Fast numerical calculation of conduction shape factors with the finite element method in the COMSOL platform

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Abstract: - The heat conduction across a collection of square modules forming a large plane wall is a one–dimensional problem, whereas the heat conduction across a collection of scalloped modules forming a large corrugated wall is a two–dimensional problem. In this work, the two dimensional heat conduction equation for three different scalloped modules derived from the square module is solved numerically with the Finite Element Method in the platform of COMSOL Multiphysics. When the temperature fields in the modules are post-processed, the conduction shape factors S to be used in the algebraic formula

$$Q = k\,S\left(T_H - T_C\right)$$

can be easily determined. The heat conductive increments provided by the derived scalloped modules are qualitatively compared with the square module, subsequently accounting for beneficial mass reductions.

Keywords: - Large plane wall, stackable square modules, large corrugated wall, stackable scalloped modules, incremental heat conduction, mass reduction

Nomenclature

A_S	surface area, m^2
k	thermal conductivity, W/mC
L	thickness of large plane wall or side of square module, m
H	height of large plane wall, m
q_S	surface heat flux, W/m^2
$\overline{q_S}$	mean surface heat flux, W/m^2
Q	heat flow, W
S	conduction shape factor, m
T	temperature, C
T_C	cold side temperature, C
T_H	hot side temperature, C
T_{max}	maximum temperature in eq. (6), C
x,y	coordinates, m

W width of large plane wall, m
δ (y) thickness of scalloped module varying with height, m

1. Introduction

Numerous studies on steady heat conduction in one–, two– and three–dimensional bodies subjected to various types of heating and cooling conditions at the surfaces have been documented in technical articles as well as in textbooks on heat conduction.

The exact analytic solutions of 2–D heat conduction problems are normally expressed in the form of Fourier infinite series (Carslaw and Jaeger [1]). These infinite series are inconvenient to use because of two factors: 1) knowledge of the eigenvalues and eigenfunctions and 2) large number of terms need to be retained to secure convergence. When the 2–D conduction problems involve irregular geometries, exact analytic solutions are impossible and numerical solutions with the Finite Element Method (FEM) are well suited (Pepper and Heinrich [2]).

During the pre-computer age, an effective way of approximate estimating conduction heat transfer through complex bodies with constant surface temperatures was based on the graphical method. The graphical method eventually evolved into an approximate mathematical method that led to the conduction shape factor. The idea behind the concept of conduction shape factor as conceived by Langmuir et al. [3] back in 1913 was to articulate it with analytical or numerical techniques. Conduction shape factors have been determined analytically for numerous configurations and compact equations have been compiled by Andrews [4], Sunderland and Johnson [5] and Hahne and Grigull [6] for relevant configurations in engineering practice. Additionally, tables in heat transfer textbooks [7-13] present a limited number of conduction shape factors. Since most of the relations for conduction shape factors are approximations to exact solutions, restrictions on their applicability have to be accounted for.

Within the framework of one–dimensional bodies the large plane wall with a hot surface and a cold surface constitutes the first example in heat conduction [7-15]. If oriented vertically, the plane wall can be conceived as a collection of stackable square modules; each square module having one hot vertical surface, one cold vertical surface and two horizontal adiabatic surfaces. Clearly, the modeling of a typical square module entails to a simple one–dimensional heat equation whose exact analytic solution is easy. On the contrary, a different state of affairs transpires when the hot and cold vertical surfaces of the square module are symmetrically curved inward to form a vertical large corrugated wall. This case implicates a two–dimensional heat equation because the heat flux vector possesses horizontal and vertical components. To our surprise, the heat conduction characteristics of large corrugated wall remains unknown and are unavailable in the specialized literature.

The underlying goal of the present paper on engineering education is to delineate the numerical calculation of a general class of conduction shape factors that emanate from a vertical large corrugated wall with scalloped modules.

The body of the paper is divided into three sections. In the first section, three modules with different degrees of scallopness along with their descriptive two–dimensional heat equations are addressed. The numerical computations with the Finite Element Method (FEM) are briefly explained in the second section. The third section discusses the numerically–determined 2–D temperature fields T(x, y) and the magnitudes of heat conduction Q across the three scalloped modules.

2. Case study: Large wall formed with scalloped modules

Consider a large vertical plane wall with finite thickness L, infinite height H (>> L) and infinite depth W (>> L). A high temperature T_H is applied at the left surface and a low temperature T_C is applied at the right surface. The thermal conductivity k of the material is constant in the temperature interval of operation $[T_C,T_H]$. Equivalently, the plane wall can be conceived as an assembly of stackable square modules of side L with T_H at the left surface, T_C at the right surface where the upper and bottom horizontal surfaces are adiabatic. Figure 1a is a sketch of the square module, named here the primary module. Accordingly, the one–dimensional heat equation in a square module is

$$\frac{d^2T}{dx^2}=0 \qquad (1)$$

and the heat flow Q through it corresponds to

$$Q=k\left(\frac{H\times W}{L}\right)(T_H-T_C) \qquad (2)$$

When the hot and cold vertical surfaces of a square module of side L are symmetrically bent inward, a family of derived modules with variable thickness δ (y) satisfying 0 < δ < L can be derived. For a "proof–of–concept" study, we chose three derived modules. First, a slightly scalloped module owing the largest thickness δ = L at the two adiabatic surfaces and the smallest thickness δ_{min} = L/2 at the horizontal mid–plane of symmetry is shown in Figure 1b. Second, a moderately scalloped module owing the largest thickness δ = L at the two adiabatic surfaces and the smallest thickness δ_{min} = L/4 at the horizontal mid–plane of symmetry is shown in Figure 1c. Third, when the curved surfaces nearly touch each other (an unreal limiting condition), a derived module depicted in Figure 1d called the severely scalloped module has the largest thickness δ = L at the two adiabatic surfaces and the smallest thickness $\delta_{min} \approx 0$ at the horizontal mid–plane of symmetry.

Framed in a Cartesian coordinate system, the three scalloped modules chosen are governed by the two–dimensional heat conduction equation

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0 \qquad (3)$$

The applicable boundary conditions are of mixed type. First, constant specified temperatures T_H and T_C (Dirichlet type) are assigned at the left and right curved surfaces, respectively. Second, the upper and lower horizontal flat surfaces are taken as planes of symmetry signifying null temperature gradients $\frac{\partial T}{\partial y}=0$ (von Neumann type).

3. Conduction shape factor

When one high temperature T_H and a low temperature T_C are specified along parts of the periphery of a two–dimensional body, the heat flow Q passing through the body can be computed by the algebraic formula conceived by Langmuir et al. [3]:

$$Q=k\ S\ (T_H-T_C) \qquad (4)$$

where S is the conduction shape factor in m and k is the constant thermal conductivity k of the material in W/mC. The quantity S depends only on the body geometry (i.e., shape and dimensions). Interestingly, this short–cut approach provides a quick estimate for the heat transfer passing through the body.

Shifting the attention to the 1–D square module shown in Figure 1a, the calculation of the conduction shape factor S is arithmetic and equates to

$$S = \frac{A_s}{L} = \frac{H \text{ x } W}{L} \qquad (5)$$

In contrast, the calculations of the conduction shape factors S for the trio of 2–D scalloped modules in Figures 1b, 1c and 1d are involved and forcibly have to be computed from the numerical solution of Eq. (3).

4. Numerical computations

For the sake of simplicity, the computations are initiated with a square module of side L = 1 m and width W = 1 m, high temperature T_H = 1 C at the left surface, low temperature T_C = 0 C at the right surface. The material has constant thermal conductivity k = 1 W/m.C in the temperature interval of operation [0, 1].

The availability of fast and inexpensive computers allows heat conduction problems that are intractable to analytic methods to be solved numerically in a relatively easy manner. While the Finite Difference Method (FDM) in its basic form is restricted to rectangular shapes and simple alterations thereof, the handling of complex bodies and/or irregular boundaries with the Finite Element Method (FEM) is straightforward [2].

Equation (3) subject to the boundary conditions was solved with FEM and the numerical computations were performed with the advanced software code COMSOL 3.1 [16], a multiphysics MATLAB–based program that possesses a wide array of modeling capabilities. The COMSOL Multiphysics simulation environment provides all steps in the modeling/calculation processes, namely 1) defining the geometry, 2) specifying the physics, 3) constructing the mesh, 4) solving the system of algebraic equations and 5) post-processing the numerical results. With regards to the meshing, COMSOL features fully automatic adaptive mesh generation with a precise size control of the mesh. Embedded into COMSOL are high–performance solvers capable of handling the large systems of algebraic equations with ease.

It is typical that the word "element" refers either to the triangles in the computational domain, or the piecewise linear basis function, or both. It should be added that FEM is not restricted to triangles, but can be defined on quadrilateral sub-domains or higher order shapes, e.g., curvilinear elements [2].

After satisfactory convergence of the 2-D temperature fields T (x,y) was reached and mesh–independence were secured for the three scalloped modules, representative constant temperature lines or isotherms were plotted for each module. The convergence criteria was overseen using the standard norm

$$\frac{1}{T_{\max}} \left[\sum_{i=1}^{N} (T_i^{n+1} - T_i^{n})^2 \right]^{1/2} \leq \varepsilon \qquad (6)$$

where ε typically varied between 10^{-4} and 10^{-6}. Upon performing a sensitivity analysis of the mesh, the optimal number of elements turned out to be 401 in the slightly scalloped, 558 in the moderate scalloped and 618 in the severely scalloped modules.

First, the surface heat flux $q_s(s)$ perpendicular the hot curved surface of a scalloped module is obtained by applying Fourier's law to the 2–D temperature field T(x,y). Second, the mean surface heat flux $\overline{q_S}$ is determined from the mean value integral

$$\overline{q_S} = \frac{1}{L} \int_0^L q_S(s)\, ds \qquad (7)$$

5. Discussion of results

In this section, we assessed the geometric effects and their bearing on the heat conduction through the three derived scalloped modules with respect to the basic square module.

The numerical 2-D temperature fields T (x,y) for the three derived scalloped modules are displayed in Figures 2b, 2c and 2d. In Figures 2b and 2c, it is seen that the curved isotherms appear toward the hot and cold curved sides, whereas vertical isotherms characterize the central part, i.e., the vertical plane of symmetry. The limiting condition in Figure 2d exhibits full curved isotherms in the entire module.

A brief discussion of the items listed in Table 1 seems to be appropriate now. First, the curved side of the slightly scalloped module is 15% larger than the straight side of the square module. This number indicates a mass reduction for the slightly scalloped module of 34% with respect to the mass of the square module. First, the conduction shape factor S across the slightly scalloped module amounts to 91% higher than the conduction shape factor S across the square module. Second, the curved side of the moderately scalloped module is 21% larger than the straight side of the square module. This is equivalent to a 41% mass reduction for the moderately scalloped module in reference to the mass of the square module. The conduction shape factor S across the moderately scalloped module amounts to 135% higher than the conduction shape factor S across the square module.

The passage from the square module with $\frac{\delta}{L}=1$ to a slightly scalloped module wit $\frac{\delta_{\min}}{L}=1/2$ results in a significant mass reduction from 1 kg to 0.66 kg and a remarkable increment in the mean heat flux $\overline{q_S}$ going from 1 W/m to 1.66 W/m. This combination of factors translates into a conduction shape factor of 1.91 m for the slightly scalloped module as compared to 1 m for the basic square module. In other words, this means that decreasing the mass of a large plane wall by one third, the heat conduction through it is almost doubled.

The passage from the slightly scalloped module with $\frac{\delta_{\min}}{L}=1/2$ to the moderately scalloped module with $\frac{\delta_{\min}}{L}=1/4$ results in a modest mass reduction from 0.66 kg to 0.59 kg and a small increment in mean heat flux $\overline{q_S}$ from 1.66 W/m to 1.94 W/m. This in turn is equivalent to a conduction shape factor of 2.35 m for the moderately scalloped module with $\frac{\delta_{\min}}{L}=1/4$ as compared to 1.91 m for the slightly scalloped module. There is not significant difference of the mass and in the heat conductance C between the slightly scalloped module with $\frac{\delta_{\min}}{L}=1/2$ and the moderately scalloped module with $\frac{\delta_{\min}}{L}=1/4$.

As far as the conduction shape factor S of the derived scalloped modules is concerned, it depends on two characteristic lengths, one is the smallest thickness $\delta_{\min}$ and the other is the side L. The correlation equation for the conduction shape factor S was computed in terms of the relative thickness $\frac{\delta_{\min}}{L}$ with Minitab [17]:

$$\frac{1}{S}=0.034+2.356\,\frac{\delta_{\min}}{L}-3.983\left(\frac{\delta_{\min}}{L}\right)^2+2.596\left(\frac{\delta_{\min}}{L}\right)^3 \qquad (8)$$

Here, the R^2 value is 0.99 and the maximum error is 3.11% at $\frac{\delta_{\min}}{L} = 1/2.$

To put the correlation equation (8) at work, we chose a representative example. For a scalloped module characterized with an intermediate the relative thickness $\frac{\delta_{\min}}{L}$ = 1/8 and active side of 1.3 m, the mass is halved from 1 kg to 0.5 kg and the conduction shape factor S ascends from 1 m to 3.69 m (more than a three-fold factor).

6. Conclusions

In this paper on engineering education, we have demonstrated the benefits of the Finite Element Method and COMSOL Multiphysics to calculate two–dimensional heat conduction in irregular bodies in a heat transfer course. Additionally, we have presented a curious concept aiming at augmenting the heat conduction in a large plane wall. That is, curving inward the opposite heated and cooled sides of a primary square module symmetrically, a family of derived scalloped modules was created in a natural way. The latter have proved to be exemplary for heat conduction intensification because the heat flux paths are less tortuous in the central regions of adjacent scalloped modules. Further, from the standpoint of fabrication, the derived scalloped modules in a large corrugated wall required less material than the counterpart primary square module in a large plane wall. The outcome of this paper may be of interest to instructors of the heat transfer course.

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Table 1. Comparison of the conduction shape factor S between the primary square module and the three derived scalloped modules

Module configuration	Size of active side	Mass (kg)	Mean heat flux (W/m)	Conduction shape factor S (m)
square	1.00	1.00	1.00	1.00
slightly scalloped $\frac{\delta_{min}}{L} = 1/2$	1.15	0.66	1.66	1.91
moderately scalloped $\frac{\delta_{min}}{L} = 1/4$	1.21	0.59	1.94	2.35
severely scalloped $\frac{\delta_{min}}{L} \approx 0$	1.49	0.32	21.73	32.33

LIST OF FIGURES

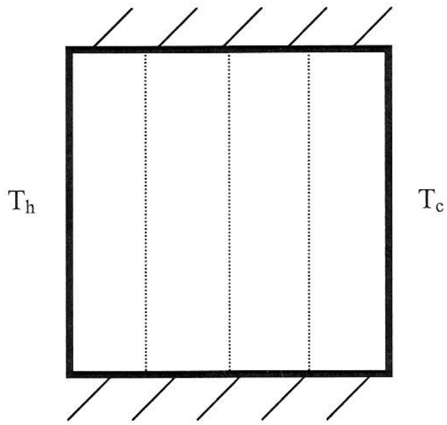

Figure 1a Square module

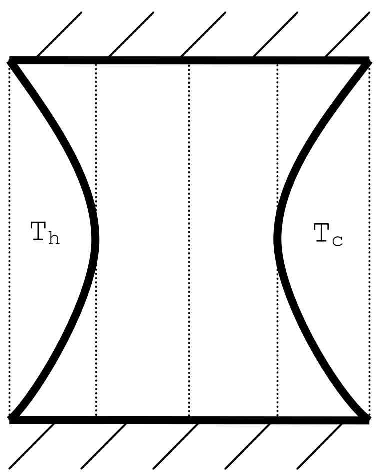

Figure 1b Slightly scalloped module with $\frac{\delta_{min}}{L}$ = 1/2

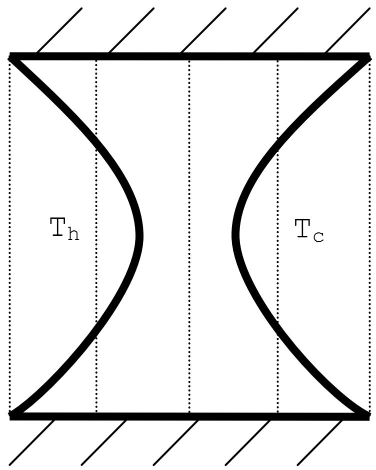

Figure 1c Moderately scalloped module with $\frac{\delta_{\min}}{L} = 1/4$

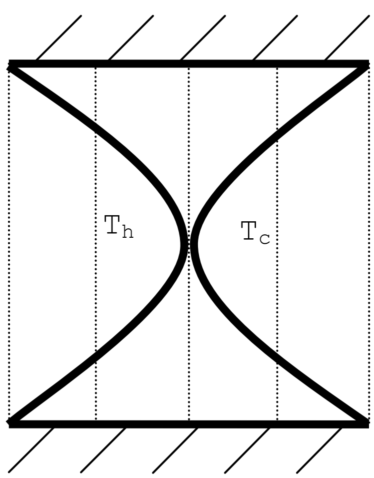

Figure 1d Severely scalloped module with $\frac{\delta_{\min}}{L} \approx 0$

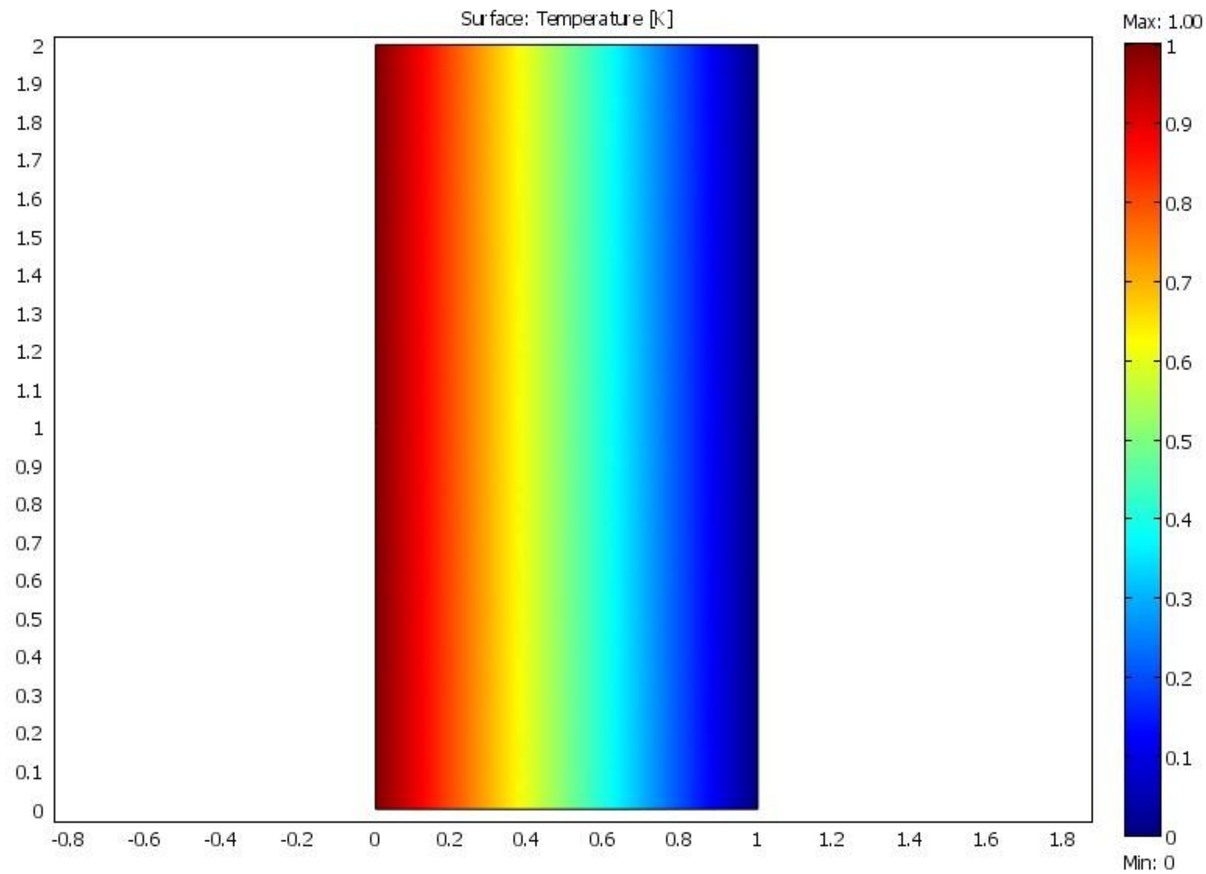

Figure 2a Isotherms plot for the square module

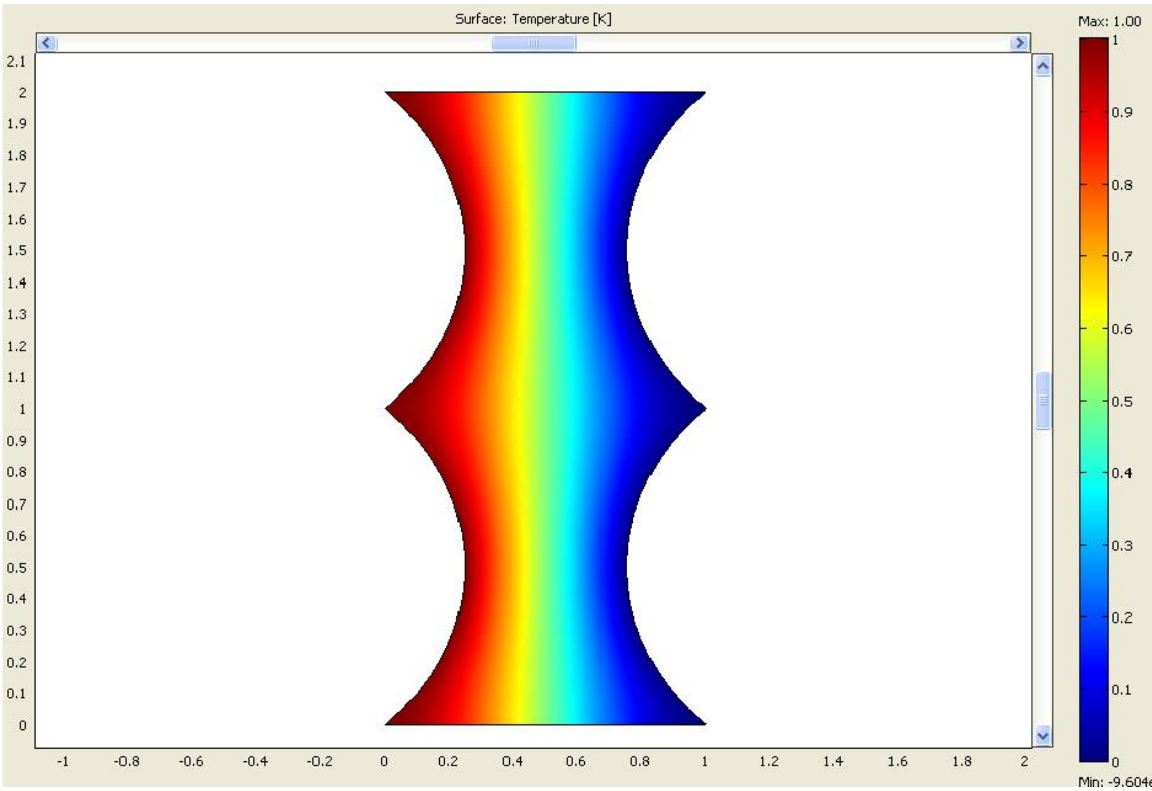

Figure 2b Isotherms plot for the slightly scalloped module with $\frac{\delta_{\min}}{L}$ = 1/2

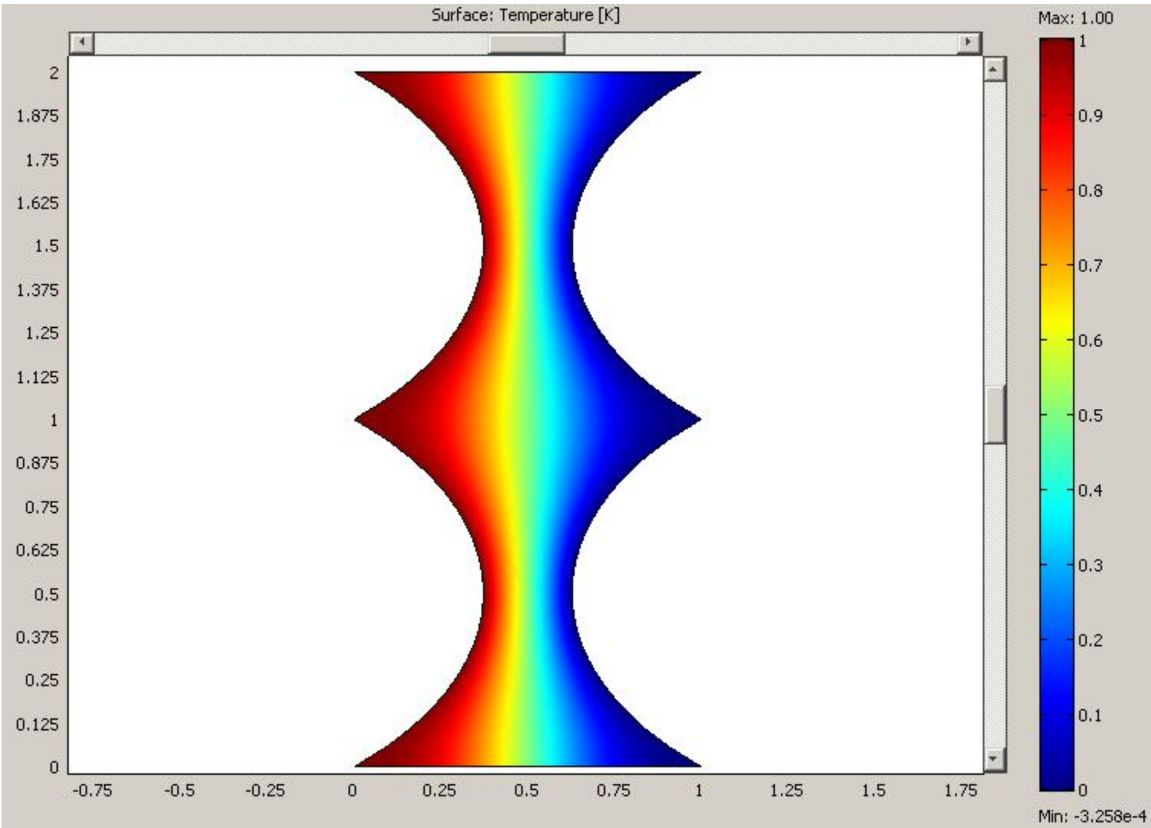

Figure 2c Isotherms plot for the moderately scalloped module with $\frac{\delta_{\min}}{L}$ = 1/4

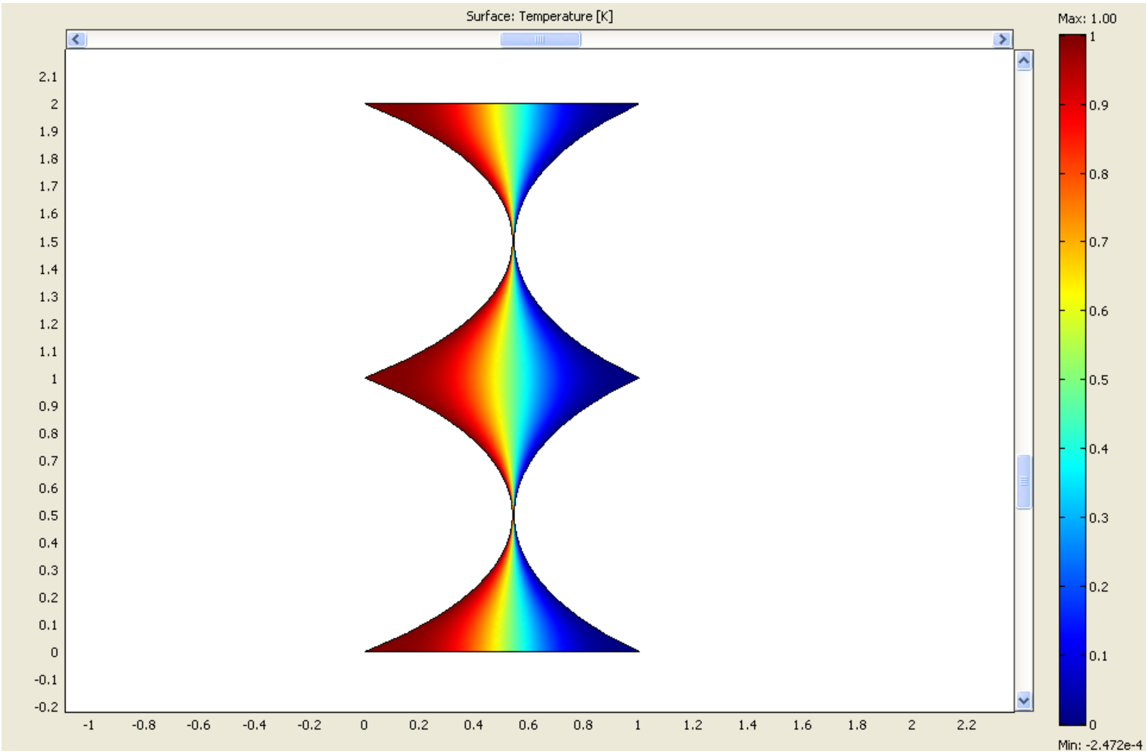

Figure 2d Isotherms plot for the severely scalloped module with $\frac{\delta_{\min}}{L} \approx 0$