substitution has been made from the identity

$$-\int \rho_{i\beta}\varepsilon_{ijk}\boldsymbol{\omega}_{i\beta}\mathbf{v}_j.\mathbf{n}dA_\beta = \int \rho_{i\beta}\varepsilon_{ijk}\boldsymbol{\omega}_{j\beta}\mathbf{v}_i.\mathbf{n}dA_\beta \quad (126)$$

By application of *Gauss's* divergence theorem to (122)-(125) one arrives at

$$\frac{\partial \rho_{i\beta}}{\partial t} + \nabla.\left(\rho_{i\beta}\mathbf{w}_\beta\right) = \Re_{i\beta} \quad (127)$$

$$\frac{\partial \varepsilon_{i\beta}}{\partial t} + \nabla.\left(\varepsilon_{i\beta}\mathbf{w}_\beta\right) = 0 \quad (128)$$

$$\frac{\partial \mathbf{p}_{i\beta}}{\partial t} + \nabla.\left(\mathbf{p}_{i\beta}\mathbf{w}_\beta\right) = -\nabla.\mathbf{P}_{ij\beta} \quad (129)$$

$$\frac{\partial \boldsymbol{\pi}_{i\beta}}{\partial t} + \nabla.\left(\boldsymbol{\pi}_{i\beta}\mathbf{w}_\beta\right) = \rho_{i\beta}\boldsymbol{\omega}_\beta.\nabla\mathbf{v}_{i\beta} \quad (130)$$

In the above formulation the flux of quantities $(\rho_{i\beta}, \varepsilon_{i\beta}, \mathbf{p}_{i\beta}, \boldsymbol{\pi}_{i\beta})$ across the system boundary occurs by convective velocity that is now expressed as the vector sum of local plus diffusion velocities as

$$\mathbf{w}_\beta = \mathbf{v}_{j\beta} + \mathbf{V}_{ij\beta} \quad (131)$$

that is different from (21c). To examine the difference between (131) and (21c) it is first noted that diffusion velocity relates to the mean of the peculiar velocity

$$\mathbf{u}_{i\beta} - \mathbf{v}_{i\beta} = \mathbf{V}'_{ij\beta} \quad (132)$$

Also, one can express the peculiar and diffusion velocities of scale EMD from particle speed profiles shown in Fig. 6 as

$$\mathbf{u}_{im} - \mathbf{v}_{im} = \mathbf{V}'_{ijm} \quad , \quad \mathbf{v}_{m,mp} \le \mathbf{v}_{im} \le \mathbf{u}_{im} \quad (132a)$$

$$\mathbf{v}_{im} - \mathbf{w}_m = \mathbf{V}_{ijm} \quad , \quad \mathbf{w}_m \le \mathbf{v}_{im} \le \mathbf{v}_{m,mp} \quad (132b)$$

From the overlap region between velocity distribution of EMD and ECD fields in Fig. 6 one identifies the equivalence of (132b) with

$$\mathbf{u}_{ic} - \mathbf{v}_{ic} = \mathbf{V}'_{ijc} \quad , \quad \mathbf{v}_{c,mp} \le \mathbf{v}_{ic} \le \mathbf{u}_{ic} \quad (133)$$

Therefore, according to (3) since the equality $\mathbf{w}_\beta = <\mathbf{v}_{j\beta}>$ leads to the inequality $\mathbf{v}_{i\beta} > \mathbf{w}_{i\beta}$ the sign of the diffusion velocity $\mathbf{V}_{ij\beta}$ will be positive and this is insured by having a negative sign in its definition

$$\mathbf{V}_{\beta hg} = -\nu_{i\beta}\nabla \ln(\mathbf{p}_{i\beta}) \quad (134)$$

In (131) on the other hand the sign of the diffusion velocity vector is not anticipated by retaining the negative sign in its definition in (134),

Substituting from (131), (21) and from (84) for the stress tensor into (127)-(130) and assuming constant transport coefficients and unity *Schmidt* and *Prandtl* numbers $Sc_\beta = Pr_\beta = 1$ result in the *invariant forms of conservation equations for chemically reactive fields*

$$\frac{\partial \rho_{i\beta}}{\partial t} + \mathbf{v}_\beta.\nabla\rho_{i\beta} = D_{i\beta}\nabla^2\rho_{i\beta} + \Re_{i\beta} \quad (135)$$

$$\frac{\partial T_{i\beta}}{\partial t} + \mathbf{v}_\beta.\nabla T_{i\beta} = \alpha_{i\beta}\nabla^2 T_{i\beta} - h_{i\beta}\Re_{i\beta}/\rho_{i\beta}c_{pi\beta} \quad (136)$$

$$\frac{\partial \mathbf{v}_{i\beta}}{\partial t} + \mathbf{v}_\beta.\nabla\mathbf{v}_{i\beta} = \nu_{i\beta}\nabla^2\mathbf{v}_{i\beta} - \frac{\nabla p_{i\beta}}{\rho_{i\beta}} + \frac{\nu_{i\beta}}{3}\nabla(\nabla.\mathbf{v}_{i\beta}) - \mathbf{v}_{i\beta}\Re_{i\beta}/\rho_{i\beta} \quad (137)$$

$$\frac{\partial \boldsymbol{\omega}_{i\beta}}{\partial t} + \mathbf{v}_\beta.\nabla\boldsymbol{\omega}_{i\beta} = \nu_{i\beta}\nabla^2\boldsymbol{\omega}_{i\beta} + \boldsymbol{\omega}_\beta.\nabla\mathbf{v}_{i\beta} - \boldsymbol{\omega}_{i\beta}\Re_{i\beta}/\rho_{i\beta} \quad (138)$$

5that appear similar to the classical forms of conservation equations except for the reactive terms in (137)-(138). It is emphasized however that even though the final results are mathematically identical there are subtle and important fundamental differences between (135)-(138) and classical conservation equations besides the fact that the continuity equation (135) contains a diffusion term even for a pure fluid that is absent in the classical continuity equation (29).

To relate (135)-(138) to classical conservation equations one must start with the most elementary question namely the definition of fluid velocity. In classical fluid mechanics the local fluid velocity $\mathbf{v} = \mathbf{v}_\beta = \mathbf{v}_m$ is usually defined as the average of molecular velocity $\mathbf{u}_\beta = \mathbf{u}_m$ by (2). Therefore, conventional fluid mechanics is considered to be associated with laminar-molecular-dynamics LMD scale (Fig. 2) with relevant velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m)$. However, it is known that the most probable molecular speed is the velocity of sound, $\mathbf{v}_{m,mp} = 358\ \mathrm{m/s}$ in air at standard conditions [69] that is a constant at constant temperature. As a result, $\mathbf{u}_c = \mathbf{v}_{m,mp}$ is random and hence the velocity of fluid mechanics should correspond to higher scale than LMD. This implies that stationary fluids should be identified with ECD and consequently conventional fluid dynamics should correspond to laminar-cluster-dynamics LCD scale with relevant velocities $(\mathbf{u}_c, \mathbf{v}_c, \mathbf{w}_c)$ as shown in Figs. 2 and 6. The evidence for the existence of intermediate scale of ECD separating the statistical field of EMD from EED is the phenomenon of *Brownian* motions as discussed in [38, 39].

The exact nature of fluid mechanic velocity is best revealed in the field of combustion where distinction between molecular specie velocity $\mathbf{u}_{im}$, mean molecular velocity of specie $\mathbf{v}_{im}$, and mass-average velocity $\mathbf{v}_o$ of all species i.e. mixture velocity become necessary [42]. In the model being described herein, one associates "atomic", "element", and "system" velocities for identification of different scales as opposed to identity of specific molecular species. Accordingly, the velocity of conventional fluid mechanics should be identified as the mass-average velocity $\mathbf{v}_o$ as noted by *Williams* [42] and is identified as system velocity of LMD scale $\mathbf{v}_o = \ll \mathbf{u}_{im} \gg = < \mathbf{v}_{im} > = \mathbf{w}_m$.

In view of the above discussions, the relevant velocities for conventional fluid mechanics should be identified as

$$\begin{cases} \mathbf{u}_{ic} & \text{"ATOMIC" VELOCITY} \\ \mathbf{v}_{ic} & \text{ELEMENT VELOCITY} \\ \mathbf{w}_{c} & \text{SYSTEM VELOCITY} \end{cases} \tag{139}$$

Hence, the conservation equations (135)-(138) at LCD scale β = c with local velocity $\mathbf{v}_c = \mathbf{w}_m$ given by *Navier-Stokes* equation (137) represents the conventional field of fluid mechanics. This is because $\mathbf{w}_m$ as system velocity of LMD cannot appear in differential equations since by definition it is not locally defined i.e. its value at any position will depend on velocity at other locations remote from this position as discussed earlier [34]. It appears therefore that due to the scale invariant nature of the problem the conservation equations (135)-(138) at LCD scale closely coincide with the classical forms of conservation equations that are conventionally conceived to correspond to the lower scale of LMD. However, as discussed in Section 5, LMD scale corresponds to internal structure of of shock waves. Many of the concepts and results described in this study are in need of further investigations.

10 Concluding Remarks

A scale-invariant model of statistical mechanics was applied to introduce invariant *Boltzmann* equation and the associated invariant *Enskog* equation of change. The invariant modified forms of mass, thermal energy, linear and angular momentum conservation equations were derived. A modified form of continuity equation with a diffusion term even in pure systems was presented and applied to describe internal hydro-thermo-diffusive structure of normal shock in pure gaseous system. Thus, the classical paradox of inapplicability of classical Navier-Stokes equation of motion to shocks due to violation of continuum approximation was resolved. The internal shock structure was shown to be governed by LMD scale representing a new continuum.

A modified form of equation of motion with distinction between convective and local velocity similar to *Carrier* equation of motion was presented. The solutions of modified equation of motion for the problems of laminar and turbulent flow over a flat plate were described. The predicted velocity profiles were shown to be in close agreement with experimental observations available in the literature.

A definition of vorticity was introduced as the mean iso-spin of particle leading to a modified form of *Helmholtz* vorticity equation that was solved for the problem of spherical flow within a liquid droplet located at the stagnation point of opposed gaseous axisymmetric finite jets. Finally, by application of integral methods classical forms of conservation equations were derived. The invariant nature of conservation equations across broad range of spatio-temporal scales described herein is in harmony with the observed universal occurrence of fractals in physical science [97].

Acknowledgments:
This research was supported by NASA grant No. NAG3-1863 and in part supported by The U.S. National Science Foundation grant No. CTS-8820077.

References:

[1] Broglie, L. de, "Interference and Corpuscular Light," *Nature* **118**, 2969, 1926, pp. 441-442; " Sur la Possibilité de Relier les Phénomènes d'Interférence et de Diffraction à la Théorie des Quanta de Lumière," *C. R. Acad. Sci. Paris,* **183**, 1927, pp. 447-448; "La Structure Atomique de la Matière et du Rayonnement et la Mécanique Ondulatoire," **184**, 1927, pp. 273-274; "Sur le Rôle des Ondes Continues en Mécanique Ondulatoire," **185**, 1927, pp. 380-382

[2] Broglie, L. de, *Non-Linear Wave Mechanics: A Causal Interpretation*, Elsevier, New York, 1960

[3] Broglie, L. de, "The Reinterpretation of Wave Mechanics," *Found. Phys.* **1**, 5, 1970, pp. 5-15.

[4] Madelung, E., "Quantentheorie in Hydrodynamischer Form," *Z. Physik.* **40**, 1926, pp. 332-326.

[5] Schrödinger, E., "Über die Umkehrung der Naturgesetze," *Sitzber Preuss Akad Wiss Phys-Math Kl*, 1931, p. 144-153.

[6] Fürth, R., "Über Einige Beziehungen zwischen klassischer Staristik und Quantenmechanik," *Z. Phys.* **81**, 1933, pp. 143-162.

[7] Bohm, D., "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I," *Phys. Rev.* **85**, 2, 1952, pp. 166-179.

[8] Takabayasi, T., "On the Foundation of Quantum Mechanics Associated with Classical Pictures," *Prog. Theor. Phys.* **8**, 2, 1952, pp. 143-182.

[9] Bohm, D., and Vigier, J. P., "Model of the Causal Interpretation of Quantum Theory in Terms of a Fluid with Irregular Fluctuations," *Phys. Rev.* **96**, 1, 1954, pp. 208-217.

[10] Nelson, E., "Derivation of the Schrödinger Equation from Newtonian Mechanics," *Phys. Rev.* **150**, 4, 1966, pp. 1079-1085.

[11] Nelson, E., *Quantum Fluctuations*, Princeton University Press, Princeton, New Jersey, 1985.

[12] de la Peña, L., "New Foundation of Stochastic Theory of Quantum Mechanics," *J. Math. Phys.* **10**, 9, 1969, pp. 1620-1630.

[13] de la Peña, L., and Cetto, A. M., "Does Quantum Mechanics Accept a Stochastic Support?" *Found. Phys.* **12**, 10, 1982, pp. 1017-1037.

[14] Barut, A. O., "Schrödinger's Interpretation of ψ as a Continuous Charge Distribution," *Ann. der Phys.* **7**, 4-5, 1988, pp. 31-36.

[15] Barut, A. O., and Bracken, A. J., "Zitterbewegung and the Internal Geometry of the Electron," *Phys. Rev. D* **23**, 10, 1981, pp. 2454-2463.

[16] Vigier, J. P., "De Broglie Waves on Dirac Aether: A Testable Experimental Assumption," *Lett. Nuvo Cim.* **29**, 14, 1980, pp. 467-475; Gueret, Ph., and Vigier, J. P., "De Broglie's Wave Particle Duality in the Stochastic Interpretation of Quantum Mechanics: A Testable Physical Assumption," *Found. Phys.* **12**, 11, 1982, pp. 1057-1083; Cufaro Petroni, C., and Vigier, J. P., "Dirac's Aether in Relativistic Quantum Mechanics," *Found. Phys.* **13**, 2, 1983, pp. 253-286; Vigier, J. P., "Derivation of Inertia Forces from the Einstein-de Broglie-Bohm (E.d.B.B) Causal Stochastic Interpretation of Quantum Mechanics," *Found. Phys.* **25**, 10, 1995, pp. 1461-1494.

[17] Arecchi, F. T., and Harrison, R. G., *Instabilities and Chaos in Quantum Optics*, Springer-Verlag, Berlin, 1987.

[18] Maxwell, J. C., "On the dynamic theory of gases," Phil. Trans. Roy. Soc. London **157**, 49 (1867).

[19] Boltzmann, L., "Weitere Studien uber das Warmegleichgewicht unter Gasmoleculen," Sitzungsberichte Akad.Wiss., Vienna, Part II, **66**¸ 275 (1872). English translation in: Brush, G., S., *Kinetic Theory*, Vol.1-3 (Pergamon Press, New York, 1965.

[20] Boltzmann, L., *Lectures on Gas Theory* (Dover, New York, 1964.

[21] Reynolds, O., "On the Dynamical Theory of Incompressible Viscous Fluid and the Determination of the Criterion," *Phil. Trans. Roy. Soc.* A **186**, 1, 1895, pp. 23-164.

[22] Enskog, D., "*Kinetische theorie der vorgange in massig verdunnten gasen*," by Almqvist and Wiksells Boktryckeri-A.B., Uppsala, 1917. English translation in Ref. [60], pp. 125-225.

[23] Taylor, G. I., "Statistical Theory of Turbulence-Parts I-IV," *Proc. Roy. Soc.* A **151**, 873, 1935, pp. 421-478.

[24] Kármán, T. von, and Howarth, L., "On the Statistical Theory of Isotropic Turbulence," *Proc. Roy. Soc.* A **164**, 917. 1938, pp. 192-215.

[25] Robertson, H. P., "The Invariant Theory of Isotropic Turbulence," *Proc. Camb. Phil. Soc.* **36**, 1940, pp. 209-223.

[26] Kolmogoroff, A. N., "Local Structure on Turbulence in Incompressible Fluid," *C. R. Acad. Sci. U. R. S. S.* **30**, 1941, pp. 301-305; "Dissipation of Energy in Locally Isotropic Turbulence," *C. R. Acad. Sci. U. R. S. S.* **32**, 1942, p. 19-21; "A Refinement of Previous Hypothesis Concerning the Local Structure of Turbulence in a Viscous Incompressible Fluid at High Reynolds Number," *J. Fluid Mech.* **13**, 1962, pp. 82-85.

[27] Obukhov, A. M., "On the Distribution of Energy in the Spectrum of Turbulent Flow," *C. R. Acad. Sci. U. R. S. S.* **32**, 1941, pp. 19-22; "Some Specific Features of Atmospheric Turbulence," *J. Fluid Mech.* **13**, 1962, pp. 77-81.

[28] Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy," *Rev. Mod. Phys*. **15**, 1, 1943, pp. 1-89.

[29] Chandrasekhar, S., *Stochastic, Statistical, and Hydrodynamic Problems in Physics and Astronomy*, Selected Papers, vol.3, University of Chicago Press, Chicago, 1989, pp. 199-206.

[30] Heisenberg, W., "On the Theory of Statistical and Isotropic Turbulence," *Proc. Roy. Soc.* A **195**, 1948, pp. 402-406; "Zur Statistischen Theorie der Turbulenz," *Z. Phys*. **124**, 7-12, 1948, p. 628-657.

[31] Batchelor, G. K., *The Theory of Homogeneous Turbulence*, Cambridge University Press, Cambridge, 1953.

[32] Landau, L. D., and Lifshitz, E. M., *Fluid Dynamics*, Pergamon Press, New York, 1959.

[33] Tennekes, H., and Lumley, J. L., *A First Course In Turbulence*, MIT Press, 1972.

[34] Sohrab, S. H., "Transport phenomena and conservation equations in multicomponent chemically-reactive ideal gas mixtures," *Proceeding of the 31st ASME National Heat Transfer Conference*, HTD-Vol. **328**, 1996, pp. 37-60.

[35] Sohrab, S. H., "A scale invariant model of statistical mechanics and modified forms of the first and the second laws of thermodynamics," *Rev. Gén. Therm.* **38**, 1999, pp. 845-853.

[36] Sohrab, S. H., "Derivation of invariant forms of conservation equations from the invariant Boltzmann equation," In: *Theoretical and Experimental Aspects of Fluid Mechanics*, S. H. Sohrab, H. C. Catrakis, and F. K. Benra (Eds.), WSEAS Press, 2008, ISBN: 978-960-6766-30-5, pp. 27-35.

[37] Sohrab, S. H., "Universality of a scale invariant model of turbulence and its quantum mechanical foundation," In: *Recent Advances in Fluid Mechanics & Aerodynamics*, S. Sohrab, H. Catrakis, and. N. Kobasko (Eds.), WSEAS Press, 2009, ISBN: 978-960-474-106-9, pp. 134-140.

[38] Sohrab, S. H., "Invariant Planck energy distribution law and its connection to the Maxwell-Boltzmann distribution function," *WSEAS Transactions on Mathematics* **6**, 2, 2007, pp. 254-262.

[39] Sohrab, S. H., "Quantum theory of fields from Planck to cosmic scales," *WSEAS Transactions on Mathematics* **9**, 8, 2010, pp. 734-756.

[40] de Groot, R. S., and Mazur, P. *Nonequilibrium Thermodynamics*, North-Holland, 1962.

[41] Schlichting, H., *Boundary-Layer Theory*, McGraw Hill, New York, 1968.

[42] Williams, F. A., *Combustion Theory*, Addison Wesley, New York, 1985.

[43] Hirschfelder, J. O., Curtiss, C. F., and Bird, R. B., *Molecular Theory of Gases and Liquids*, Wiley, New York, 1954.

[44] Chapman, S., and Cowling, T. G., *The Mathematical Theory of Non-uniform Gases*, Cambridge University Press, Cambridge, 1953.

[45] Townsend, A. A., "The measurement of double and triple Correlation derivatives in isotropic turbulence," *Proc. Camb. Phil. Soc.* **43**, 560 , 1947, pp. 560-570.

[46] Planck, M., *The Theory of Heat Radiation*, Dover, New York, 1991.

[47] Planck, M., "On the Law of the Energy Distribution in the Normal Spectrum," *Ann. der Phys.* **4**, 1901, pp. 553-558.

[48] Van Atta, C. W. and Chen, W. Y., "Measurements of spectral energy transfer in gridturbulence," *J. Fluid Mech.* **38**, 1969, pp. 743-763.

[49] Landahl, M. T., and Mollo-Christensen, E., *Turbulence and Random Processes in Fluid Mechanics*, Cambridge University Press, Cambridge, 1992, p. 62.

[50] Marusic, I. , Mathis, R., and Hutchins, N., "Predictive model for wall-bounded turbulent flow" *Science* **329**, July 2010, pp. 193-196.

[51] McComb, W. D., and Shanmugasundaram, V. , "Numerical calculation of decaying isotropic turbulence using the LET theory," *J. Fluid Mech.* **143**, 1984, pp. 95-123.

[52] Saddoughi, G. S., and Veeravalli, V. S., "Local isotropy in turbulent boundary layers at high Reynolds number," *J. Fluid Mech.* **268**, 1994, pp. 333-372.

[53] Lin, C. C., *Turbulent Flows and Heat Transfer*, Princeton University Press, Princeton, 1959.

[54] Ling, L. S. C., and Huang, T. T., "Decay of weak turbulence," *J. Phys. Fluids.* **13**, 12, 1970, pp. 2912-2924.

[55] Lumley, L. J., Berkooz, G. , Elezgaray, J. , Holmes, P., Poje, A. , and Volte, C., "Fundamental aspects of incompressible and compressible turbulent flows," In: *Simulation and Modeling of Turbulent Flows.* Gatski, B. T., Hussaini, M. Y., Lumley, L. J. (Eds.), Oxford University Press 1996, pp. 5-78.

[56] Sohrab, S. H., "Some implications of a scale invariant model of statistical mechanics to transport phenomena," In: *Recent Advances in Systems*, N. Mastorakis, V. Mladenov, Z.

Bojkovic, S. Kartalopoulos, A. Varonides, and M. Jha (Eds.), WSEAS Press, 2009, ISBN: 978-960-474-097-0, pp. 557-568.

[57] Lugt, H. J., *Vortex Flow in Nature and Technology*, Wiley, New York, 1983.

[58] Bird, R. B. , Stewart, E. W., and Lightfoot, N. E., *Transport Phenomena*, Wiley, New York, 1960.

[59] Carrier, G., "On slow viscous flow," Nonr-653-00-1, 1953.

[60] Olmstead, W. E., "Reciprocal relationships in viscous hydrodynamics," *Acta Mech.* **21** (1975), pp. 289-300.

[61] Roper, M., and Brenner, M. P., "A nonperturbative approximation for the moderate Reynolds number Navier-Stokes equations," PNAS **106**, 9, 2009, pp. 2977-2982.

[62] Sohrab, S. H., "A modified hydro-thermo-diffusive theory of shock waves," *WSEAS Transactions on Heat and Mass Transfer* **1** (5), pp: 606-611 (2007).

[63] Sherman, F. S., "A low-density wind tunnel study of shock wave structure and relaxation phenomena in gases," *NACA Tech. Notes* 3298 (1955).

[64] Alsmeyer, H., "Density profiles in argon and nitrogen shock waves measured by the absorption of an electron beam," *J. Fluid. Mech.* **74** (3) 497 (1976).

[64] Mott-Smith, H. M., "The solution of the Boltzmann equation for a shock wave," *Phys. Rev*. **82**, 885 (1951).

[65] Bird, G. A., "The velocity distribution function within a shock wave," *J.Fluid Mech.* **30** (3) 479 (1967).

[66] Bird, G. A., "Aspects of the structure of strong shock waves," *Phys. Fluids.* **13** (5) 1172 (1970).

[67] Elizarova, T. G., Shirokov, I. A., and Montero, S. , "Shock wave structure for argon and nitrogen," AIP Conference Proceedings; 2005, Vol. 762 Issue 1, p1253.

[68] Taniguchi, S., Arima, T., Ruggeri, T., and Sugiyama, M., "Thermodynamic theory of the shock wave structure in a rarified polytropicc gas: Beyond the Bethe-Teller theory," *Phys. Rev. E.* **89**, 013025 (2014).

[69] Sohrab, S. H., "Some implications of a scale invariant model of statistical mechanics to classical and relativistic thermodynamics," in *Recent Researches in Electric Power and Energy Systems*, Ki Young Kim, Dario Assante, Marian Ciontu, and Jana Jirickova (Eds.), pp: 298-313, WSEAS press, 2013, ISBN: 978-960-474-328-5.

[70] Clausius, R., "Ueber die Art der Bewegung, welche wir Wärme nennen", Ann. der Phyz. **100**, 353 (1957). *Phil. Mag.* **14**, 108 (1957).

[71] Sohrab, S. H., "Modified theory of laminar boundary layer flow over a flat plate," *IASME Transactions* **2** (8), pp. 1389-1394 (2005).

[72] Darrigol, O., "Between hydrodynamics and elasticity theory: The first five births of the Navier-Stokes equation," *Arch. Hist. Exact Sci.* **56** (2002), pp. 95-150.

[73] Darrigol, O., *World of Flow*, Oxford University Press, New York, 2005, p. 123.

[74] Bahatia, A. B., and, Singh, R. N., *Mechanics of Deformable Media*, IOP publication, Boston, 1986, p. 153.

[75] Dhawan, S., "Direct measurements of skin friction," *NACA* TN 2567 (1952).

[76] Blasius, H., "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Z. Math. Phys.* **56**, 1 (1908). English translation in NACA TM 1256 (1950).

[77] Burgers, J. M., and van der Hegge Zijnen, B. G., "Preliminary measurements of the distribution of the velocity of a fluid in the immediate neighborhood of a plane smoooth surface," *Verhand. Akad. v. Wetensch*. Amsterdam (eertste sectie) Deel XIII, No.3, 1924.

[78] van der Higge Zijnen, B. G., "Measurements of the velocity distribution in the boundary layer along a plane surface," Thesis Delft, 1924.

[79] Hansen, M., "Velocity distribution in the boundary layer of a submerged plate," NACA TM, N^o 585, 1930.

[80] Büttner, L. and Czarske, J., "Investigation of the influence of spatial coherence of a broad-area laser diode on the interference fringe system of a Mach–Zehnder interferometer for highly spatially resolved velocity measurements," *Applied Optics*, 44, 9 (2005).

[81] Inkman, M. J., and Sohrab, S. H., "Comparisons between velocity profiles according to the modified and the Navier-Stokes equations of motion and the experimental measurements for laminar boundary layer over a flat plate," in *Computing and Computational Techniques in Sciences*, Jose M^a Zamanillo Sainz de la Maza, and Pablo Luis Lopez Espi (eds.), WSEAS Press, 2008.

[82] Nikuradse, J., "Laminare Reibungsschichten an der längsangetrömten Platte," Zentrale f. wiss. Berichtswesen, Berlin, 1942.

[83] Sohrab, S. H., "A modified theory of turbulent flow over a flat plate," in *Fluid Mechanics and Aerodynamics*, S. H. Sohrab, H. Catrakis, and N. Kobasko, and S. Necasova (Eds), pp: 71-79, WSEAS Press, 2007, ISBN: 978-960-6766-99-7.

[84] Martinelli, R. C., "Heat transfer to molten metals," *Transactions*, ASME 69, 947 (1947).

[85] Ueda, H., and Hinze, J. O., "Fine-structure in the wall region of a turbulent boundary layer," *J. Fluid Mech.* **67**, 57 (1989).

[86] Zagarola, M. V., Perry, A. E., and, Smits, A. J., "Log laws or power laws: The scaling in the overlap region," *Phys. Fluids* **9** (7), 2094 (1997)

[87] Lancien, E. Lajeunesse, and F. Metivier, "Near wall velocity measurements by particle-tracking," arXiv:physics/0701131v1 [physics.flu-dyn], 11 January 2007.

[88] Meinhart, C. D., Wereley, S. T., and, Santiago, J. G., "PIV measurements of a micro channel flow," *Experiments in Fluids* **27**, 414 (1999).

[89] Kundu, P. K., *Fluid Mechanics*, Academic Press, New York, 1990.

[90] Sohrab, S. H., "Modified theories of axi-symmetric stagnation-point laminar boundary layer flow and counterflow jets," *IASME Transactions* **2** (7), pp: 1097- 1105 (2005).

[91] Sohrab, S. H., "Hydrodynamics of spherical flows and geometry of premixed flames near the stagnation-point of axisymmetric viscous counterflows," *Fifth International Microgravity Combustion Workshop*, NASA, May 18-20, 1999, Cleveland, Ohio, pp.85-88.

[92] Sohrab, S. H., "Modified form of the Helmholtz vorticity equation and its solution for spherical flow within a droplet in uniform or counter flow streams," *IASME Transactions.* Issue 4,Vol **1,** pp.634-640 (2004).

[93] Chandrasekhar, S., "Turbulence- A physical theory of astrophysical interest," in Ref, 29 (1949).

[94] Hill, M. J. M., "On a spherical vortex," *Phil. Trans. Roy. Soc.* A **185**, 213 (1894).

[95] Panton, R. L., *Incompressible Flow*, Wiley, New York, 1996, p. 333.

[96] Elsasser, W. M., "The earth as a dynamo," *Scientific American*, May, (1958).

[97] Takayasu, H., *Fractals in Physical Science*, Wiley, New York, 1990.