On the Analytical Determination of the Contour of Well - Styeamlined Bodies

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Abstract. - The problem is solved with the help of a modified Prandtl equation applied to the case under study. This is a two-dimensional problem of flowing around a flat body when the essential factor is to take into account the limitation of its dimensions in the longitudinal and transverse directions. Thanks to the above Prandtl equation it was possible to reduce the problem to a self-similar equation. An analytical solution has been found. Thanks to this solution the shape of the body is analytically determined when the resistance is at its lowest. An analysis of the solution of the problem for different Reynolds numbers is carried out. The resulting equation is solved numerically for different values of its included parameters. With the help of a graphic illustration the different shapes of such contours are shown.

Key-Words: - a continuity equation, the Navier–Stokes equation, a non-linear equation, a self-similar solution, Prandtl equation, Reynolds number, numerical equation.

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1 Introduction

This article is devoted to the issue related to the analytical determination of the shape of bodies with the least resistance force.

As it is known, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], the shape of a raindrop also belongs to such bodies. This is quite understandable because due to the lack of turbulence in its tail section, the resistance force is greatly reduced. At the same time, we are not aware of any work where the drop shape would be described analytically.

In this regard, we have set ourselves another task namely to find a body shape for which the resistance force will be significantly less compared to other bodies. As follows from this statement this is about the application of methods of the variational calculus. At the same time, the question comes to the fore: what should be chosen as the functional extremum we must findThe answer is obvious since the role of the desired function must be assigned to the resistance force. Let us start with its calculation.

2 The Function of the Resistance Force

We will represent the shape of the body in the form of a flat two-dimensional figure of finite thickness h which corresponds to the longitudinal section of the spatial body similar to the task of N.E. Zhukovskiy on calculating the lift on the wing.

To find the full resistance force value that it experiences the resistance force not only on the end part of the body must be taken into account but also on both side surfaces. Considering the body symmetrical concerning the direction of the streamline flow let us use a general expression for the resistance force which we will write in the form, [1]:

$$F_i = \int_S \sigma_{ik} ds^k ,$$

where $S-$ is the body's total surface, $\sigma_{ik} -$ is the viscous stress tensor.

Having expanded the integral we have

$$F_i = \int_{2S_0+S_1} \sigma_{ik} ds^k = \int_{2S_0} \sigma_{iy} dxdz + \int_{S_1} \sigma_{in} ds^n .$$

Projecting this force onto the axis x along which the flow is directed we will get

$$F_x = \int\limits_{2S_0} \sigma_{xy} dxdz + \int\limits_{S_1} \sigma_{xn} ds^n = = 4\int\limits_{x_0}^{x_1} \sigma_{xy}\Big|_{y=\pm\frac{h}{2}} \eta(x)\,dx + h\oint\limits_{\Gamma^+} \sigma_{xn} dL,$$

where $\Gamma^+ -$ is the flow contour. Or

$$F_c = -4\int\limits_{x_0}^{x_1} \sigma_{xy}\Big|_{y=\pm\frac{h}{2}} \eta(x)\,dx - -2h\int\limits_{x_0}^{x_1} \sigma_{xn}\Big|_{\Gamma^+} \sqrt{1+\eta'^2(x)}dx., \quad (1)$$

where the components of the viscous resistance tensor are

$$\sigma_{xy} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial y} + \frac{\partial \mathrm{v}_y}{\partial x}\right) \approx \mu\frac{\partial \mathrm{v}_x}{\partial y} \quad (2)$$

$$\sigma_{xn} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial x_n} + \frac{\partial \mathrm{v}_n}{\partial x}\right) \quad (3)$$

where $\mu -$ is the dynamic viscosity.

Note that a quite obvious condition was taken into account [1] in formula (2) $\mathrm{v}_y \ll \mathrm{v}_x$, $\delta \ll a$ The second inequality means that the typical variation range of the velocity argument along the axis y should be of the order δ and along the axis z it is the order of the average linear size of the body $\sqrt{ab}$.

The normal derivative can be calculated as follows

$$\frac{\partial \mathrm{v}_x}{\partial x_n} = \frac{\partial \mathrm{v}_x}{\partial x}\sin\alpha - \frac{\partial \mathrm{v}_x}{\partial z}\cos\alpha \quad (4)$$

Therefore we get the following from the expression (3)

$$\sigma_{xn} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial x_n} + \frac{\partial \mathrm{v}_n}{\partial x}\right)\Bigg|_{\Gamma^+} = = \mu\Bigg[\; 2\frac{\partial \mathrm{v}_x}{\partial x}\sin\alpha - \left(\frac{\partial \mathrm{v}_x}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial x}\right)\cos\alpha + +\left(\mathrm{v}_x\cos\alpha + \mathrm{v}_z\sin\alpha\right)\alpha' \;\Bigg]\Bigg|_{z=\eta(x)} \quad (4)$$

Considering that

$$\cos\alpha = \frac{1}{\sqrt{1+\eta'^2}}, \sin\alpha = \frac{\eta'}{\sqrt{1+\eta'^2}}, (\sin\alpha)' = \frac{\eta''}{\left(1+\eta'^2\right)^{\frac{3}{2}}}, (\cos\alpha)' = -\frac{\eta'\eta''}{\left(1+\eta'^2\right)^{\frac{3}{2}}}, \quad (5)$$

the functional (1) can be rewritten as follows:

$$F_c = -4\mu\int\limits_{x_0}^{x_1} \frac{\partial \mathrm{v}_x}{\partial y}\Bigg|_{y=\pm\frac{h}{2}} \eta(x)\,dx - -2\mu h\int\limits_{x_0}^{x_1} \left[\left(2\frac{\partial \mathrm{v}_z}{\partial x}\eta' - \left(\frac{\partial \mathrm{v}_x}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial x}\right) + \mathrm{v}_x\eta''\right)\right]\Bigg|_{z=\eta} dx \quad (6)$$

To calculate the velocity distributions appearing in (6) we can use the modified Prandtl equation.

Indeed, near the flow surface, the Navier–Stokes equation taking into account the finite dimensions of the body can be brought to the following equation (for a detailed analysis, [1])

$$\mathrm{v}_x\frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{v}_z\frac{\partial \mathrm{v}_x}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 \mathrm{v}_x}{\partial x^2} + \frac{\partial^2 \mathrm{v}_x}{\partial z^2}\right) \quad (7)$$

where $\nu -$ is the kinematic viscosity, $P -$ is the pressure, $\rho -$ is the fluid density.

Equation (7) is valid if $\mathrm{v}_x \gg \mathrm{v}_z$.

It should be emphasized that equation (7) describes the flow around the end part of the body the thickness of which is h. The longitudinal dimensions of the body are finite and that is why both second-order partial derivatives had to be taken into account in the Laplace operator in contrast to the equation solved by Blasius, [1], who considered the body semi-infinite.

Further, it will be seen that it is the account of the two-dimensionality of the Laplace operator that allows us to find a self-similar solution that differs significantly from the solution found by Blasius.

If we use the Bernoulli's principle

$$P+\frac{\rho \mathrm{v}_x^2}{2}=\frac{\rho U^2}{2} \tag{8}$$

and consider the velocity U to be constant, equation (7) will be simplified, and using the continuity equation we come to the following

$$\begin{cases} \mathrm{v}_z \dfrac{\partial \mathrm{v}_x}{\partial z}=\nu\left(\dfrac{\partial^2 \mathrm{v}_x}{\partial x^2}+\dfrac{\partial^2 \mathrm{v}_x}{\partial z^2}\right), \\ \dfrac{\partial \mathrm{v}_x}{\partial x}+\dfrac{\partial \mathrm{v}_z}{\partial z}=0. \end{cases} \tag{9}$$

We will look for the solution of the continuity equation in the following form

$$\begin{cases} \mathrm{v}_x=u+\dfrac{\partial \psi}{\partial z}, \\ \mathrm{v}_z=-\dfrac{\partial \psi}{\partial x}. \end{cases} \tag{10}$$

where the function $\psi=\psi(x,z)$ is to be found.

Both velocity components in (10) must satisfy two conditions

$$\mathrm{v}_x \to u, \tag{11}$$

$$\mathrm{v}_z \to 0. \tag{12}$$

If we now substitute formula (10) into the upper equation of system (9) we get

$$\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial \psi}{\partial z}+\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial z^2}=0 \tag{13}$$

With the introduction of a dimensionless function

$$\varphi=\frac{\psi}{u\sqrt{ab}} \tag{14}$$

equation (13) is brought to the form:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial \varphi}{\partial z}+\mathrm{R}\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial z^2}=0 \tag{15}$$

where $\mathrm{R}=\dfrac{u\sqrt{ab}}{\nu}$ is the Reynolds number.

It is easy to see that equation (15) admits a self-similar solution. Indeed, if a new argument is entered

$$\xi=\frac{z}{x} \tag{16}$$

then equation (15) is brought to the following equation

$$\left(\xi^2+1\right)\varphi'''+4\xi\varphi''+2\varphi'-\mathrm{R}\,\varphi'\varphi''=0 \tag{17}$$

According to (10) we have as a result

$$\begin{cases} \mathrm{v}_x=u\left(1+\dfrac{\sqrt{ab}}{x}\varphi'\right), \\ \mathrm{v}_z=\dfrac{u\sqrt{ab}\xi}{x}\varphi'. \end{cases} \tag{18}$$

If a notation that reduces the order of the equation is entered

$$G=\varphi' \tag{19}$$

we will get

$$\left(\xi^2+1\right)G''+4\xi G'+2G-\mathrm{R}\,GG'=0 \tag{20}$$

Or

$$\left[\left(\xi^2+1\right)G\right]''-\mathrm{R}\,GG'=0.$$

The first integral is

$$\left[\left(\xi^2+1\right)G\right]'=\frac{\mathrm{R}}{2}G^2+C_1 \tag{21}$$

where C_1- is the constant of integration.

Assuming $C_1=0$ we come to Bernoulli's equation the solution of which can be represented as

$$G(\xi)=\frac{1}{C_2\left(\xi^2+1\right)-\frac{\mathrm{R}}{4}\left[\xi+\left(\xi^2+1\right)arctg\xi\right]} \quad (22)$$

where C_2- is another dimensionless constant of integration.

Thus the velocity distribution for arbitrary values of the Reynolds number according to (10) and (22) can be represented as follows

$$\begin{cases} \mathrm{v}_x=u\left(1+\frac{\sqrt{ab}}{x}G\right)= \\ =u\left\{1+\frac{\sqrt{ab}}{x}\frac{1}{C_2\left(\xi^2+1\right)-\frac{\mathrm{R}}{4}\left[\xi+\left(\xi^2+1\right)arctg\xi\right]}\right\}, \\ \mathrm{v}_z=\frac{u\sqrt{ab}\xi}{x}G= \\ =\frac{u\sqrt{ab}\xi}{x}\frac{1}{C_2\left(\xi^2+1\right)-\frac{\mathrm{R}}{4}\left[\xi+\left(\xi^2+1\right)arctg\xi\right]}, \\ \xi=\frac{z}{x}. \end{cases} \quad (23)$$

Taking into account solutions (23) functional (6) eventually becomes as follows

$$\begin{aligned} F_c=&-4\mu\int_{x_0}^{x_1}\left.\frac{\partial \mathrm{v}_x}{\partial y}\right|_{y=\pm\frac{h}{2}}\eta(x)dx+ \\ &+2\mu hu\sqrt{ab}\int_{x_0}^{x_1}\left[\frac{2\eta'(x)}{x^2}\left(G(\xi)+\frac{\eta(x)}{x}G'(\xi)\right)+\right. \\ &+\left(\frac{G'(\xi)}{x^2}-\frac{\eta(x)}{x^3}\left(2G(\xi)+\frac{\eta(x)}{x}G'(\xi)\right)\right)- \\ &\left.\left.-\left(\frac{1}{a}+\frac{G(\xi)}{x}\right)\eta''(x)\right]\right|_{z=\eta(x)} \end{aligned} \quad (24)$$

By introducing an abbreviated notation for the second integral F_{2c} we obtain the following for it as a result of simple transformations

$$F_{2c}=2\mu hu\sqrt{ab}\int_{x_0}^{x_1}Hdx, \quad (25)$$

where the sub-integral function is

$$\begin{aligned} H=&\frac{1}{x^2}\left[\left(2G+\frac{\eta}{x}G'\right)\eta'+G'\left(1-\frac{\eta^2}{x^2}\right)-\right. \\ &\left.\left.-\frac{2\eta}{x}G-\frac{x^2}{a}\left(1+\frac{a}{x}G\right)\eta''\right]\right|_{z=\eta(x)} \end{aligned} \quad (26)$$

And thus for functional (25) the Euler-Poisson equation (see [21]) has the form:

$$H_\eta-\frac{d}{dx}H_{\eta'}+\frac{d^2}{dx^2}H_{\eta''}=0 \quad (27)$$

leads us to the equation

$$\left.\left[\eta''-\frac{\eta'}{x}\left(2+\frac{\eta}{x}\frac{G''}{G'}\right)+\frac{3\eta}{x}-\frac{1}{x}\left(1-\frac{2\eta^2}{x^2}\right)\frac{G''}{G'}\right]\right|_{z=\eta}= \\ =0 \quad (28)$$

The solution of equation (28) allows us to find out the shape of the contour $\eta(x)$ we are interested in.

3 The Solution of Equation (28)

With the accordance decision (22) at $\mathrm{R}\ll 1$ we have

$$G(\xi)\approx\frac{1}{C_2\left(\xi^2+1\right)}$$

It means that equation (28) is somewhat simplified and brought to the following

$$\begin{aligned} &\eta''-\frac{\eta'}{x}\frac{3x^2-\eta^2}{x^2+\eta^2}+\frac{3\eta}{x^2}- \\ &-\frac{1}{\eta}\frac{x^2-3\eta^2}{x^2+\eta^2}\left(1-\frac{2\eta^2}{x^2}\right)=0 \end{aligned} \quad (29)$$

where it is taken into account that

$$\frac{G''}{G'}=\frac{x}{\eta}\left(\frac{x^2-3\eta^2}{x^2+\eta^2}\right)$$

Due to substitution

$$f=\frac{\eta}{x} \quad (30)$$

equation (29) is transformed into the following

$$f''x^2\left(1+f^2\right)+f'x\left(3f^2-1\right)+\frac{5f^2-2f^4-1}{f}=0\cdot \tag{31}$$

Its stationary solution ($f'=f''=0$) leads to four "fixed points":

$$\overline{f}^2=\frac{5\pm\sqrt{17}}{4}, \tag{32}$$

The numerical solution of equation (31) according to (30) is shown in Fig. 1-3.

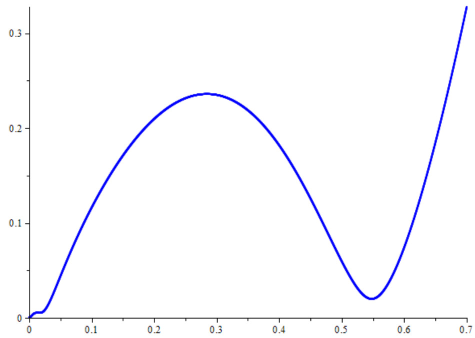

Fig. 1: The contour shape for the case $f_2(0,01)=\sqrt{\dfrac{5-\sqrt{17}}{4}}$

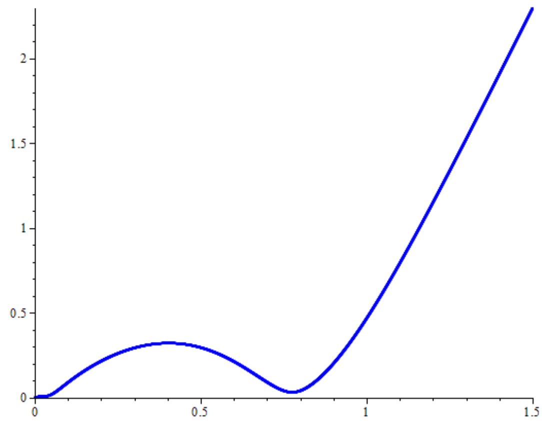

Fig. 2: The contour shape for the case $f_2(0,01)=\sqrt{\dfrac{5-\sqrt{17}}{4}}$ but on a larger scale

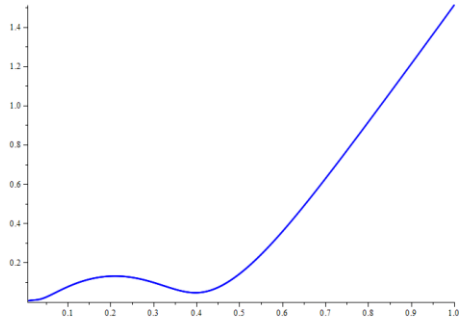

Fig. 3: The flow contour shape in the case when $f_1(0,01)=\sqrt{\dfrac{5+\sqrt{17}}{4}}$

As can be seen in these figures the shape of a fish is one of the possible well-streamlined bodies.

4 Arbitrary Reynolds numbers

In this case, solution (22) and equation (28) can be written as a system

$$\begin{cases} G(\xi)=\dfrac{1}{C_2\left(1+\xi^2\right)-\dfrac{\mathrm{R}}{4}\left[\xi+\left(1+\xi^2\right)arctg\xi\right]}, \\ \eta''-\dfrac{\eta'}{x}\left(2+\dfrac{\eta}{x}\dfrac{G''}{G'}\Big|_{\xi=\frac{\eta}{x}}\right)+\dfrac{3\eta}{x^2}-\dfrac{1}{x}\left(1-\dfrac{2\eta^2}{x^2}\right)\dfrac{G''}{G'}\Big|_{\xi=\frac{\eta}{x}}=0, \end{cases} \tag{33}$$

Where

$$G'=\frac{dG}{d\xi},\quad G''=\frac{d^2G}{d\xi^2}.$$

And new argument $\xi=\dfrac{\eta}{x}$. After differentiation of the function G we find the following expression

$$p(\mathrm{R},\xi)=\frac{G''}{G'}=\frac{q_1q_2'-2q_2^2}{q_1q_2}, \tag{34}$$

where the functions are

$$q_1=C_2\left(\xi^2+1\right)-\frac{\mathrm{R}}{4}\left[\xi+\left(\xi^2+1\right)arctg\xi\right],$$
$$q_2=q_1'=2C_2\xi-\frac{\mathrm{R}}{2}\left(1+\xi arctg\xi\right). \tag{35}$$

Therefore the lower equation in system (33) according to the replacement $\eta = \xi x$ is brought to the form

$$\xi'' x^2 - \xi\xi' x p(\mathrm{R}, \xi) + \xi - (1 - \xi^2) p(\mathrm{R}, \xi) = 0 \quad , \quad (36)$$

where $p(\mathrm{R}, \xi)$ is given by formulas (34), (35).

The asymptotic behaviour of the function $p(\mathrm{R}, \xi)$ is the following.

If it is $x \to 0$ then it is $\lim_{x \to 0} \frac{p}{x} = -3$ and if it is $x \to \infty$ then it is $\lim_{x \to \infty} \frac{p}{x} = \frac{b}{\eta}$ where

$$b = \frac{\mathrm{R}^2 - 4C_2^2}{C_2 \mathrm{R}} \quad (37)$$

The numerical solution of equation (36) is shown in Fig. 4-7. The boundary conditions were assumed as follows

$$G(0) = 1, \; G'(0) = 0 \quad (38)$$

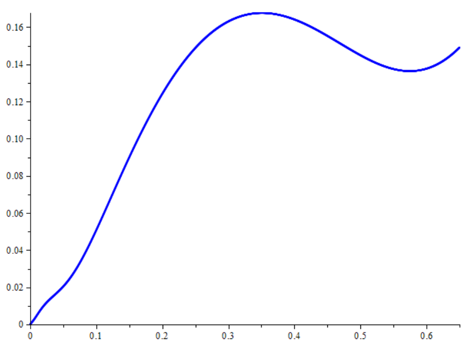

Fig. 4: The contour shape for arbitrary Reynolds numbers and in the case when $f_1(0,01) = \sqrt{\frac{5 + \sqrt{17}}{4}}$ (small scale)

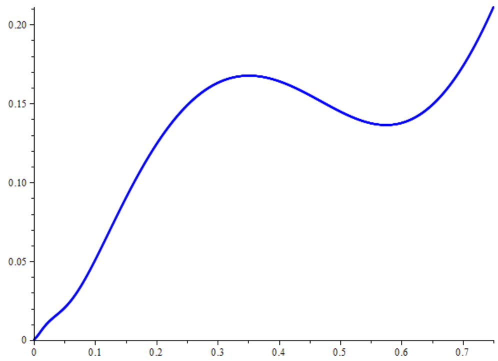

Fig. 5: The same as in Fig. 8 that is, with $f_1(0,01) = \sqrt{\frac{5 + \sqrt{17}}{4}}$ but in a more distant coordinates of the abscissa

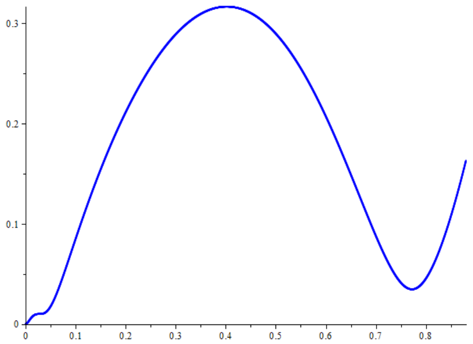

Fig. 6: The flow contour shape for the case $f_2(0,01) = \sqrt{\frac{5 - \sqrt{17}}{4}}$

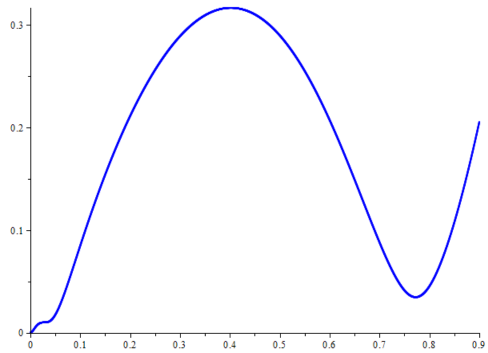

Fig. 7: The count shape for $f_2(0,01) = \sqrt{\frac{5 - \sqrt{17}}{4}}$ with the abscissa axis prolongation

5 Conclusion

1. A mathematical model is described that allows analytically determining the most optimal type of contour of a well-streamlined body;
2. A numerical solution of the problem for arbitrary Reynolds numbers is obtained.
3. In the next paper we will considering in more detail the influence of the Prandtl number.

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