Linear and Nonlinear Splitting Schemes Conserving Total Energy and Mass in the Shallow Water Model

YURI N. SKIBA
Instituto de Ciencias de la Atmósfera y Cambio Climático,
Universidad Nacional Autónoma de México ,
Av. Universidad #3000, CU/UNAM, Coyoacán, Mexico City, 04510,
MEXICO

Abstract: - Two linear and one nonlinear implicit unconditionally stable finite-difference schemes of the second-order approximation in all variables are given for a shallow-water model including the rotation and topography of the earth. The schemes are based on splitting the model equation into two one-dimensional subsystems. Each of the subsystems conserves the mass and total energy in both differential and discrete (in time and space) forms. One of the linear schemes contains a smoothing procedure not violating the conservation laws and suppressing spurious oscillations caused by the application of central-difference approximations of spatial derivatives. The unique solvability of the linear schemes and convergence of iterations used to find their solutions are proved.

Key-Words: - Shallow-water model, splitting method, linear and nonlinear numerical schemes, conservative laws, unique solvability of the linear schemes, convergence of the iterative process.

Received: December 11, 2022. Revised: October 18, 2023. Accepted: November 13, 2023. Published: December 14, 2023.

1 Introduction

The shallow-water equations describe a thin layer of fluid of constant density in hydrostatic balance, bounded from below by the bottom topography and from above by a free surface. They exhibit a rich variety of features, because they have various conservation laws. Shallow-water equations can be used to model Rossby and Kelvin waves in the atmosphere, rivers, lakes, and oceans as well as gravity waves in a smaller domain (e.g. surface waves in a bath), [1], [2], [3], [4], [5], [6].

The equations are derived from depth-integrating the Navier-Stokes equations, in the case where the horizontal length scale is much greater than the vertical length scale. Under this condition, conservation of mass implies that the vertical velocity scale of the fluid is small compared to the horizontal velocity scale. It can be shown from the momentum equation that vertical pressure gradients are nearly hydrostatic, and that horizontal pressure gradients are due to the displacement of the pressure surface, implying that the horizontal velocity field is constant throughout the depth of the fluid. Vertically integrating allows the vertical velocity to be removed from the equations. The shallow-water equations are thus derived.

The classic system of the shallow-water equations taking account of the Coriolis force and topography can be written as

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-g\frac{\partial h}{\partial x}\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-g\frac{\partial h}{\partial y}\\
&\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}\left(uH\right)+\frac{\partial}{\partial y}\left(vH\right)=0
\end{aligned} \tag{1}$$

In, [7], [8], [9], where $u(x,y,t)$ and $v(x,y,t)$ are the components of the velocity vector, $f=f_0+\beta y$ is the Coriolis parameter, $h(x,y,t)$ is the free surface height,

$$H(x,y,t)=h(x,y,t)-h_T(x,y) \tag{2}$$

and $h_T(x,y)$ is the topography height (Figure 1).

System (1) is considered with initial conditions

$$\begin{aligned}
&u(x,y,0)=u^0(x,y),\ v(x,y,0)=v^0(x,y),\\
&H(x,y,0)=H^0(x,y)
\end{aligned} \tag{3}$$

in a channel $D=\{(x,y):0\le x\le X;0\le y\le Y\}$ with periodic conditions in x:

$$u(X,y,t)=u(0,y,t),\quad v(X,y,t)=v(0,y,t)$$
$$H(X,y,t)=H(0,y,t) \tag{4}$$

In addition, at the boundaries the *y*-component of the velocity vector must vanish:

$$v(x,0,t)=v(x,Y,t)=0 \tag{5}$$

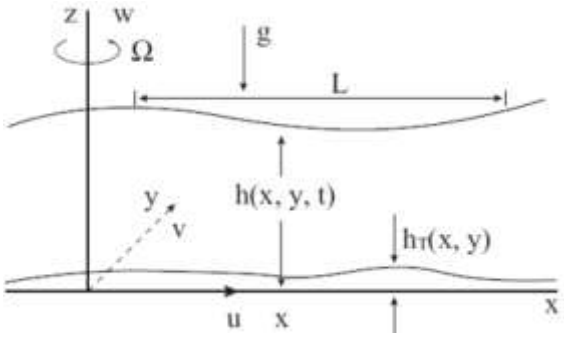

Fig. 1: Shallow-water model.

Numerical simulation of the problem of atmospheric dynamics, [9], [10], [11], [12], [13]. [14], shows that the best difference schemes are those that retain the most important properties of the original differential model. Full conservatism is one of such properties, [15]. As is known, [8], system (1)-(5) has several conservation laws, in particular, it preserves the mass $M=\int_D h\,dD$ and the total (kinetic plus potential) energy $E(t)=K(t)+P(t)$:

$$\frac{d}{dt}M(t)=0 \tag{6}$$
$$\frac{d}{dt}E(t)=0 \tag{7}$$

where

$$K=\int_D\left\{\frac{1}{2}\left(u^2+v^2\right)H\right\}dD,\quad P=\int_D\left\{\frac{1}{2}g\left(h^2-h_T^2\right)\right\}dD$$

are the kinetic and potential energies of the system, respectively, and $dD=dxdy$. Note that

$$\frac{d}{dt}\int_D g\left(h^2-h_T^2\right)dD=\frac{d}{dt}\int_D g\,h^2\,dD$$

[10], proposed a spatial discretization of the shallow-water equations that precisely conserves the energy and potential enstrophy in semi-discrete (discrete in space) system. Their algorithm is important because the simultaneous conservation of energy and potential enstrophy is known to prevent the spurious cascade of energy to high wavenumbers.

However, the numerical schemes proposed in, [9], [10], [11], [16], conserve the energy and enstrophy only when the time derivatives are taken in the differential form. The discretization of these derivatives leads to the violation of these laws in a completely discrete system. In this paper, three implicit schemes are proposed that ensure the conservation of mass and energy in a completely discrete (in space and time) system (1)-(5). One of the schemes is nonlinear, and the other two are linear. Since the integral in (7) can be considered as the square of the norm of the solution, the constructed schemes are unconditionally stable. The linear schemes have a second-order approximation in all spatial and temporal variables.

It is well known that implementing multi-dimensional implicit schemes requires a large amount of computation. To simplify the calculations at each small-time interval, a geometric division of the original two-dimensional problem into two one-dimensional ones was used. Every split differential system satisfies laws (6) and (7). The corresponding discrete one-dimensional systems and therefore the entire finite-difference scheme also conserve these two laws in discrete form.

2 Modified Shallow-Water Equation

Assuming that in the domain *D* the height of the atmospheric layer $H(x,y,t)$ is always positive, we introduce new dependent variables, [14].

$$Z=\sqrt{H}\ ,\quad U=Zu\ ,\quad V=Zv \tag{8}$$

Then according to the formulas

$$\frac{\partial U}{\partial t}=Z\frac{\partial u}{\partial t}+u\frac{\partial Z}{\partial t}\quad\text{and}\quad\frac{\partial V}{\partial t}=Z\frac{\partial v}{\partial t}+v\frac{\partial Z}{\partial t}$$

system (1) can be rewritten as

$$\frac{\partial U}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial x}(uU)+u\frac{\partial U}{\partial x}\right]+\frac{1}{2}\left[\frac{\partial}{\partial y}(vU)+v\frac{\partial U}{\partial y}\right]$$
$$-fV=-gZ\frac{\partial h}{\partial x}\ ,$$

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial x}(uV)+u\frac{\partial V}{\partial x}\right]+\frac{1}{2}\left[\frac{\partial}{\partial y}(vV)+v\frac{\partial V}{\partial y}\right]$$

$$+fU=-gZ\frac{\partial h}{\partial y}\ ,$$

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}(ZU)+\frac{\partial}{\partial y}(ZV)=0 \tag{9}$$

Multiplying the equations of system (9) by U, V, and gh, respectively, integrating the obtaining equations over D, and summing the results, we arrive at the energy conservation law (7)

$$\frac{d}{dt}\int_D\left[\left(\frac{U^2+V^2}{2}\right)H+\frac{g\,h^2}{2}\right]dD=0 \tag{10}$$

Law (6) is also valid for system (9). The main advantage of system (9) over system (1) is "the divergent form" of its equations. Indeed, for example, the term

$$\frac{\partial}{\partial x}(uU)+u\frac{\partial U}{\partial x} \tag{11}$$

when multiplied by U, takes the divergent form, which after integration over x and using periodic conditions (4) gives

$$\int_D\frac{\partial}{\partial x}(uU^2)\,dD=0 \tag{12}$$

The same is true for the remaining terms of the system (9) enclosed in brackets. In addition, the sum of the terms $gZ(\partial h/\partial x)$ and $\partial(ZU)/\partial x$ previously multiplied by U and gh, respectively, also leads to the divergent form $\partial(gZUh)/\partial x$. To explain the usefulness of transforming system (1) to the divergent form (9), consider the central-difference approximation

$$\frac{u_{i+1}U_{i+1}-u_{i-1}U_{i-1}}{2\Delta x}+u_i\frac{U_{i+1}-U_{i-1}}{2\Delta x} \tag{13}$$

of the divergent form (11), where the fixed index j of the grid functions $u_{ij}=u(x_i,y_j)$ and $U_{ij}=U(x_i,y_j)$ is omitted for simplicity. Then it is easy to see that multiplying (13) by U_i with subsequent summation over all internal nodes of the grid from $i=1$ to $i=I$ leads to

$$\frac{1}{2\Delta x}\sum_{i=1}^{I}\left\{(u_{i+1}U_iU_{i+1}-u_iU_{i-1}U_i)\right\}$$
$$-\frac{1}{2\Delta x}\sum_{i=1}^{I}\left\{(u_{i-1}U_{i-1}U_i-u_iU_iU_{i+1})\right\}=0$$

due to periodic conditions

$$u_{I+1}=u_1\ ,\ u_0=u_I\ ,\ U_{I+1}=U_1\ ,\ U_0=U_I \tag{14}$$

approximating (4). Thus, the finite-difference form (13) preserves the property (12) of the divergent form (11). This property is of great importance for constructing a scheme that conserves the energy of the system.

3 Splitting of the Model Operator

As mentioned above, our goal is to construct a numerical scheme that preserves laws (6) and (7) in the discrete form (both in time and space) and has a second-order approximation in all independent variables. For this reason, only implicit schemes should be used. However, it is known that in the case of a multi-dimensional problem the use of implicit schemes leads to numerical algorithms, the implementation of which is too expensive. Therefore, to construct an accessible numerical algorithm, we use a geometric splitting of the original two-dimensional system into two one-dimensional subsystems that admit a simple solution. To illustrate the main idea of the splitting method, [17], [18], [19], [20], consider the linear homogeneous problem:

$$\frac{\partial\Phi}{\partial t}+A_1\Phi+A_2\Phi=0\,,\quad \Phi(t_1)=g \tag{15}$$

in the domain D and on a sufficiently small-time interval (t_1,t_2). Here $\Phi(t,x,y)$ is the solution with the initial value of $g(x,y)$, and the operators A_1 and A_2 are assumed to be positive semidefinite:

$$\int_D\Phi A_i\Phi\,dD\geq 0\quad (i=1,2) \tag{16}$$

According to the splitting method, the first split subsystem

$$\frac{\partial \Phi_1}{\partial t} + A_1 \Phi_1 = 0, \quad \Phi_1(t_1) = g \tag{17}$$

is solved within (t_1, t_2), and its final solution $\Phi_1(t_2)$ is used as the initial condition for solving the second subsystem

$$\frac{\partial \Phi_2}{\partial t} + A_2 \Phi_2 = 0, \quad \Phi_2(t_1) = \Phi_1(t_2) \tag{18}$$

on the same interval (t_1, t_2). Then the final solution $\Phi_2(t_2)$ of subsystem (18) will approximate the solution $\Phi(t_2)$ of the original problem (15): $\Phi_2(t_2) \cong \Phi(t_2)$. In particular, if A_1 and A_2 are the one-dimensional operators (in the *x*- and *y*-directions, respectively), then the two-dimensional problem (15) reduces to the sequential solution of one-dimensional problems (17) and (18), which greatly simplifies the calculations. The smaller the interval (t_1, t_2), the closer $\Phi_2(t_2)$ to $\Phi(t_2)$. Thus, by choosing a sufficiently small interval (t_1, t_2), the error of the splitting procedure can be made arbitrarily small. The same approach can also be applied to nonlinear problems.

Let us split system (9) on a small-time interval (t_1, t_2) in accordance with algorithms (17), (18). The subsystem

$$\frac{\partial U}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial x}(uU) + u\frac{\partial U}{\partial x}\right] - \frac{1}{2} fV = -gZ\frac{\partial h}{\partial x} \tag{19}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial x}(uV) + u\frac{\partial V}{\partial x}\right] + \frac{1}{2} fU = 0 \tag{20}$$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(ZU) = 0 \tag{21}$$

is first solved with the initial conditions $U(x, y, t_1)$, $V(x, y, t_1)$ and $H(x, y, t_1)$. Then its solution $U(x, y, t_2)$, $V(x, y, t_2)$ and $H(x, y, t_2)$ at $t = t_2$ is used as the initial condition at $t = t_1$ to solve the subsystem

$$\frac{\partial U}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial y}(vU) + v\frac{\partial U}{\partial y}\right] - \frac{1}{2} fV = 0 \tag{22}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial y}(vV) + v\frac{\partial V}{\partial y}\right] + \frac{1}{2} fU = -gZ\frac{\partial h}{\partial y} \tag{23}$$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial y}(ZV) = 0 \tag{24}$$

on the same time interval (t_1, t_2). The solution of this subsystem obtained at time $t = t_2$, approximates the solution of the original non-split system (9). Obviously, the mass conservation law (6) is satisfied for each of the split systems (19)-(20) and (22)-(24). Now let's multiply the equations (19) and (22) by *U*, (20) and (23) by *V*, and (21) and (24) by *gh*, sum the results, and integrate the final equation over D. Using the divergent form of terms and boundary conditions (4) and (5), it is easy to show that the conservation law (7) is also valid for each of split systems (19)-(21) and (22)-(24).

4 Linear and Nonlinear Implicit Schemes Conserving the Mass and Total Energy

Due to a certain symmetry in the structure of subsystems (19)-(21) and (22)-(24), the numerical scheme constructed for the first subsystem can be easily modified and applied to the second. Let us introduce net functions at the grid nodes (x_i, y_j, t_n), where (x_i, y_j) is the internal point of the domain *D* ($1 \le i \le I$, $1 \le j \le J$), t_n is a discrete moment in time ($n \ge 1$), and denote

$$\tau = t_{n+1} - t_n, \quad \Delta x = x_{i+1} - x_i, \quad \Delta y = y_{j+1} - y_j,$$
$$f_j = f(y_j), \quad R_{ij}^n = R(x_i, y_j, t_n)$$

where R can be one of the functions u, v, h, Z, U, V or H, and τ, Δx and Δy are small steps of regular grids. Moreover, we will use the notation

$$R_{ij} = \frac{1}{2}(R_{ij}^{n+1} + R_{ij}^n) \tag{25}$$

At each interval (t_n, t_{n+1}) and for each fixed y_j, system (19)-(21) is solved using the implicit scheme

$$\frac{U_i^{n+1} - U_i^n}{\tau} + \frac{1}{2}\left[\frac{\bar{u}_{i+1}U_{i+1} - \bar{u}_{i-1}U_{i-1}}{2\Delta x} + \bar{u}_i\frac{U_{i+1} - U_{i-1}}{2\Delta x}\right] - \frac{1}{2} f_j V_i = -g\bar{Z}_i\frac{h_{i+1} - h_{i-1}}{2\Delta x} \tag{26}$$

$$\frac{V_i^{n+1}-V_i^n}{\tau}+\frac{1}{2}\left[\frac{\bar{u}_{i+1}V_{i+1}-\bar{u}_{i-1}V_{i-1}}{2\Delta x}+\bar{u}_i\frac{V_{i+1}-V_{i-1}}{2\Delta x}\right]+\frac{1}{2}f_jU_i=0 \quad (27)$$

$$\frac{H_i^{n+1}-H_i^n}{\tau}+\frac{\bar{Z}_{i+1}U_{i+1}-\bar{Z}_{i-1}U_{i-1}}{2\Delta x}=0 \quad (28)$$

with periodic boundary conditions. The functions *U*, *V*, and *h* are defined here by formula (25), and the fixed index *j* is omitted for simplicity of notation. Further, for each fixed x_i, the net functions U_{ij}^{n+1}, V_{ij}^{n+1} and H_{ij}^{n+1}, obtained from (26)-(28), are used as initial conditions U_{ij}^n, V_{ij}^n and H_{ij}^n to solve system (22)-(24) on the same time interval (t_n,t_{n+1}) according to the implicit scheme

$$\frac{U_j^{n+1}-U_j^n}{\tau}+\frac{1}{2}\left[\frac{\bar{v}_{j+1}U_{j+1}-\bar{v}_{j-1}U_{j-1}}{2\Delta y}+\bar{v}_j\frac{U_{j+1}-U_{j-1}}{2\Delta y}\right]-\frac{1}{2}f_jV_j=0 \quad (29)$$

$$\frac{V_j^{n+1}-V_j^n}{\tau}+\frac{1}{2}\left[\frac{\bar{v}_{j+1}V_{j+1}-\bar{v}_{j-1}V_{j-1}}{2\Delta y}+\bar{v}_j\frac{V_{j+1}-V_{j-1}}{2\Delta y}\right]+\frac{1}{2}f_jU_j=-g\bar{Z}_j\frac{h_{j+1}-h_{j-1}}{2\Delta y} \quad (30)$$

$$\frac{H_j^{n+1}-H_j^n}{\tau}+\frac{\bar{Z}_{j+1}U_{j+1}-\bar{Z}_{j-1}U_{j-1}}{2\Delta y}=0 \quad (31)$$

(the fixed index *i* is also omitted here) with the boundary conditions

$$\frac{1}{2}(V_0+V_1)=0,\quad \frac{1}{2}(V_J+V_{J+1})=0$$
$$\frac{1}{2}(\bar{v}_0+\bar{v}_1)=0,\quad \frac{1}{2}(\bar{v}_J+\bar{v}_{J+1})=0$$
$$\bar{Z}_0=\bar{Z}_1,\quad h_0=h_1,\quad \bar{Z}_{J+1}=\bar{Z}_J,\quad h_{J+1}=h_J$$

Then this procedure is repeated on the interval (t_{n+1},t_{n+2}), while the functions U_{ij}^{n+1}, V_{ij}^{n+1} and H_{ij}^{n+1} found on the previous interval are taken as initial conditions when solving system (26)-(28).

Note that, depending on the choice of the functions $\bar{u}_{ij}$, $\bar{v}_{ij}$ and $\bar{Z}_{ij}$, at least three different schemes can be constructed:

1) If

$$\bar{u}_{ij}=u_{ij},\quad \bar{v}_{ij}=v_{ij},\quad \bar{Z}_{ij}=Z_{ij} \quad (32)$$

where u_{ij}, v_{ij} and Z_{ij} are defined by (25), both schemes (26)-(28) and (29)-(31) are nonlinear.

2) If

$$\bar{u}_{ij}=u_{ij}^n,\quad \bar{v}_{ij}=v_{ij}^n,\quad \bar{Z}_{ij}=Z_{ij}^n \quad (33)$$

where u_{ij}^n, v_{ij}^n, and Z_{ij}^n are the solutions obtained on the previous time interval (t_{n-1},t_n), schemes (26)-(28) and (29)-(31) are linear.

3) If

$$\bar{u}_{ij}=\sum_{k,m}\alpha_{km}^n(ij)u_{km}^n,\quad \bar{v}_{ij}=\sum_{k,m}\beta_{km}^n(ij)v_{km}^n,\quad \bar{Z}_{ij}=\sum_{k,m}\gamma_{km}^n(ij)Z_{km}^n \quad (34)$$

where

$$\sum_{k,m}\alpha_{km}^n(ij)=1,\quad \sum_{k,m}\beta_{km}^n(ij)=1,\quad \sum_{k,m}\gamma_{km}^n(ij)=1$$

the schemes (26)-(28) and (29)-(31) are also linear. Here u_{ij}^n, v_{ij}^n, and Z_{ij}^n are the same as in (33), and $\alpha_{km}^n(ij)$, $\beta_{km}^n(ij)$ and $\gamma_{km}^n(ij)$ are certain coefficients of the interpolation used to smooth the functions $\bar{u}_{ij}$, $\bar{v}_{ij}$ and $\bar{Z}_{ij}$. The latter procedure can be effective for suppressing spurious oscillations caused by the use of central difference approximations.

Schemes (26)-(28) and (29)-(31) conserve mass and total energy regardless of the choice of (32), (33), or (34) for the functions $\bar{u}_{ij}$, $\bar{v}_{ij}$ and $\bar{Z}_{ij}$. Indeed, if we multiply (28) by $\tau\Delta x\Delta y$ and sum over all internal grid nodes (x_i,y_j) $(1\le i\le I,\ 1\le j\le J)$. Then:

$$\Delta x\Delta y\sum_{i,j}H_{ij}^{n+1}=\Delta x\Delta y\sum_{i,j}H_{ij}^n \quad (35)$$

since

$$\sum_j[\bar{Z}_{I+1,j}U_{I+1,j}-\bar{Z}_{1,j}U_{1,j}]=0$$

due to the periodic conditions $\bar{Z}_{I+1,j}=\bar{Z}_{1,j}$ and $U_{I+1,j}=U_{1,j}$ (see (14)). In complete analogy with (28), equation (31) leads to (35) due to the conditions $V_{i,0}=V_{i,J+1}=0$ (see (5)). Next, if we multiply (26), (27) and (28) by $\tau\Delta x\Delta y U_i$, $\tau\Delta x\Delta y V_i$ and $\tau\Delta x\Delta y\, g h_i$, respectively, and sum the resulting equations over all internal grid nodes (x_i,y_j) of the domain D then the use of (2), (25) and the boundary conditions leads to the conservation of the total energy $E^n=E(t_n)$ in the finite-difference system

$$E^{n+1}\equiv\Delta x\Delta y\sum_{i,j}\frac{1}{2}\left\{\left[U_{ij}^{n+1}\right]^2+\left[V_{ij}^{n+1}\right]^2+g\left[h_{ij}^{n+1}\right]^2\right\}$$
$$=\Delta x\Delta y\sum_{i,j}\frac{1}{2}\left\{\left[U_{ij}^{n}\right]^2+\left[V_{ij}^{n}\right]^2+g\left[h_{ij}^{n}\right]^2\right\}\equiv E^n \tag{36}$$

Relation (36) is also fulfilled for system (29)-(31). Defining the solution norm as

$$\left\|\Phi^n\right\|=\left(E^n\right)^{1/2}$$

we get

$$\left\|\Phi^{n+1}\right\|=\left\|\Phi^n\right\|$$

This means that the difference scheme (26)-(31) is unconditionally stable.

Note that the linear scheme (26)-(31) has the second order of accuracy in Δx and Δy, but the first order of accuracy in τ, since the matrices approximating the dynamic operators A_1 and A_2 of the splitting algorithm (17), (18) do not commute:

$$A_1A_2\neq A_2A_1$$

However, a second-order approximation in τ can be achieved using the symmetrized algorithm, [18], when the original 2D problem is solved on each double interval (t_{n-1},t_{n+1}). Moreover, first the scheme (26)-(28) and then the scheme (29)-(31) are solved on (t_{n-1},t_n). After this, on the interval (t_n,t_{n+1}) the order changes and first the scheme (29)-(31) is solved, and then the scheme (26)-(28). Thus, within the framework of the symmetric algorithm, scheme (29)-(31) can be immediately resolved on the double interval (t_{n-1},t_{n+1}). Note that when using the symmetric algorithm on the double time interval (t_{n-1},t_{n+1}), conditions (33) must be replaced by solutions obtained on the previous double time interval (t_{n-3},t_{n-1}):

$$\bar{u}_{ij}=u_{ij}^{n-1},\quad \bar{v}_{ij}=v_{ij}^{n-1},\quad \bar{Z}_{ij}=Z_{ij}^{n-1}$$

5 Unique Solvability of the Linear Schemes

Schemes (26)-(28) and (29)-(31) are solved using an iterative method. Let us now show that in cases (33) and (34) when these schemes are linear, their solutions are unique. Indeed, each of these schemes can be written as a vector equation

$$\vec{\Phi}^{n+1}-\vec{\Phi}^n+\tau\Lambda\vec{\Phi}=0 \tag{37}$$

where $\vec{\Phi}=\frac{1}{2}(\vec{\Phi}^{n+1}+\vec{\Phi}^n)$, [21], the components of vector $\vec{\Phi}^n$ are values of net functions U_{ij}^n, V_{ij}^n and $h_{ij}^n\sqrt{g}$, and the time-dependent matrix $\Lambda=\Lambda(t_n)$ is skew-symmetric due to the energy conservation law (36):

$$\vec{\Phi}^T\Lambda\vec{\Phi}=0 \tag{38}$$

Here the superscript "T" means transposition. Therefore, there is a complete orthogonal system of eigenvectors $\vec{\varphi}_k$ of matrix Λ:

$$\Lambda\vec{\varphi}_k=\lambda_k\vec{\varphi}_k,\quad \vec{\varphi}_k^T\vec{\varphi}_m=\delta_{km} \tag{39}$$

where δ_{km} is the Kronecker symbol and the eigenvalue λ_k is pure imaginary for each k. Let's rewrite (37) as

$$B\vec{w}=\vec{F} \tag{40}$$

where $\vec{w}=\vec{\Phi}^{n+1}$ is the solution to be found, $\vec{F}$ is a known vector depending on t_n,

$$B=(E+\frac{1}{2}\tau\Lambda)$$

and E is the unit matrix. By virtue of (38), the matrix B is positive definite: $\vec{\Phi}^T B\vec{\Phi}>0$. Moreover, it has the same eigenvectors $\vec{\varphi}_k$ as the matrix Λ with eigenvalues $\mu_k=1+\frac{1}{2}\tau\lambda_k$:

$B\vec{\varphi}_k = \mu_k \vec{\varphi}_k$. Since every λ_k is purely imaginary,

$$|\mu_k| = \left(1 + \tfrac{1}{4}\tau^2 |\lambda_k|^2\right)^{1/2} > 1,$$

the matrix B is nonsingular, and problem (40) (and therefore the linear scheme (26)-(31)) has a unique solution $\vec{w}$ at each time step (t_n, t_{n+1}).

6 Convergence of the Iterative Process

We now derive the condition under which the iterative process:

$$\vec{w}_{p+1} = \vec{w}_p - \omega(B\vec{w}_p - \vec{F}) \qquad (41)$$

used to solve (40) converges. Here ω is the relaxation parameter. The error vector $\vec{\psi}_p = \vec{w}_p - \vec{w}$ satisfies the equation:

$$\vec{\psi}_{p+1} = T_\omega \vec{\psi}_p \qquad (42)$$

where $T_\omega = E - \omega B$ is the transition matrix from $\vec{\psi}_p$ to $\vec{\psi}_{p+1}$. Using Fourier series:

$$\vec{\psi}_p = \sum_m \psi_m(p)\vec{\varphi}_m \ , \quad \psi_m(p) = \vec{\varphi}_m^T \vec{\psi}_p \ ,$$

eigenvectors (39) and equation (42) we obtain:

$$\psi_m(p+1) = (1 - \omega\mu_m)\psi_m(p)$$
$$= [(1-\omega) - \frac{\omega}{2}\tau\lambda_m]\psi_m(p)$$

Therefore, the Fourier coefficients $\psi_m(p)$ tend to zero as $p \to \infty$ if and only if:

$$\max_m \left|(1-\omega) - \frac{\omega}{2}\tau\lambda_m\right| < 1$$

or

$$\omega < \frac{2}{1 + \frac{1}{4}\tau^2\nu^2} \qquad (43)$$

where $\nu = \max_m |\lambda_m|$. Thus, the iterative method (41) is convergent under condition (43).

7 Conclusion

One nonlinear and two linear implicit unconditionally stable finite-difference schemes of the second-order approximation in all variables are given for a shallow-water model that includes the rotation and topography of the earth. The schemes are developed using the splitting of the model equation into two one-dimensional subsystems. Each subsystem conserves the mass and total energy in differential and discrete (in time and space) forms. One of the linear schemes contains a smoothing procedure not violating the conservation laws and suppressing spurious oscillations caused by the application of central-difference approximations of spatial derivatives. Unique solvability of the linear schemes is shown and convergence of iterations used to find their solutions is proved.

Acknowledgement:
This work was partially supported by the National System of Researchers of Mexico (SNI, CONACYT) through the grant 14539.

References:

[1] Anderson D., M. Harris, H. Hartle, D. Nicolsky, E. Pelinovsky, A. Raz and A. Rybkin, Run-up of long waves in piecewise sloping U-shaped bays, *Pure and Applied Geophysics*, Vol. 174, No. 8, 2017, pp. 3185-3207.

[2] Battjes, J.A. and R.J. Labeur, *Unsteady Flow in Open Channels*, Cambridge University Press, 2017.

[3] Carrier G. F. and H. Yeh, Tsunami propagation from a finite source, *Computer Modeling in Engineering & Sciences*, Vol. 10, No. 2, 2005, pp. 113–122.

[4] Didenkulova I. and E. Pelinovsky, Rogue waves in nonlinear hyperbolic systems (shallow-water framework), *Nonlinearity*, Vol. 24, No. 3, 2011, pp. R1–R18.

[5] Lindborg E. and A.V. Mohanan, A two-dimensional toy model for geophysical turbulence, *Physics of Fluids*, Vol. 29, No. 11, 2017, ID 111114.

[6] Skiba Yu.N. and D.M. Filatov, Numerical simulation of coastal flows in open multiply-connected irregular domains. In: *Simulation and Modeling Methodologies, Technologies and Applications*. *Advances in Intelligent Systems and Computing*, Springer, Vol. 256, pp. 71-84, 2014.

[7] Batchelor G.K., *An Introduction to Fluid Dynamics*, Cambridge University Press, 1970.
[8] Pedlosky J., *Geophysical Fluid Dynamics*, Springer-Verlag, 1979.
[9] Sadourny R., The dynamics of finite-difference models of the shallow-water equations, *Journal of Atmospheric Sciences*, Vol. 32, No. 4, 1975, pp. 680-689.
[10] Arakawa A. and V.R. Lamb, A potential enstrophy and energy conserving scheme for the shallow-water equations, *Monthly Weather Review*, Vol.109, No. 1, 1981, pp. 18-36.
[11] Salmon R., Poisson-bracket approach to the construction of energy- and potential enstrophy-conserving algorithms for the shallow-water equations, *Journal of Atmospheric Sciences*, Vol. 61, 2004, pp. 2016-2036.
[12] Skiba, Yu.N., *Mathematical Problems of the Dynamics of Incompressible Fluid on a Rotating Sphere*. Springer International Publishing AG, Switzerland, 2017.
[13] Vreugdenhil C.B., *Numerical Methods for Shallow-Water Flow*, Kluwer, 1994.
[14] Weiyan Tan, *Shallow-Water Hydrodynamics*, Elsevier, 1992.
[15] Popov Yu.P. and A.A. Samarskii, Complete conservative difference schemes, *USSR Computational Mathematics and Mathematical Physics*, Vol. 9, No. 4, 1969, pp. 296-305.
[16] Ringler T.D. and D.A. Randall, A potential enstrophy and energy conserving numerical scheme for solution of the shallow-water equations on a geodesic grid, *Monthly Weather Review*, Vol. 130, 2002, pp. 1397-1410.
[17] Douglas J. and H.H. Rachford, On the numerical solution of heat conduction problems in two and three space variables, *Transactions of the American Mathematical Society*, Vol. 82, No. 2, 1956, pp. 421-439.
[18] Marchuk G.I., *Methods of Numerical Mathematics*, Springer-Verlag, 1982.
[19] Skiba Yu.N. and D. Filatov, Conservative schemes based on the separation method, for the numerical simulation of vortices in the atmosphere, *Interciencia*, Vol. 3, No. 1, 2006, pp. 16-21.
[20] Yanenko N. N., *Method of Fractional Steps*, Springer-Verlag, 1971.
[21] Crank J. and P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type, *Proceedings of the Cambridge Philosophical Society*, Vol. 43, 1947, pp. 50-67.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work was partially supported by the National System of Researchers of Mexico (SNI, CONACYT) through the grant 14539.

Conflict of Interest

The author has no conflicts of interest to declare.