Stability Analysis of Drill Strings with Initial Curvature in a Fluid Flow under Environmental Uncertainty

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Abstract: - The paper explores the nonlinear dynamics of drill strings under stochastic factors and external influences by providing its comprehensive analysis. These studies are motivated by the need to improve the safety and efficiency of drilling operations in complicated conditions. The authors model the dynamics of drill strings from the perspective of the randomness of the processes involved, enabling a broader range of operational modes for drilling equipment to be considered, as well as enhancing the accuracy and efficiency of optimal decision-making. The nonlinear dynamics of the drill string are examined under the influence of random factors and environmental conditions. The mathematical model of the drill string describes its planar vibrations accounting for the initial curvature of the string, external loads, fluid flow around the string, and frictional forces against the wellbore walls. The randomness of the effects is taken into account through the friction coefficient. Stochastic terms derived using the prior probability distribution, which allows the application of Shannon's maximum entropy principle, are employed in the study. A numerical experiment is conducted to identify instability zones of the drill string motion. The study relies on the Galerkin method and partial discretization method. The influence of random factors on the drill string stability is revealed.

Key-Words: - dynamics, nonlinearity, stochastic, vibrations, stability, drill strings.

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1 Introduction

Drilling is a key stage in the oil and gas industry. This process involves complex dynamics, as it depends not only on the components of the equipment but also on external factors such as rock properties, the drilling fluid used, wellbore characteristics, etc. During the drilling of wells, the drill string is subject to undesirable vibrations, which can lead to reduced efficiency and, in extreme cases, can compromise the integrity of the drill string.

One of the common methods to minimize vibrations is the proper selection of drilling parameters, such as rotational speed and bit load, along with the adjustment of drilling fluid properties. This approach can be implemented quickly and, in most cases, does not require additional equipment. However, due to the complexity of the drilling process, its mathematical modeling is not limited to nonlinearities, discontinuities, and other challenges. Drilling operations are subject to uncertainties of various nature like imperfections in the drill string, the complex interaction between the bit and the rock, borehole wall properties, fluid-equipment interactions, and more. To account for such uncertain effects on the drill string, it is advisable to employ stochastic analysis.

For instance, in [1], the coupled lateral and torsional vibrations of the drill string taking into account the interaction between the drill bit and the rock, as well as the contact of the drill string and drilling fluid with the borehole walls, are investigated. Stochastic models were developed for the cases under consideration, and the results were compared with deterministic ones.

Several studies utilize stochastic methods to optimize model parameters. The authors of [2] apply the Bayesian approach to the parameter estimation and selection of an appropriate model for drill bit-rock interaction in the case of torsional vibrations. During the parameter estimation process, randomness effects were also incorporated. Moreover, the authors conducted the stability analysis in the context of uncertainty. In [3], the Bayesian approach was employed using the Markov method to estimate the parameters of drill bit-rock interaction. Gaussian noise was used to simulate randomness. The reliability of the approach in the context of drill string torsional vibrations was demonstrated by comparison with pseudo-experimental data. The application of the Gaussian noise was also noted in studies [4], [5], [6]. The results showed that the intensity of random perturbations had a significant impact on the system's response. Increased noise led to vibrations with larger amplitudes, underscoring the importance of accounting for the randomness when modeling drill string vibrations.

The nonlinear stochastic dynamics of horizontal drill strings were examined in [7]. Based on the beam theory, the authors developed a model that accounted for rotational inertia, shear deformations, friction forces, impact effects, and torque. In addition to the mechanical analysis of the system, the study formulated a methodology for addressing uncertainties in dynamic systems. Stochastic analysis of the system parameter randomness was conducted using a probabilistic approach based on the maximum entropy principle. Stochastic modeling of vertical and inclined drilling was investigated in [8]. The application of stochastic analysis was justified by the spatial heterogeneity of the mechanical properties of rock formations, allowing for more accurate predictions of plastic deformation zones around the wellbore and assessments of the probability of the wellbore collapse or drilling equipment sticking. The stick-slip effect in systems with dry friction using a stochastic model was explored in [9]. The authors treated the duration and frequencies of stick-slip events as stochastic variables, and their statistics were estimated from the system dynamic equations. Statistical analysis of stick-slip phenomena was also conducted in [10]. Furthermore, this study focused on modeling random disturbances during drill bit-rock interaction. The proposed model treated variations of the drill bit torque as a random process dependent on the bit rotational speed, implemented via the multiplicative Gaussian stochastic process calibrated on real data. In [11], stochastic disturbances in drill bit-rock interactions and their impact on the performance of two vibration control controllers were studied. The model was calibrated with field data, and an intensity map of torsional vibrations was constructed. Uncertainties in drill bit-rock interactions and applied maximum speeds were analyzed in [12]. The random nature of these interactions was further examined in [13], where the authors developed a lumped-mass model to analyze the torsional and axial vibrations of drill strings. The study demonstrated that high formation heterogeneity resulted in greater vibration amplitudes.

The influence of rock strength variations in the context of torsional vibrations was analyzed in [14]. The drill bit-rock interaction model considered the coupling between axial and torsional dynamics, while the stochastic dynamics was based on the Itô equation, which allowed the description of the cutting component of the torque.

The stability analysis of the model considering uncertainties, as well as the development of a control methodology for drilling systems, was presented in [15]. According to the study's findings, variability in the drill bit-rock interaction, implemented through random friction, significantly influenced the stability and performance of the drilling system.

The application of statistical and stochastic methods to dynamic systems was discussed in [16]. The study examined linear dynamic systems subject to random influences using the stochastic Galerkin method. The formulation and solution of partial differential equations considering uncertainties in input data were presented in [17]. The authors identified uncertainties arising from natural randomness, which can be described by random fields with covariance functions, as well as uncertainties caused by a lack of knowledge about input parameters. For the latter, an algorithm was developed to improve the efficiency of the solution. In [18], a methodology was proposed for reliability assessment and design optimization of structures considering uncertainties of both probabilistic and interval nature. Moreover, the consideration of the random factor goes beyond the study of drill strings, thus demonstrating its versatility. For example, in [19], stochastic modeling was used in the context of ecological systems. The dynamics of the stochastic Gilpin-Ayala model with a white noise overlay were considered. In this way, fluctuations in the concentration of the pollutant were taken into account. In [20], a stochastic epidemiological model, which accounts for the effects of transportation movements, media exposure, and time delays on the spread of infectious diseases, was introduced. Random processes allowed a more realistic modeling of the influence of external factors on the infection dynamics.

Thus, it can be noted that the rock formation often serves as a source of uncertainty in the model. In this study, this uncertainty is accounted for by the application of a probabilistic model for the friction coefficient. This approach is based on the fact that real rock formations can be heterogeneous, containing various types of inclusions, fractures, and variations in mineral composition, which affect their mechanical properties.

2 Deterministic Model

Based on the structural features of the drill string, it is modeled as a rotating elastic isotropic rod. In contrast to known models, in particular [21], only lateral vibrations are considered as dominant. Additionally, the imperfections of the drill string such as its initial curvature, [22], along with external loads and the influence of fluid flow, are taken into account. In accordance with [12], the effect of friction forces against the borehole walls is also considered. The cross-section of the drill string is assumed to be constant and symmetric.

$$\begin{aligned}&\left(\rho A+\rho_l A_l\right)\frac{\partial^2 v}{\partial t^2}+EI_x\frac{\partial^4 v}{\partial z^4}-\rho I_x\frac{\partial^4 v}{\partial z^2\partial t^2}\\&+\frac{\partial}{\partial z}\left(N(z,t)\frac{\partial\left(v+v_0\right)}{\partial z}\right)-\frac{EA}{1-\nu}\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)^3\\&-\left(\rho A+\rho_l A_l\right)\omega^2 v=-\mu\rho A\frac{\partial^2 v}{\partial t^2},\end{aligned} \tag{1}$$

where E is Young's modulus, I_x axial inertia moments, ρ the drill string density, ν Poisson's ratio, A the cross-sectional area of the drill string, ρ_f the fluid density, A_f the internal cross-sectional area of the drill string, μ the friction coefficient.

Let us define the boundary and initial conditions as follows:

$$\begin{aligned}&v(z,t)=0,\quad EI_x\frac{\partial^2 v(z,t)}{\partial z^2}=0,\\&v(z,0)=0,\quad EI_x\frac{\partial v(z,0)}{\partial t}=1\qquad(z=0,\,z=l).\end{aligned} \tag{2}$$

The axial compressive load is of a conservative nature and follows a periodic law:

$$N=N_0+N_t\cos\overline{\Omega}t, \tag{3}$$

where N_0 and N_t represent the constant and time-varying components of the compressive load, respectively; $\overline{\Omega}$ is the frequency of external influence.

To set the initial curvature, the following expression is considered:

$$v_0(z)=f_0\sin\left(\frac{\pi z}{l}\right). \tag{4}$$

For further analysis, the model has to be reduced to an equation with generalized parameters using the Galerkin method. The displacement of the drill string is expressed as a series:

$$v(z,t)=\sum_{i=1}^{n}f_i(t)\sin\left(\frac{i\pi z}{l}\right). \tag{5}$$

The basis functions $\sin\left(\frac{i\pi z}{l}\right)$, $i=\overline{1,n}$ fully satisfy the boundary conditions (2).

For transition to the relative time variable, τ, a nondimensionalization operation is performed with respect to *t*:

$$\tau=t\cdot\Omega_0, \tag{6}$$

where Ω_0 is the natural vibration frequency of the drill string.

Considering the lateral vibrations of the drill string in its basic form of bending with respect to the dimensionless function $f(\tau)$, the equation in generalized parameters takes the form:

$$\frac{d^2 f}{d\tau^2}+\left(1-2\beta\cos\Omega\tau\right)f+\alpha f^3=F_0+F_1\cos\Omega\tau, \tag{7}$$

where

$$\beta=\frac{\beta_2}{\beta_1},\quad \Omega=\frac{\overline{\Omega}}{\Omega_0},\quad \alpha=\frac{\tilde{\alpha}}{\Omega_0^2},\quad F_i=\frac{\tilde{F}_i}{\Omega_0^2},\quad i=0,1;$$

$$\beta_1=\frac{l}{2}\left(EI_x\left(\frac{\pi}{l}\right)^4-N_0\left(\frac{\pi}{l}\right)^2-\left(\rho A+\rho_l A_l\right)\omega^2\right),$$

$$\beta_2=\frac{N_t\pi^2}{4l},\quad \Omega_0=\sqrt{\frac{\beta_1}{\delta_1}},$$

$$\delta_1=\frac{l}{2}\left(\rho A(1+\mu)+\rho_l A_l+\rho I_x\left(\frac{\pi}{l}\right)^2\right),$$

$$\tilde{\alpha}=\frac{3EA\pi^4}{8(1-\nu)l^3\delta_1},$$

$$\tilde{F}_0 = f_0 \frac{N_0 \pi^2}{2l\delta_1}, \quad \tilde{F}_1 = f_0 \frac{N_t \pi^2}{2l\delta_1}.$$

To analyze the stability of the dynamic system, let us represent the function $f(\tau)$ as the sum of some periodic solution $f_0(\tau)$ and a small perturbation from the equilibrium state δf. We exclude the main solution and the terms containing the deviation from the solution to powers higher than the first since this value is sufficiently small.

$$\frac{d^2\delta f}{d\tau^2} + \left(\gamma - 2\beta\cos\Omega\tau + 3\alpha f_0^{\,2}\right)\delta f = 0. \tag{8}$$

For the case of main resonance $f_0 = r_1\cos(\Omega\tau - \varphi_1)$, the equation for the perturbed state with respect to the small deviation takes the form of the well-known Hill equation, [23], [24]:

$$\frac{d^2\delta f}{d\tau^2} + \left(\theta_0 + \theta_{1c}\cos\Omega\tau + \theta_{2s}\sin 2\Omega\tau + \theta_{2c}\cos 2\Omega\tau\right)\delta f = 0. \tag{9}$$

In this case, the stability of the system is characterized by the behavior of the small deviation δf:

1) if δf uncontrollably increases at $\tau \to \infty$, then the system is unstable;
2) if δf remains bounded at $\tau \to \infty$, then the system is stable. It corresponds to the Lyapunov stability.

To determine the instability zones, according to the Floquet theory, a particular solution of equation (9) is sought in the form of a vibration spectrum:

$$\delta f = e^{\eta\tau}\sum_k b_k\cos(k\Omega\tau - \psi_k), \quad k = 1, 3, 5, \ldots, \infty \quad \text{or} \quad k = 0, 2, 4, \ldots, \infty. \tag{10}$$

Here η is the characteristic exponent, which governs the behavior of the variation δf:

1) $\mathrm{Re}(\eta) < 0$: $\delta f \to 0$ at $\tau \to \infty$; hence, the solution is stable;
2) $\mathrm{Re}(\eta) = 0$: the solution at the boundary of stability zones.
3) $\mathrm{Re}(\eta) > 0$: $\delta f \to \infty$ at $\tau \to \infty$; then, the solution is unstable.

Let us focus on the first instability zone:

$$\delta f = e^{\eta\tau} b_1\cos(\Omega\tau - \psi_1). \tag{11}$$

To obtain the equation describing the boundaries of the first instability zone, the harmonic balance method is applied.

The transition to the characteristic determinant is performed as follows:

$$\Delta(\eta) = \begin{vmatrix} \eta^2 - \Omega^2 + \theta_0 + \dfrac{\theta_{2c}}{2} & 2\eta\Omega + \dfrac{\theta_{2s}}{2} \\ -2\eta\Omega + \dfrac{\theta_{2s}}{2} & \eta^2 - \Omega^2 + \theta_0 - \dfrac{\theta_{2c}}{2} \end{vmatrix} \tag{12}$$

To construct the boundaries of the first instability zone the condition $\Delta(\eta = 0) = 0$ is introduced:

$$\left(A_0 - \Omega^2\right)^2 + \left(B_0 + B_1\Omega^2\right)r_1^{\,2} + C_0 r_1^{\,4} = 0, \tag{13}$$

where $A_0 = \gamma, \quad B_0 = 3\alpha\gamma, \quad B_1 = -3\alpha, \quad C_0 = \dfrac{27}{16}\alpha^2.$

For a more detailed quantitative analysis of the system's response, the partial discretization method is applied. The idea of the partial discretization method is to discretize the nonlinear terms of the equation, which complicate the process of solving the problem. Since the nonlinear term becomes constant at each discretization step, it allows one to obtain a quasi-analytical solution. Its general form is given by:

$$F(z) = \frac{1}{2}\sum_{i=1}^{n}\left(z_i + z_{i+1}\right)\left[F(z_i)\delta(z - z_i) - F(z_{i+1})\delta(z - z_{i+1})\right], \tag{14}$$

where $F(z)$ is the nonlinear function that must be discretized, $F(z_i)$ is the discrete form of the function $F(z)$ at $z = z_i$, $i = \overline{1,n}$; n is the number of divisions of z, $\delta(z - z_i)$ is the Dirac's delta function.

Applying the partial discretization method to Eq. (7), one obtains:

$$\frac{d^2\delta f}{d\tau^2} + \frac{1}{2}\sum_{i=1}^{n}\left(\tau_i + \tau_{i+1}\right)\left[C_i\,\delta f(\tau_i)\delta(\tau - \tau_i) - C_{i+1}\,\delta f(\tau_{i+1})\delta(\tau - \tau_{i+1})\right] = 0, \tag{15}$$

where

$$C_i = \left(\theta_0 + \theta_{1c}\cos\Omega\tau_i + \theta_{2s}\sin 2\Omega\tau_i + \theta_{2c}\cos 2\Omega\tau_i\right).$$

Using the method of mathematical induction, a recurrence formula is derived from the quasi-analytical solution of Eq. (15):

$$\delta f(\tau_k) = -\frac{1}{2}[(\tau_1+\tau_2)C(\tau_1)\delta f(\tau_1)(\tau_k-\tau_1)+ \\ +\sum_{i=2}^{k-1}(\tau_{i+1}-\tau_{i-1})C(t_i)\delta f(\tau_i)(\tau_k-\tau_i)] \\ +\delta\dot{f}_0\tau_k+\delta f_0. \quad (16)$$

3 Uncertainty Modeling

The friction force is assumed to be of a random nature. The influence of randomness is accounted for through the friction coefficient, [12].

The random field $\{v(x): x\in[0,L]\}$ is defined as a set of real-valued random variables over the probability space $(\Omega, \mathrm{F}, \mathrm{P})$, where Ω is the set of elementary events, F is the σ-algebra, and P is the probability measure.

It is assumed that v is the stationary truncated Gaussian random field on the interval [0, L] with the exponential autocorrelation function:

$$R(x_1,x_2) = \sigma^2 \exp\left(-\frac{|x_2-x_1|}{b}\right). \quad (17)$$

The stochastic field is transformed using the Karunen-Loeve expansion, [25]:

$$v(x,\xi) = \underline{\mu}(x) + \sum_{k=1}^{N}\sqrt{\lambda_k}Z_k(\xi)\varphi_k(x) \quad (18)$$

where $\underline{\mu}$ is the mean value of the friction coefficient, λ_k and φ_k are the k-th eigenvalue and eigenvector of the autocorrelation function, respectively, Z_k are independent random variables with the standard Gaussian distribution, N determines the accuracy of the expansion (the larger N, the more accurate the approximation).

4 Results and Discussions

Parameters used for numerical simulations are presented in Table 1. To obtain statistically significant results, 100 simulations were conducted for each stochastic experiment. If the amplitude-frequency characteristics fall within the instability region for the given system parameters, the system undergoes primary resonance, where the amplitude begins to increase uncontrollably. This can lead to adverse consequences.

Table 1. Parameters of the drilling system.

System parameter	**Value**
Drill string length, l	100m
Young's modulus, E	2.1×10^{11} Pa
Drill string density, ρ	$7800\,\mathrm{kg/m^3}$
Poisson's ratio, ν	0.28
The outer diameter of the drill string, D	0.2m
The inner diameter of the drill string, d	0.12m
Longitudinal load, constant part, N_0	1.7×10^3 N
Expected value of the friction coefficient, μ	0.4

As can be seen from the results obtained (Figure 1 and Figure 2), the instability zone is limited to the average value of the friction coefficient. The addition of stochastic influence causes the expansion of the boundaries of the instability zones, but no qualitative changes are observed. The same can be said about instability zones at $\omega = \frac{30}{60}$ (Figure 3 and Figure 4). Although there is a significant difference between the shapes of the zones at different values of angular velocity, no qualitative difference between the deterministic and stochastic models is observed.

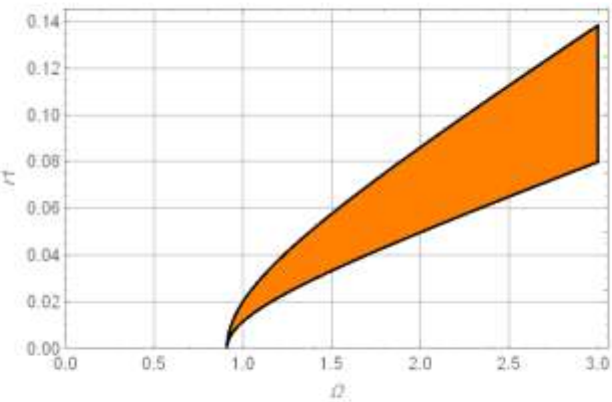

Fig. 1: The deterministic first instability zone at $\rho_l = 890$ kg/m^3, $\omega = \frac{10}{60}$.

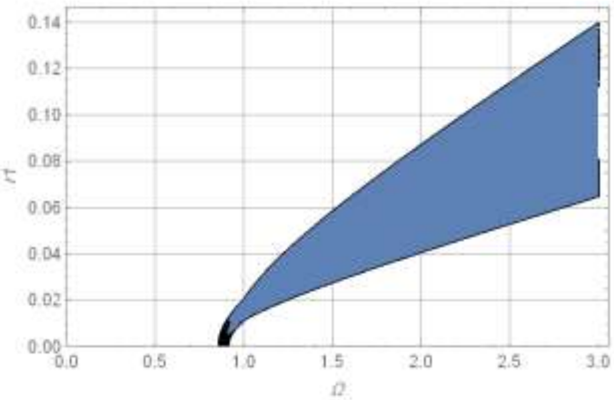

Fig. 2: The stochastic first instability zone at $\rho_l = 890$ kg/m^3, $\omega = \frac{10}{60}$.

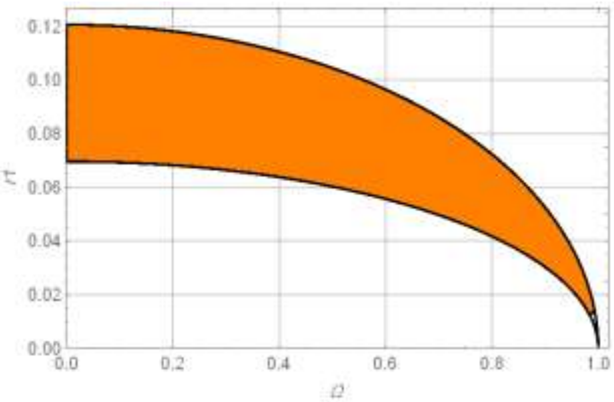

Fig. 3: The deterministic first instability zone at $\rho_l = 1500$ kg/m^3, $\omega = \frac{30}{60}$.

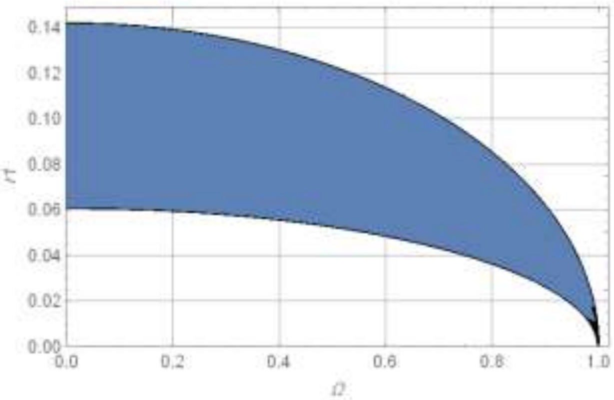

Fig. 4: The stochastic first instability zone at $\rho_l = 1500$ kg/m^3, $\omega = \frac{30}{60}$.

For better understanding of the effect of random friction, the results using the partial discretization method are considered (Figure 5, Figure 6, Figure 7, Figure 8, Figure 9 and Figure 10).

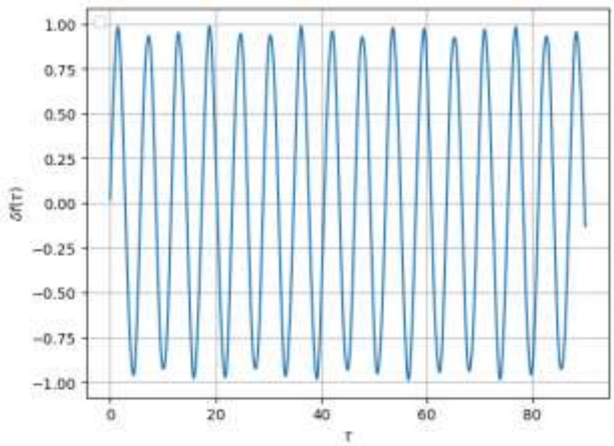

Fig. 5: The deterministic stable system at $\rho_l = 890$ kg/m^3, $r_1 = 0.02$, $\Omega = 2$.

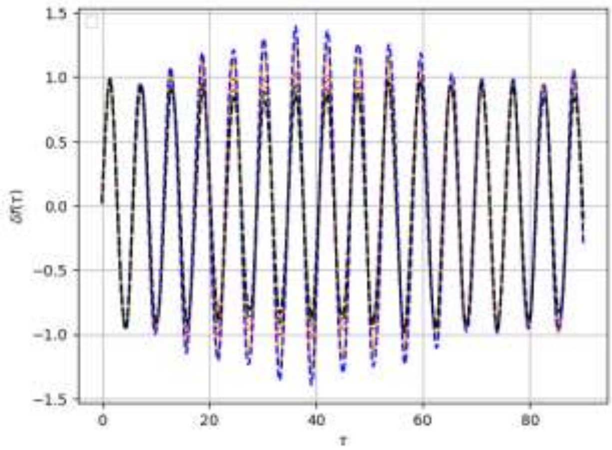

Fig. 6: The stochastic stable system at $\rho_l = 890$ kg/m^3, $r_1 = 0.02$, $\Omega = 2$.

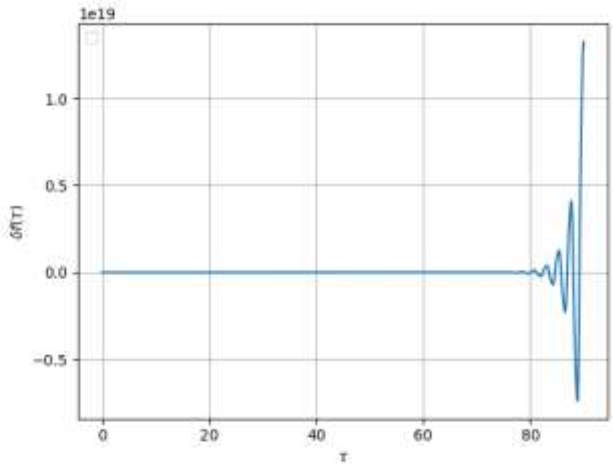

Fig. 7: The deterministic unstable system at $\rho_l = 890$ kg/m^3, $r_1 = 0.1$, $\Omega = 2.75$.

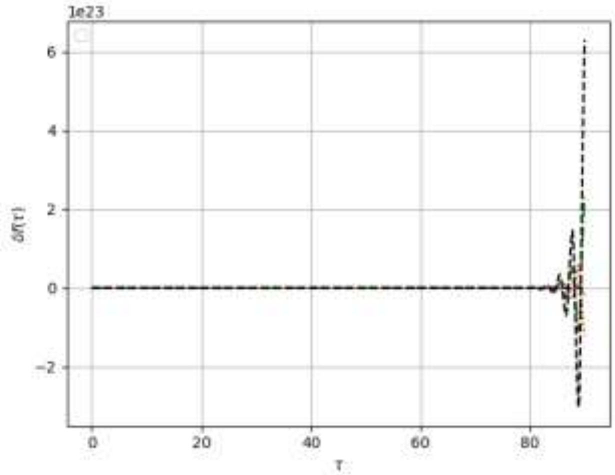

Fig. 8: The stochastic unstable system at $\rho_l = 890$ kg/m^3, $r_1 = 0.1$, $\Omega = 2.75$.

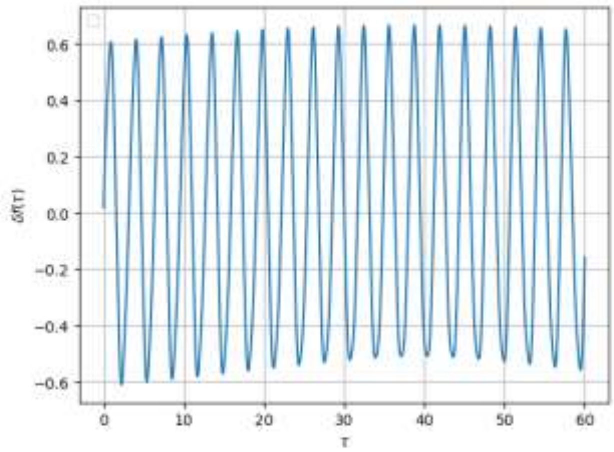

Fig. 9: The stochastic stable system at $\rho_l = 1500$ kg/m^3, $r_1 = 0.05$, $\Omega = 3$.

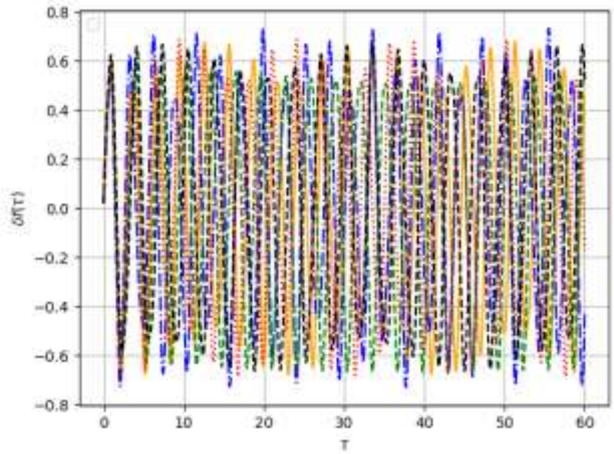

Fig. 10: The stochastic stable system at $\rho_l = 1500$ kg/m^3, $r_1 = 0.05$, $\Omega = 3$.

For the given system parameters, a good agreement of the results was obtained. Namely, the system with parameters outside the instability zone remains restricted, while parameters within the instability zone result in the sharp growth of perturbations. Stochastic effects cause an increase in amplitude over specific intervals, indicating significant quantitative changes.

When analyzing the displacements (Figure 11 and Figure 12), since the largest vibrations of the rod are observed at the center and significantly decrease toward the ends, the vibrations in the cross-section $z = 0.5l$ are considered, [26].

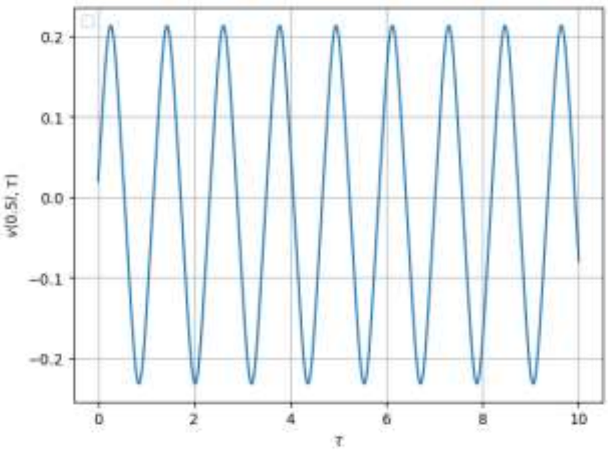

Fig. 11: The deterministic system's displacement values

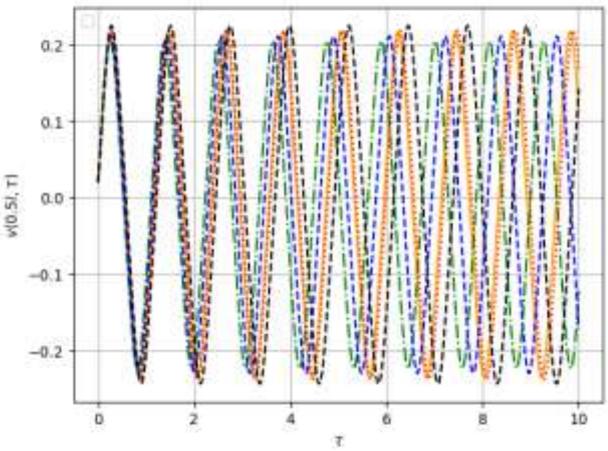

Fig. 12: The stochastic system's displacement values

According to the results of the analysis, in the case of random impact, the expansion of the instability zones is observed, and the results of the study using the Floquet theory are in good agreement with those obtained by the partial discretization method. In the case of the stochastic model, the overall structure of the vibration remains predictable. The increase in amplitude is due to the proximity of the system to the resonant frequency zone, which requires careful consideration in design and operation.

It should be noted that in the discretization case considered here, the discrete model corresponds to the studied processes, unlike the results of [27], where a potential mismatch between dynamic models after their discretization and re-continuation occurs, which can lead to a radical change in their behavior. The work presents examples of deterministic discretization and chaotic re-continuation of dynamic systems. To preserve the properties of the model, the authors adjusted the dynamics of the phenomenon based on the characteristic parameters of chaotic systems, such as Lyapunov exponents and dimensions.

5 Conclusion

The findings confirmed the significant influence of stochastic factors on the drill string behavior. By accounting for random factors, in particular the coefficient of friction, the study provided a deeper understanding of the drill string behavior under environmental uncertainty. Using the Galerkin method, Floquet theory, and partial discretization method, critical zones of instability were identified.

The results showed that the introduction of stochastic effects led to a significant increase in the range of critical values of the system parameters. No structural changes in the dynamics were observed; therefore, the behavior of the system remained predictable. These results emphasized the importance of considering random factors when modeling drilling processes.

Identification of instability zones is important for the optimization of drilling processes, as it allows determining in advance the operating modes where potential risks of equipment failure may arise. More accurate consideration of the random factors reproduces the unpredictable dynamics of the real process. Thus, based on the forecasts obtained on the basis of the presented methodology, either equipment operators can adjust the intensity of work, or modifications can be introduced at the stage of equipment design, thereby reducing the risks of emergencies as well as financial and time losses. In practice, this leads to safer, more efficient drilling.

Moreover, the shown methodology can be applied to a wide class of mechanical engineering problems beyond the oil and gas industry and can be adopted for the use of other similar numerical methods, allowing for further flexibility in application. Given the unpredictability inherent in real systems, this approach allows for more accurate modeling and prediction of processes. Such methods can be applied in aerospace, automotive, and construction fields. This approach will improve models of structural behavior by introducing factors that are difficult to account for with traditional methods.

Further development of the work is seen in the development of models complicated by uncertainties of other nature such as randomness of initial curvature, random impact of gas or liquid flows, and their subsequent solution by the partial discretization and stability analysis. For each case, it will be necessary to derive its own probability distribution functions based on statistical data.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the authors used ChatGPT in order to improve the readability and language of the manuscript. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

- L.A. Khajiyeva was responsible for conceptualization and methodology of the studied problem.
- S.Kh. Efendiyev incorporated uncertainty in the model, reduced the solution to a form ready for numerical simulations.
- A.K. Kudaibergenov developed the nonlinear mathematical model of the drill string vibrations.
- Sh.M. Gabayev done numerical experiments.

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Conflict of Interest

The authors have no conflicts of interest to declare.