

Thermal and Flow Characteristics of Dusty Hybrid Nanofluids on an Inclined Permeable Surface with Slip Conditions and Viscous Dissipation Effects

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Abstract: - The heat transfer problem over an inclined surface has emerged as a significant topic of interest among researchers. Its application has grown and has been highlighted particularly because of its durability in engineering developments. Hence, in accordance with the current phase of development, a mathematical model of steady flow and heat transmission involving an alumina-copper dusty hybrid nanofluid across inclined, stretched, and shrunk surfaces by implementing viscous dissipation and slip conditions has been proposed. Similarity transformations were utilized in reducing the governing equations to a set of ordinary differential equations (ODEs), and the numerical results were achieved by employing a bvp4c solver in MATLAB. The findings highlighted that dual solutions occur in both cases for shrinking surfaces. Moreover, both the skin friction coefficient and the rate of heat transmission decrease with increasing inclination angle. In contrast, the rise in Eckert number values, which implies viscous dissipation, depicts an upsurging pattern for the coefficient of skin friction while diminishing the rate of heat transmission between the fluid flow and the surface.

Key-Words: - Dusty hybrid nanofluids, inclined surface, shrinking/stretching surface, thermal slip, velocity slip, viscous dissipation.

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1 Introduction

[1] introduces nano-sized particles with dimensions ranging from 1 to 100 nm into traditional fluids known as nanofluids. Nanofluids containing diminutive solid particles ensure an alternative working fluid for use in energy devices [2]. Further,

the properties of fluids are notably improved when high-conductivity solid nanoparticles are dispersed in a low-conductivity base, leading to their application in numerous fields including heat exchangers, solar collectors, radiators, and more [3]. The two most widely used models in nanofluid

research are Buongiorno and Tiwari and Das. The validation of nanofluid behavior was determined by analyzing the nanoparticle volume fraction as defined in the Tiwari and Das model [4] while the Buongiorno model is classified as the two-phase non-homogeneous model that incorporates the effects of thermophoresis and Brownian motion within the laminar fluid flow [5].

The attempt to develop a more efficient fluid to improve fluid heat transmission efficiency led to the discovery of a hybrid nanofluid, an advanced variation of nanofluids that integrates multiple nanoparticle types in a base fluid. [6] performed a study on the preparation, thermophysical properties, and significance of hybrid nanofluids throughout numerous engineering areas. Multiple appropriate nanoparticle combinations were evaluated for utilization across different industries. The main discovery of their study was that hybrid nanofluids demonstrate markedly enhanced thermal conductivity in comparison to traditional fluids and mono-nanofluids. The flow properties were optimized through the incorporation of hybrid nanoparticles, and the heat transmissions were improved through the augmentation of nanoparticle concentration. Further, following the discovery of a hybrid nanofluid, another innovative development has emerged, namely dusty hybrid nanofluid, which takes into account the presence of suspended microscopic dust particles in the hybrid nanofluids. In addition, the occurrence of dusty fluid has been thoroughly investigated by researchers, as it encompasses various fields such as the cooling effects of air conditioning systems, the reduction of emissions from vehicles, and pollution of the environment [7]. [8] numerically examined the impact of temperature-dependent viscosity (TDV) over a vertically stretched sheet utilizing the dusty Eyring Powell fluid. The Keller-box method was employed to compute the numerical results, and it was noted that the existence of dust particles affects the fluid velocity, leading to a deceleration effect. The profiles of temperature and velocity of the fluid phase consistently exceed those of the dust phase, and the temperature profile does not exhibit minimal fluctuation due to the high viscosity of the fluid. Moreover, the profiles of temperature and velocity of the dust and fluid phases have opposing impacts based on the variations of buoyancy force parameters. Recently, [9] discussed the flow of dusty fluid across a vertically stretched surface, taking note of the radiation and viscous dissipation impacts. The fluid thermal behavior improves with an upsurge in the Eckert number value, and a similar pattern was observed in the momentum profiles for

each phase. However, the dust phase temperature profile shows the reverse pattern. Furthermore, the consideration of the buoyancy parameter diminishes the thermal boundary layer, hence improving the surface cooling. [10] researched the impact of magnetohydrodynamic (MHD) and convective boundary conditions on a moving plate utilizing copper-water and alumina-water nanofluids with dust particles. Adding to that, a multitude of studies have been conducted on dusty hybrid nanofluids, including [7], [11], [12], [13], [14], and [15].

The inclusion of viscous dissipation in numerous studies has developed recently due to the presence of internal friction in fluids, particularly within the boundary layer, as well as affecting the temperature distribution [16]. Viscous dissipation, commonly referred to as frictional heating and defined by the Eckert number, represents the transformation of kinetic energy into thermal energy [17]. Additionally, it is said that the upsurge in the Eckert number elevates the temperature of the fluid while reducing the heat transmission rate [18]. [16] explored the steady Jeffrey viscoelastic polymer flow, investigating the implications of viscous dissipation and chemical reaction on an inclined plane. The problem was solved mathematically using the implicit finite difference approach, leading to the conclusion that growth in viscous dissipation values enhances both the profiles of temperature and velocity, while the profile of concentration showed a contrary trend. Additionally, as the inclination parameter elevates, the profiles of concentration and temperature rise. The unsteady mixed convection flow and heat transfer of a non-homogeneous nanofluid were discussed by [2] concerning viscous dissipation and magnetic field. An upsurge in the value of the Eckert number, which indicates viscous dissipation, significantly enhances the temperature profile. Furthermore, rising the value of the Eckert number causes a drop in the Nusselt number and skin friction coefficient, while augmenting the Sherwood number. [18] evaluated the convective MHD flow and heat transfer of a viscous incompressible fluid concerning viscous heating and ohmic heating past an inclined porous surface. It was determined that the existence of viscous dissipation causes an upsurge of Nusselt number and Sherwood number values while concurrently leading to a diminished skin friction coefficient.

An inclined surface is placed at a specified angle, which ranges from 0° and 90° with respect to a horizontal or vertical surface [19]. Inclined surfaces are crucial in engineering applications as they examine the stability, efficiency, and safety of a system before its implementation. [20] considered

the impacts of suction and buoyancy force of water-hybrid nanofluid across an inclined stretched or shrunk sheet. The findings indicate that augmenting the suction and angle of inclination leads to an enhancement in the local Nusselt number and skin friction coefficient. Furthermore, a greater volume fraction of nanoparticles and inclination angle postponed the boundary layer separation. The comparative heat transfer characteristics of three distinct nanoparticles, which are copper, alumina, and titania, embedded in water over a porous inclined plane were analyzed by [21]. The impact of various pertinent parameters, including magnetic force, mixed convection, and thermal radiation, was considered, revealing that the velocity profile of copper-water nanofluid is greater than that of alumina-water and titania-water as the value of the Grashof number increases. The temperature profile of titania-water is higher compared to copper-water and alumina-water. [22] discussed the three-dimensional heat and mass transmission of MHD nanofluid flow over a porous stretched sheet under the influence of chemical reaction and an inclined magnetic field. Numerous studies have delved into inclined planes, exploring various flow characteristics and fluids; for example, [23], [24], [25], [26], [27], [28], etc.

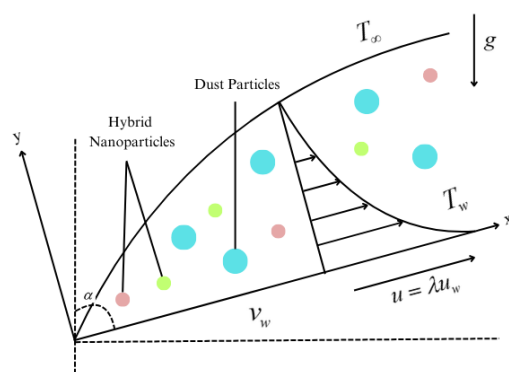
The impact of slip conditions is significant compared to no-slip conditions, as practical applications presume the fluid possesses a non-zero velocity in relation to the surface. The exploration into the effects of slip parameters, magnetic field, and viscous dissipation on a moving plate, as studied by [29], was further explored by [30]. Several parameters are taken into account, including slip, Brownian motion, and thermophoresis, as well as Lewis number, Prandtl number, and Eckert number. The analysis revealed that the incorporation of viscous dissipation and a magnetic field enhance the temperature profile. Moreover, the temperature profile increases, whereas the velocity profile diminishes when the parameter of velocity slip rises, and conversely, the temperature profile declines with a greater parameter of thermal slip. [31] Applied the Caputo fractional derivative to examine the unsteady flow and heat transfer across a vertical oscillating inclined plane. [32] scrutinized the impact of slip conditions and non-linear thermal radiation of the boundary layer flow and heat transmission characteristics of viscous nanofluids on a porous stretching sheet. The extensive use of thermal transmission fluids in industry and engineering has drawn significant interest from researchers. Nevertheless, basic or conventional

fluids, including ethylene glycol, water, kerosene, and others, exhibit limited thermal conductivity.

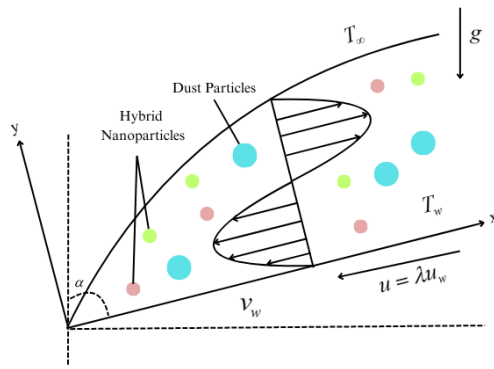
Based on the literature, there is no research being undertaken on the flow and heat transmission over an inclined surface utilizing the dusty hybrid nanofluid with the presence of thermal and velocity slip parameters and viscous dissipation effects. Thus, this study will advance by addressing the gap found in the previous research by referring to [33] who analyzed the flow and heat transfer characteristics passing a horizontal permeable shrinking/stretching surface of dusty hybrid nanofluid, considering the implications of slip conditions. The built-in MATLAB function referred to as the bvp4c solver will be implemented to generate numerical results, and copper, alumina, and water were examined as potential hybrid nanofluids. Further, the findings are presented in tabular and graphical form.

2 Mathematical Formulation

A two-dimensional boundary layer flow passing an inclined permeable shrunk or stretched surface of a Al_2O_3 -Cu/water hybrid nanofluid containing dust particles is explored, considering the slip conditions and viscous dissipation impacts. The physical illustration pertaining to the problem in this study is exhibited in Figure 1. The surface is linearly shrunk or stretched along the x -axis while the y -axis is normal to it. Assuming the surface velocity is $U_w = ax$, where a is a positive constant. Two distinctive kinds of nanoparticles, specifically copper Cu and alumina Al_2O_3 were incorporated with water, producing a hybrid nanofluid. In addition, the dust particles are assumed to be uniform, spherical in shape, and have a constant density [34].



a) Stretching Surface



b) Shrinking Surface

Fig. 1: Physical model of stretching and shrinking surfaces

Source: Created by authors.

Based on the boundary layer flow problem mentioned in this study, the partial differential equations of continuity, momentum, and energy for both fluid and dust phases are assumed to be as follows ([4], [20], and [33]):

Hybrid Nanofluid Phase:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{hnf}}{\rho_{hnf}} \frac{\partial^2 u}{\partial y^2} + \frac{KN}{\rho_{hnf}} (u_p - u) + g \beta_{hnf} (T - T_\infty) \cos \alpha \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{hnf}}{(\rho C_p)_{hnf}} \frac{\partial^2 T}{\partial y^2} + \frac{NC_{pf}}{(\rho C_p)_{hnf} \tau_T} (T_p - T) + \frac{N}{(\rho C_p)_{hnf} \tau_v} (u_p - u)^2 + \frac{\mu_{hnf}}{(\rho C_p)_{hnf}} \left(\frac{\partial u}{\partial y} \right)^2 \quad (3)$$

Dust Phase:

$$\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} = 0 \quad (4)$$

$$u_p \left(\frac{\partial u_p}{\partial x} \right) + v_p \left(\frac{\partial u_p}{\partial y} \right) = \frac{K}{m} (u - u_p) \quad (5)$$

$$u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} = \frac{C_{pf}}{C_{mf} \tau_T} (T - T_p) \quad (6)$$

subjected to boundary conditions:

$$u = \lambda U_w(x) + L_s \left(\frac{\partial u}{\partial y} \right), v = v_w(x),$$

$$T = T_w + L_T \left(\frac{\partial T}{\partial y} \right)$$

at $y = 0$

(7)

$$u \rightarrow 0, u_p \rightarrow 0, v_p \rightarrow v, T_p \rightarrow T_\infty, T \rightarrow T_\infty$$

as $y \rightarrow \infty$

where u, u_p , and v, v_p are the hybrid nanofluid and dust particle velocity components in the x and y directions, μ_{hnf} , ρ_{hnf} and β_{hnf} are the dynamic viscosity, density, and volumetric coefficient of thermal expansion of the hybrid nanofluid, respectively. Further, K, N, g , and α are the Stokes drag coefficient, dust particle number, gravitational acceleration, and inclination angle, respectively. Moreover, k_{hnf} implies the hybrid nanofluid's thermal conductivity, while c_{pf} and c_{mf} signify the specific heat of hybrid nanofluids and dust particles, respectively, $(\rho C_p)_{hnf}$ is the specific heat capacity of a hybrid nanofluid, and m is the mass of dust particles.

Furthermore, T and T_p are the temperatures of hybrid nanofluids and dust particles, while τ_v and τ_T represent the dust particles' relaxation time, thermal equilibrium time and λ implies the stretching or shrinking surface, where $\lambda > 0$ denote as the stretching surface, $\lambda < 0$ denote as the shrinking surface and, $\lambda = 0$ is a static surface. Since the slip condition is considered, L_s and L_T are the velocity and thermal slip lengths, respectively, and $L_s = L_T = 0$ shows the absence of slip conditions in the boundary layer. The thermophysical properties of nanofluid, hybrid nanofluid, and nanoparticles, and base fluid are depicted in Table 1 and Table 2, respectively.

Table 1. Thermophysical properties of nanofluid and hybrid nanofluid

Properties	Nanofluid	Hybrid Nanofluid
Density	$\rho_{nf} = (1 - \phi_1) \rho_f + \phi_1 \rho_{s1}$	$\rho_{hnf} = (1 - \phi_2) \left[(1 - \phi_1) \rho_f + \phi_1 \rho_{s1} + \phi_2 \rho_{s2} \right]$
Heat capacity	$(\rho C_p)_{nf} = (1 - \phi_1) (\rho C_p)_f + \phi_1 (\rho C_p)_{s1}$	$(\rho C_p)_{hnf} = (1 - \phi_2) \left[(1 - \phi_1) (\rho C_p)_f + \phi_1 (\rho C_p)_{s1} + \phi_2 (\rho C_p)_{s2} \right]$
Dynamic viscosity	$\mu_{hnf} = \frac{\mu_f}{(1 - \phi_1)^{2.5}}$	$\mu_{hnf} = \frac{\mu_f}{(1 - \phi_1)^{2.5} (1 - \phi_2)^{2.5}}$
Thermal conductivity	$\frac{k_{nf}}{k_f} = \frac{k_{s1} + 2k_f}{k_{s1} + 2k_f} - 2\phi_1 \frac{(k_f - k_{s1})}{k_f + \phi_1 (k_f - k_{s1})}$	$\frac{k_{hnf}}{k_{nf}} = \frac{k_{s2} + 2k_{nf}}{k_{s2} + 2k_{nf}} + 2\phi_2 \frac{(k_{nf} - k_{s2})}{k_{nf} + \phi_2 (k_{nf} - k_{s2})}$
Thermal expansion coefficient	$(\rho\beta)_{nf} = (1 - \phi_1) (\rho\beta)_f + \phi_1 (\rho\beta)_{s1}$	$(\rho\beta)_{hnf} = (1 - \phi_2) \left[(1 - \phi_1) (\rho\beta)_f + \phi_1 (\rho\beta)_{s1} + \phi_2 (\rho\beta)_{s2} \right]$

Source: [33]

Table 2. Thermophysical properties of nanoparticles and base fluid

Physical properties	Al ₂ O ₃	Cu	Base fluid (water)
ρ (kg/m ³)	3970	8933	997.1
C_p (J/kgK)	765	385	4179
K (W/mK)	40	400	0.613
$\beta \times 10^{-5}$ (1/K)	0.85	1.67	21

Source: [33]

For the similarity transformation process, the appropriate similarity variables are introduced as follows:

$$\eta = \left(\frac{a}{v_f} \right)^{\frac{1}{2}} y, u = axf'(\eta), v = -(av_f)^{\frac{1}{2}} f(\eta),$$

$$u_p = axf'_p(\eta), v_p = -(av_f)^{\frac{1}{2}} f_p(\eta), \quad (8)$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \theta_p(\eta) = \frac{T_p - T_\infty}{T_w - T_\infty},$$

$$v_w = -\sqrt{av_f} s,$$

where prime refers to differentiation with respect to η , and η is denoted as the similarity variable, v_f is the kinematic viscosity, while s is the rate of transpiration constant rate. It is noted that $s > 0$ and $s < 0$ imply suction and injection, respectively, and $s = 0$ implies an impermeable plate. Further, f, f_p , refers to the dimensionless function for the fluid phase, while θ, θ_p , refers to a dimensionless function for the dust phase and v_w is the mass transfer velocity at the surface. The surface temperature T_w is assumed to be:

$$T_w = T_\infty + bx^2 \quad (9)$$

where b is a constant, while T_∞ is the ambient temperature.

Further, Eqs. (1) and (4) are satisfied by implementing Eq. (8), while Eqs. (2), (3), (5), and (6) are reduced to

Hybrid Nanofluid Phase:

$$\begin{aligned} & \frac{1}{(1 - \phi_1)^{2.5} (1 - \phi_2)^{2.5} (1 - \phi_2)} \left[(1 - \phi_1) + \phi_1 \frac{\rho_1}{\rho_f} \right] + \phi_2 \frac{\rho_2}{\rho_f} \\ & f'''(\eta) - f''^2(\eta) + f(\eta) f''(\eta) + \\ & \frac{L\beta_v}{(1 - \phi_2) \left[(1 - \phi_1) + \phi_1 \frac{\rho_1}{\rho_f} \right] + \phi_2 \frac{\rho_2}{\rho_f}} \left[f'_p(\eta) - f'(\eta) \right] \\ & + \frac{\beta_{hnf}}{\beta_f} \lambda^* \theta(\eta) \cos \alpha = 0 \end{aligned} \quad (10)$$

$$\left[\frac{k_2 + 2k_{nf} - 2\phi_2(k_{nf} - k_2)}{k_2 + 2k_{nf} + \phi_2(k_{nf} - k_2)} \right] \left[\frac{k_1 + 2k_f - 2\phi_1(k_f - k_1)}{k_1 + 2k_f + \phi_1(k_f - k_1)} \right]$$

$$(1 - \phi_2) \left[(1 - \phi_1) + \phi_1 \frac{(\rho C_p)_1}{(\rho C_p)_f} \right] + \phi_2 \left[\frac{(\rho C_p)_2}{(\rho C_p)_f} \right]$$

$$\left(\frac{1}{Pr} \right) \theta''(\eta) + f(\eta) \theta'(\eta) - 2f'(\eta) \theta(\eta) +$$

$$\frac{L\beta_T}{m(1 - \phi_2) \left[(1 - \phi_1) + \phi_1 \frac{(\rho C_p)_1}{(\rho C_p)_f} \right] + \phi_2 \left[\frac{(\rho C_p)_2}{(\rho C_p)_f} \right]}$$

$$\left[\theta_p(\eta) - \theta(\eta) \right] +$$

$$\frac{L\beta_v Ec}{m(1 - \phi_2) \left[(1 - \phi_1) + \phi_1 \frac{(\rho C_p)_1}{(\rho C_p)_f} \right] + \phi_2 \left[\frac{(\rho C_p)_2}{(\rho C_p)_f} \right]}$$

$$\left[f'_p(\eta) - f'(\eta) \right]^2$$

$$+ \frac{\mu_{hnf} / \mu_f}{(\rho C_p)_{hnf} / \rho_f} Ec \left[f''(\eta) \right]^2 = 0 \quad (11)$$

Dust Phase:

$$f_p(\eta) f_p''(\eta) - f_p'^2(\eta) + \beta_v \left[\frac{f'(\eta)}{f'_p(\eta)} - \right] = 0 \quad (12)$$

$$f_p(\eta) \theta'_p(\eta) - 2f'_p(\eta) \theta_p(\eta) + \varsigma B_T$$

$$\left[\theta(\eta) - \theta_p(\eta) \right] = 0 \quad (13)$$

where L is the mass concentration, β_T and, β_v are the temperature and velocity of fluid-particle interaction parameters, respectively, λ^* defines the mixed convection parameter, where $\lambda^* = Gr_x / Re_x^2$, Gr_x is the local Grashof number, $\lambda^* > 0$ represents the assisting buoyant force and $\lambda^* < 0$ represents the opposing buoyant force. The term λ^* contains a variable x that generates a local similarity solution. Further, Re_x is the local Reynolds number, which is then denoted as $Gr_x = g\beta_f(T_w - T_\infty)x^3/\nu_f^2$ and, $Re_x = U_w x / \nu_f$, Pr is the Prandtl number, Ec is the Eckert number, ς is the ratio of specific heat parameters, and they are defined by

$$K = \frac{m}{\tau_v}, \beta_v = \frac{1}{a\tau_v}, \beta_T = \frac{1}{a\tau_T}, L = \frac{Nm}{\rho_f},$$

$$\varsigma = \frac{C_{pf}}{C_{mf}}, Pr = \frac{(\mu C_p)_f}{k_f}, \quad (14)$$

$$Ec = \frac{U_w^2}{C_{pf}(T_w - T_\infty)} = \frac{(ax)^2}{C_{pf}bx^2} = \frac{a^2}{C_{pf}b}$$

The boundary conditions are:

$$f(0) = s, f'(0) = \lambda + \sigma_1 f''(0), \theta(0) = 1 + \sigma_2 \theta'(0),$$

$$f'(\eta) \rightarrow 0, f'_p(\eta) \rightarrow 0, f_p(\eta) \rightarrow f(\eta), \theta(\eta) \rightarrow 0,$$

$$\theta_p(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \quad (15)$$

where $\sigma_1 = L_s \sqrt{a/\nu_f}$, and $\sigma_2 = L_T \sqrt{a/\nu_f}$ are the velocity and thermal slip parameters, respectively.

The physical quantities are:

$$C_f = \frac{\mu_{hnf}}{\rho_f U_w^2} \left(\frac{\partial u}{\partial y} \right)_{y=0},$$

$$Nu_x = - \frac{x k_{hnf}}{k_f (T_w - T_\infty)} \left(\frac{\partial T}{\partial y} \right)_{y=0} \quad (16)$$

where C_f is the skin friction coefficient and Nu_x is the local Nusselt number. Further, by applying Eqs. (8) and (16), we obtain:

$$C_f Re_x^{1/2} = \frac{\mu_{hnf}}{\mu_f} f''(0), Nu_x Re_x^{-1/2} = - \frac{k_{hnf}}{k_f} \theta'(0). \quad (17)$$

3 Results and Discussion

The boundary layer problem of a hybrid nanofluid with dust particles passing a permeable inclined stretching and shrinking surface was explored. The analysis highlighted thermal, velocity slip parameters, and viscous dissipation as significant influences. The ordinary differential equations Eqs. (10) - (13), subjected to boundary condition Eq. (15), were computed utilizing the bvp4c solver, a built-in function in MATLAB software, as implemented by [35]. As this study extends the work of [33], the incorporation of 1% of alumina (Al_2O_3), ϕ_1 and copper (Cu), ϕ_2 nanoparticle volume fraction embedded in water as the base fluid was formulated to develop the hybrid nanofluids. The Prandtl number has been defined as 6.2, implying the base fluid is water. Besides that, to

obtain the numerical results, a set of appropriate initial guesses was selected based on the boundary layer thickness η_∞ within a specified range $8 < \eta < 13$ for each parameter considered. For convergence results, the relative tolerance was fixed at 10^{-10} to improve the accuracy of the results.

The findings of the skin friction coefficient $C_f \text{Re}_x^{1/2}$, local Nusselt number $Nu_x \text{Re}_x^{-1/2}$, fluid phase $f'(\eta)$ and the dust phase $f'_p(\eta)$ velocity profile, and fluid phase $\theta(\eta)$ and the dust phase $\theta_p(\eta)$ temperature profile is portrayed in graphical form. Table 3 and Table 4 display the comparable outcomes of the skin friction coefficient $f''(0)$ for numerous values of the velocity slip parameter σ_1 and heat transfer $-\theta'(0)$ for diverse Prandtl number values. The comparable outcomes illustrated in both tables demonstrate a high degree of correlation and validate the reliability of the methodology employed in this current study.

Table 3. Comparable outcomes of $f''(0)$ for a stretched surface ($\lambda = 1$) when $\phi_1 = \phi_2 = L = \beta_v = \beta_T = \sigma_2 = 0$ and $s = 0$

σ_1	Hayat et al. [36]	Ibrahim and Shankar [37]	Anuar et al. [33]	Present Results
0	-1.000000	-1.0000	-1.000000	-1.000000056
0.1	-0.872082	-0.8721	-0.872083	-0.872082478
0.2	-0.776377	-0.7764	-0.776377	-0.776377073
0.5	-0.591195	-0.5912	-0.591196	-0.591210751
2.0	-0.283981	-0.2840	-0.283981	-0.284005145
5.0	-0.144841	-0.1448	-0.144842	-0.144895434

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Table 4. Comparable outcomes of $-\theta'(0)$ for a stretched surface ($\lambda = 1$) when $\phi_1 = \phi_2 = L = \beta_v = \beta_T = s = \zeta = \sigma_1 = \sigma_2 = 0$

Pr	Gireesha et al. [38]	Naramgari and Sulochana [11]	Punith Gowda et al. [39]	Anuar et al. [33]	Present Results
0.72	1.0885	1.088561	-	1.088527	1.088536003
1	1.3333	1.333333	1.3333	1.333333	1.333333338
10	4.7968	4.796817	4.7969	4.796873	4.796873257

Source: Created by authors.

Figure 2 and Figure 3 show the outcomes of the $C_f \text{Re}_x^{1/2}$ and $Nu_x \text{Re}_x^{-1/2}$ with distinctive values of λ and inclination angle parameter ($\alpha = \pi/6, \pi/4, \pi/3$), respectively. The values of pertinent parameters employed are $\phi_1 = \phi_2 = 0.01$, $\sigma_1 = \sigma_2 = 0.2$, $m = \zeta = 1.0$, $\text{Ec} = s = 2.0$, $\beta_v = \beta_T$

$= 0.5$, $\lambda^* = 0.001$. Based on Figure 2 and Figure 3, there is a decline in both the $C_f \text{Re}_x^{1/2}$ and $Nu_x \text{Re}_x^{-1/2}$ values as inclination angles rise for both the first and second solutions. The buoyancy effect grows as α upsurges, while the shear stress is reduced, diminishing the friction between the dusty hybrid nanofluid and the surface and consequently reducing the heat transmission rate. Dual solutions were identified when $\lambda < -0.5$, while the occurrence of a unique solution is when $\lambda > -0.5$. Furthermore, λ_c symbolizes the critical point at which the first and second solutions intersect, and boundary layer separation delays as it develops as α increases.

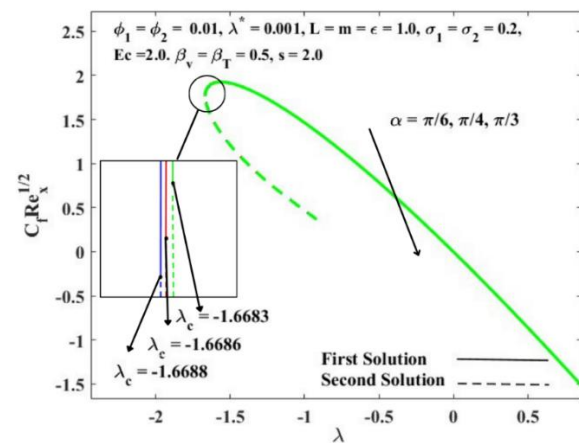


Fig. 2: $C_f \text{Re}_x^{1/2}$ with λ for distinctive values of α
Source: Created by authors.

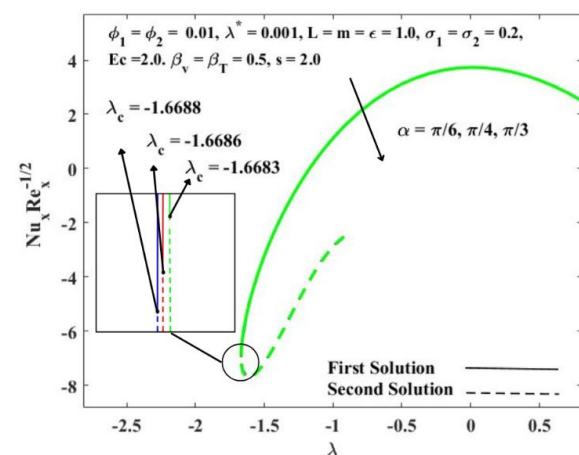


Fig. 3: $Nu_x \text{Re}_x^{-1/2}$ with λ for distinctive values of α
Source: Created by authors.

The corresponding velocity and temperature profiles for the fluid-phase $f'(\eta), \theta(\eta)$ and dust-phase $f'_p(\eta), \theta_p(\eta)$ of Figure 2 and Figure 3 are portrayed in Figure 4 and Figure 5, respectively. From Figure 4, it is observed that the velocity profile shows an increase in the momentum boundary layer thickness for the first solution for both phases, resulting in a decreased velocity gradient; however, an opposing trend can be seen in the second solution. Thus, it can be concluded that the drag force within the surface and the fluid diminishes as the angle of inclination rises. Furthermore, Figure 5 illustrates that the temperature profile of the first solution of both phases rises, whereas it declines for the second solution. The thermal boundary layer thickness of the first solution in both phases becomes greater, indicating a reduction in heat transmission efficiency. Additionally, the profiles depicted in Figure 4 and Figure 5 comply with the boundary condition outlined in Eq. (15), and the occurrence of the dual solution justifies the findings presented in Figure 2 and Figure 3.

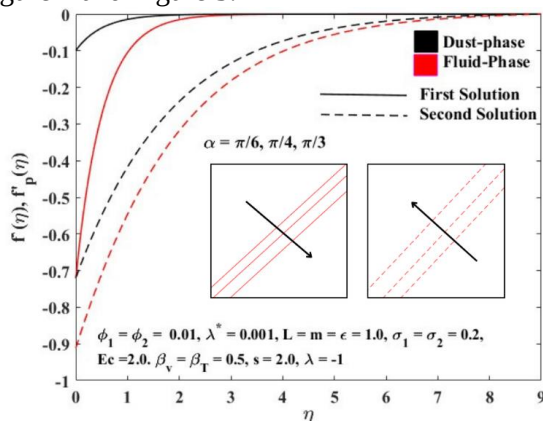


Fig. 4: Velocity profile for distinctive values of α
Source: Created by authors.

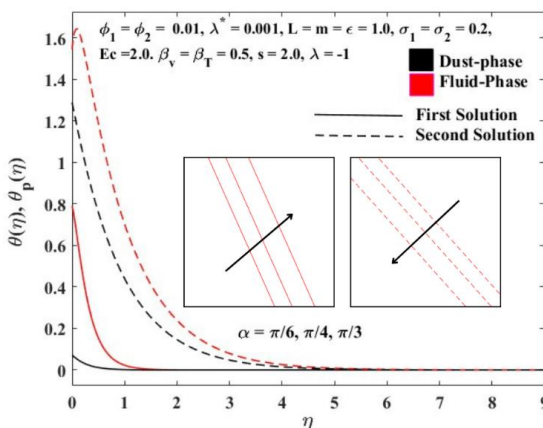


Fig. 5: Temperature profile for distinctive values of α
Source: Created by authors.

Figure 6 and Figure 7 elucidate the outcomes of the $C_f Re_x^{1/2}$ and $Nu_x Re_x^{-1/2}$ with various values of λ and Ec number when $\phi_1 = \phi_2 = 0.01$, $\sigma_1 = \sigma_2 = 0.2$, $\varsigma = m = 1.0$, $s = 2.0$, $\beta_v = \beta_T = 0.5$, $\lambda^* = 0.001$ and $\alpha = \pi/4$. Figure 6 shows that a greater Ec number enhances the coefficient of skin friction in the first and second solutions. This signifies that an upsurge in the value of the Ec number corresponds with the rise in the transformation of kinetic energy to thermal energy, which subsequently causes a greater surface shear stress. The separation of the boundary layer is postponed with the increase of the Ec number values. According to Figure 7, it is observed that the increment in values of the Ec number leads to a decline in the heat transmission rate in both solutions. This arises due to the insignificant thermal gradient within the fluid and the surface, resulting in less efficient heat transmission.

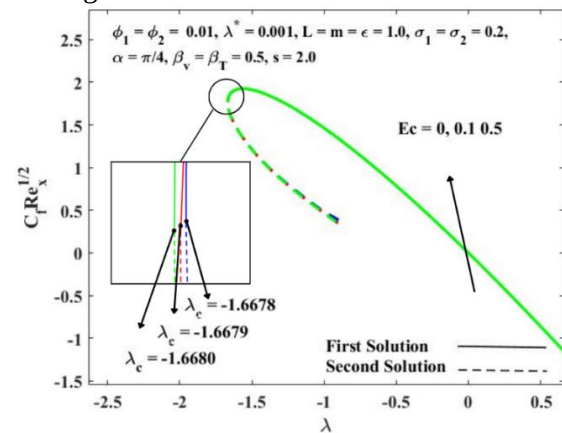


Fig. 6: $C_f Re_x^{1/2}$ with λ for distinctive values of Ec number
Source: Created by authors.

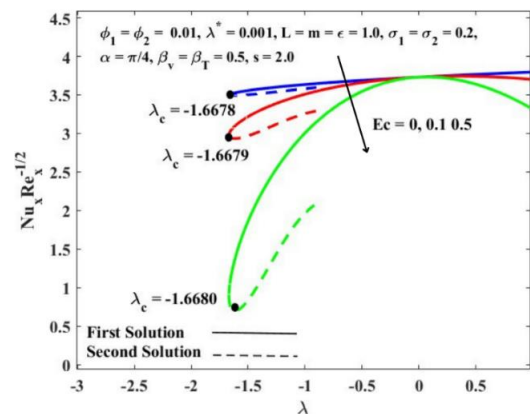


Fig. 7: $Nu_x Re_x^{-1/2}$ with λ for distinctive values of Ec number
Source: Created by authors.

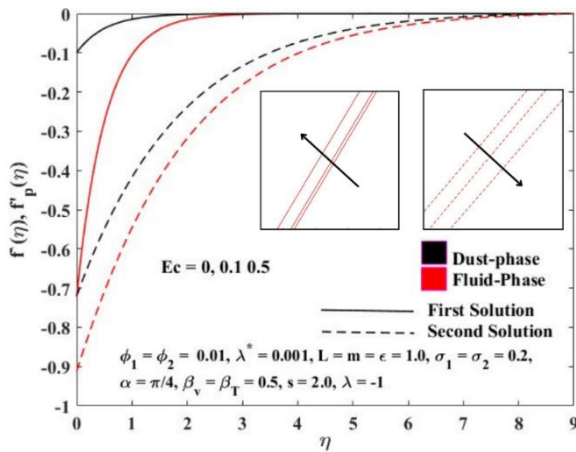


Fig. 8: Velocity profile for distinctive values of Ec number

Source: Created by authors.

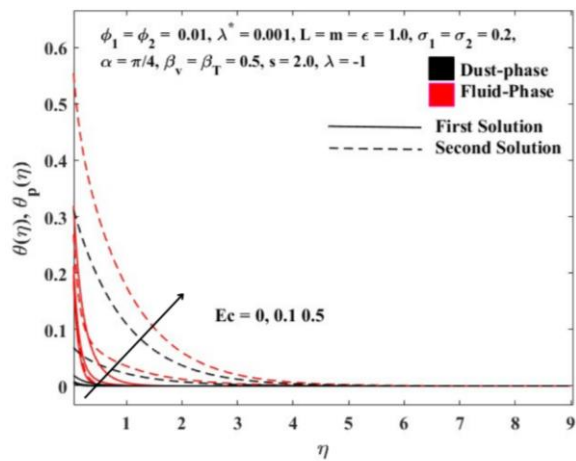


Fig. 9: Temperature profile for distinctive values of Ec number

Source: Created by authors.

Figure 8 and Figure 9 illustrate the velocity and temperature profiles for the fluid-phase $f'(\eta)$, $\theta(\eta)$ and dust-phase $f'_p(\eta)$, $\theta_p(\eta)$ of Figure 6 and Figure 7, correspondingly. Based on Figure 8, the momentum boundary layer thickness for the velocity profile of both phases for the first solution diminishes, while it increases for the second solution. Besides that, increasing the values of the Ec number reduced the boundary layer thickness of the first solution relative to the second solution, resulting in diminished fluid flow due to the increased drag force. Besides that, as referred to Figure 9, the temperature profile increases for both solutions of both phases, indicating a rise in the thermal boundary layer thickness. The first solution of both phases exhibits a narrower boundary layer thickness compared to the second solution, showing that the heat transmission in the first solution is more efficient due to a steeper temperature gradient.

Figure 10 and Figure 11 represent the impact of σ_2 on the values of $Nu_x Re_x^{-1/2}$ when $\alpha = \pi/4$ and the corresponding temperature profiles, respectively. It exhibits the rise of σ_2 reducing the values of $Nu_x Re_x^{-1/2}$. This situation occurs because the wall surface transmits less momentum to the fluid and weakens the thermal transport between the wall and the fluid. From Figure 11, it is observed that the thermal boundary layer of the fluid phase and the dust phase decreases with an increase in σ_2 for the first solution, and the opposite observation exists for the second solution.

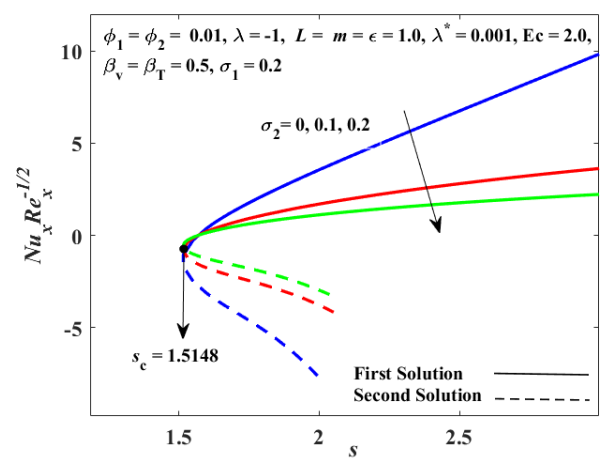


Fig. 10: $Nu_x Re_x^{-1/2}$ with s for distinctive values of σ_2

Source: Created by authors.

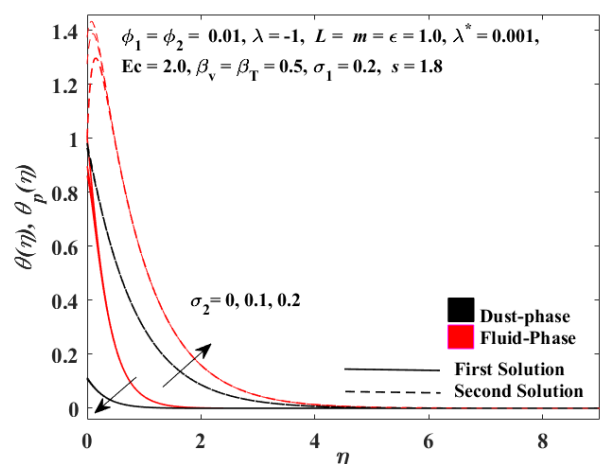


Fig. 11: Temperature profile for distinctive values of σ_2

Source: Created by authors.

4 Conclusions

A computational investigation was conducted to investigate the impact of inclination angle, viscous dissipation, and slip conditions on the flow and heat transmission of alumina-copper/water dusty hybrid nanofluid over a permeable inclined shrunk or stretched surface. The results were obtained by computing the reduced ordinary differential equations using a bvp4c built-in solver in MATLAB. The conclusions derived from the computed findings are as follows:

- An augmentation in the inclination angle results in a reduced skin friction coefficient, which subsequently elevates the fluid velocity as the drag force between the fluid and surface reduces, while simultaneously reducing the heat transmission rate.
- A greater Eckert number, indicating enhanced viscous dissipation, correlates with an increase in the skin friction coefficient, enhancing the shear stress, followed by a decrease in heat transmission rate due to a reduction in the temperature gradient at the surface.
- A dual solution exists for the shrinking surface $\lambda < -0.5$ as the inclination angle parameter and Eckert number increase.
- The separation of the boundary layer is delayed with the rising values of the inclination angle and Eckert number.

This work provides a framework for researchers to enhance their understanding of fluid dynamics and heat transfer, facilitating future investigations by allowing modifications of pertinent parameters considered, such as extending the investigations to unsteady or oscillatory stretching/shrinking surface or other surfaces, considering the influences of various thermophysical properties including magnetic fields, thermal radiation, as well as other variations of fluids.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified,

and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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