Magneto-Rotatory Convection in Couple-Stress Fluid

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Abstract: - **Background**: Thermal convection is the most convective instability when crystals are produced from a single element like silicon and the thermal instability of a fluid layer heated from below plays an important role in geophysics, oceanography, atmospheric physics, etc. The flow through porous media is of considerable interest for petroleum engineers, for geophysical fluid dynamicists and has importance in chemical technology and industry. Many of the flow problems in fluids with couple-stresses indicate some possible experiments, that could be used for determining the material constants, and the results are found to differ from those of Newtonian fluid. Keeping this in view, the present work was to study the effect of a uniform vertical magnetic field on the couple-stress fluid heated from below in the presence of a uniform vertical rotation through permeable media. **Methodology**: The present problem is studied using the linearized stability theory, Boussinesq approximation, normal mode analysis, and the dispersion relation is obtained. **Results**: The stationary convection, stability of the system, and oscillatory modes are discussed. In the case of stationary convection, the rotation postpones the onset of convection. The magnetic field and couple-stress may hasten the onset of convection in the presence of rotation while in the absence of rotation; they always postpone the onset of convection. The medium permeability hastens the onset of convection in the absence of rotation while in the presence of rotation, it may postpone the onset of convection. The rotation and magnetic field are found to introduce oscillatory modes in the system which was non-existent in their absence. A sufficient condition for the non-existence of overstability is also obtained.

Key-Words: - Couple-stress fluid, Porous medium, Thermal convection, Uniform vertical magnetic field, Uniform vertical rotation

Received: October 22, 2022. Revised: August 11, 2023. Accepted: September 21, 2023. Published: October 6, 2023.

1 Introduction

A comprehensive account of thermal convection (Be'nard convection) in a fluid layer, in the absence and presence of rotation and magnetic field has been summarized in the celebrated monograph by, [1]. The use of the Boussinesq approximation has been made throughout, which states that the variations of density in the equations of motion can safely be ignored everywhere except in its association with the external force. The approximation is well justified in the case of incompressible fluids. The study, [2], has studied the influence of Rayleigh-number in the turbulent and laminar region in parallel-plate vertical channels. The influence of radiation on the unsteady free convection flow of a viscous incompressible fluid past a moving vertical plate with Newtonian heating has been investigated theoretically by, [3]. The study, [4], has considered the unsteady free convection flow near the stagnation point of a three-dimensional body.

The flow through porous media is of considerable interest for petroleum engineers, for geophysical fluid dynamicists and has importance in chemical technology and industry. An example in the geophysical context is the recovery of crude oil from the pores of reservoir rocks. The derivation of the basic equations of a layer of fluid heated from below in a porous medium, using Boussinesq approximation, has been given by, [5]. The study of a layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering disciplines. Among the applications in engineering disciplines, one can find the food process industry, chemical process industry, solidification, and centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in porous medium. The study, [6], has studied the stability of convective flow in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by, [7]. When the fluid slowly percolates through the pores of the rock, the gross effect is represented by the well-known Darcy's law. An extensive and updated account of convection in porous media has been given by, [8]. The forced convection in the fluid-saturated porous medium channel has been studied by, [9]. The studies, [10], [11], have remarked that the length scales characteristic of double-diffusive convecting layers in the ocean may be sufficiently large so that the Earth's rotation might be important in their formation. Moreover, the rotation of the Earth distorts the boundaries of a hexagonal convection cell in a fluid through a porous medium and the distortion plays an important role in the extraction of energy in the geothermal regions. The study, [12], explained a double-diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose. The study, [13], found that this instability, which is deleterious to certain biochemical separations, can be suppressed by rotation in the ultracentrifuge. The effect of a magnetic field on the stability of flow is of interest in geophysics, particularly in the study of Earth's core where the Earth's mantle, which consists of conducting fluid, behaves like a porous medium that can become convectively unstable as a result of differential diffusion. The other application of the results of flow through a porous medium in the presence of a magnetic field is in the study of the stability of a convective flow in the geothermal region. The fluid has been considered to be Newtonian in all the above studies.

The theory of couple-stress fluid has been formulated by, [14]. One of the applications of couple-stress fluid is its use in the study of the mechanisms of lubrication of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing that has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee, and hip joints are the loaded–bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear. Normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. According to the theory of, [14], couple-stresses appear in noticeable magnitudes in fluids with very large molecules.

Many of the flow problems in fluids with couple-stresses, discussed by Stokes, indicate some possible experiments, that could be used for determining the material constants, and the results are found to differ from those of Newtonian fluid. Couple-stresses are found to appear in noticeable magnitudes in polymer solutions for force and couple-stresses. This theory is developed in an effort to examine the simplest generalization of the classical theory, which would allow polar effects. The constitutive equations proposed by, [14] are:

$$T_{(ij)} = (-p + \lambda D_{kk})\delta_{ij} + 2\mu D_{ij} ,$$
$$T_{[ij]} = -2\eta \vec{W}_{ij,kk} - \frac{\rho}{2}\vec{\varepsilon}_{ijs}G_s ,$$

and

$$M_{ij} = 4\eta\vec{\omega}_{j,i} + 4\eta'\vec{\omega}_{i,j} ,$$

where

$$D_{ij} = \frac{1}{2}\left(V_{i,j} + V_{j,i}\right), \vec{W}_{ij} = -\frac{1}{2}\left(V_{i,j} - V_{j,i}\right)$$

and $\vec{\omega}_i = \frac{1}{2}\vec{\varepsilon}_{ijk}V_{k,j}$.

Here $T_{ij}, T_{(ij)}, T_{[ij]}, M_{ij}, D_{ij}$, $\vec{W}_{i,j}, \vec{\omega}_i, G_s, \vec{\varepsilon}_{ijk}$, V, ρ and $\lambda, \mu, \eta, \eta'$, are stress tensor, symmetric part of T_{ij}, anti-symmetric part of T_{ij}, the couple-stress tensor, the deformation tensor, the vorticity tensor, the vorticity vector, the body couple, the alternating unit tensor, the velocity field, and the density and material constants respectively. The dimensions of λ and μ are those of viscosity whereas the dimensions of η and η' are those of momentum.

Since the long-chain hyaluronic acid molecules are found as additives in synovial fluids, [15], modeled the synovial fluid as a couple-stress fluid. The synovial fluid is the natural lubricant of the joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is, due to its content of the hyaluronic acid, a fluid of high viscosity, near to a gel. The study, [16], has studied the hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients. The study, [17], have considered a couple-stress fluid with suspended particles heated from below. They have found that for stationary convection, couple-stress has a stabilizing effect whereas suspended particles have a destabilizing effect. The study, [18], has considered the thermal instability of a layer of a couple–stress fluid acted on by a uniform rotation, and has found that for stationary convection, the rotation has a stabilizing effect whereas couple-stress has both stabilizing and destabilizing effects. The study, [19], have investigated the transport of vorticity in couple-stress fluid in the presence of suspended particles. Another study, [20], studied the thermosolutal convection in a couple-stress fluid in the presence of uniform rotation.

Darcy's law governs the flow of Newtonian fluid through isotropic and homogeneous porous medium. However, to be mathematically compatible and physically consistent with the Navier-Stokes equations, [21], heuristically proposed the introduction of the term $\frac{\mu}{\varepsilon}\nabla^2\vec{q}$ (now known as Brinkman term) in addition to Darcian term $-\left(\frac{\mu}{k_1}\right)\vec{q}$. But the main effect is through the Darcian term and the Brinkman term contributes a very little effect, for flow through porous medium. Therefore, Darcy's law is proposed to govern the flow of this non-Newtonian couple-stress fluid through porous medium heuristically.

Keeping in mind the importance of geophysics, soil sciences, groundwater hydrology, astrophysics, chemical technology, industry, and various applications mentioned above, the present paper, therefore, deals with the combined effect of uniform vertical magnetic field and uniform rotation on the couple-stress fluid heated from below in porous medium.

2 Structure of the Problem and Basic Equations

Consider an infinite, horizontal, incompressible, electrically conducting couple-stress fluid layer of thickness d, heated from below so that, the temperatures and densities at the bottom surface $z = 0$ are T_0 and ρ_0 and at the upper surface $z = d$ are T_d and ρ_d respectively and that a uniform temperature gradient $\beta(= |dT/dz|)$ is maintained. The gravity field $\vec{g} = (0, 0, -g)$, a uniform vertical magnetic field $\vec{H}(0, 0, H)$ and a uniform vertical rotation $\vec{\Omega}(0, 0, \Omega)$ pervade the system. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and of medium permeability k_1.

Let ρ, p, T, η, μ_e and $\vec{q}(u, v, w)$ denote, respectively, the fluid density, pressure, temperature, resistivity, magnetic permeability, and filter velocity. Then the momentum balance, mass balance, and energy balance equations of couple-stress fluid through a porous medium [1], [5], [14], are

$$\frac{1}{\varepsilon}\left[\frac{\partial\vec{q}}{\partial t} + \frac{1}{\varepsilon}(\vec{q}.\nabla)\vec{q}\right] = -\nabla\left(\frac{p}{\rho_0} - \frac{1}{2}\left|\vec{\Omega}\times\vec{r}\right|^2\right) + \vec{g}\left(1 + \frac{\delta\rho}{\rho_0}\right) - \frac{1}{k_1}\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\vec{q} + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\vec{H})\times\vec{H} + \frac{2}{\varepsilon}(\vec{q}\times\vec{\Omega}), \quad (1)$$

$$\nabla.\vec{q} = 0, \quad (2)$$

$$E\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \kappa\nabla^2 T, \quad (3)$$

$$\nabla.\vec{H} = 0, \quad (4)$$

$$\varepsilon\frac{d\vec{H}}{dt} = (\vec{H}.\nabla)\vec{q} + \varepsilon\eta\nabla^2\vec{H}, \quad (5)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon^{-1}\vec{q}.\nabla$$

stands for the convective derivative.

The equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_0)], \quad (6)$$

where the suffix zero refers to values at the reference level $z = 0$. In writing equation (1), use has been made of the Boussinesq approximation, which states that the density variations are ignored in all terms in the equation of motion except the external force term. The kinematic viscosity ν, couple-stress viscosity μ', magnetic permeability μ_e, thermal diffusivity κ, electrical resistivity η, and coefficient of thermal expansion α are all assumed to be constants. Here $E = \varepsilon + (1 - \varepsilon)\left(\frac{\rho_s c_s}{\rho_0 c_v}\right)$, is a constant, while ρ_s, c_s and ρ_0, c_v stand for density and heat capacity of the solid (porous matrix) material and the fluid, respectively and $\vec{r} = (x, y, z)$.

The steady-state solution is;

$$\vec{q} = (0,0,0), \quad T = -\beta z + T_0, \quad \rho = \rho_0(1 + \alpha\beta z). \quad (7)$$

Here we use the linearized stability theory and the normal mode analysis method. Consider a small perturbation on the steady state solution, let $\delta\rho, \delta p, \theta, \vec{q}(u, v, w)$ and $\vec{h}(h_x, h_y, h_z)$ denote, respectively, the perturbations in fluid density ρ, pressure p, temperature T, velocity $\vec{q}(0,0,0)$ and the magnetic field $\vec{H}(0, 0, H)$. The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature, is given by

$$\delta\rho = -\alpha\rho_0\theta. \quad (8)$$

Then the linearized perturbation equations of the couple-stress fluid become

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t} = -\frac{1}{\rho_0}(\nabla\delta p) - \vec{g}\alpha\theta - \frac{1}{k_1}\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\vec{q} + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\vec{h})\times\vec{H} + \frac{2}{\varepsilon}(\vec{q}\times\vec{\Omega}), \quad (9)$$

$$\nabla.\vec{q} = 0, \tag{10}$$

$$E\frac{\partial \theta}{\partial t} = \beta w + \kappa \nabla^2 \theta, \tag{11}$$

$$\nabla.\vec{h} = 0, \tag{12}$$

$$\varepsilon \frac{\partial \vec{h}}{\partial t} = (\vec{H}.\nabla)\vec{q} + \varepsilon\eta\nabla^2\vec{h}. \tag{13}$$

3 The Dispersion Relation

For obtaining the dispersion relation, we now analyze the disturbances into normal modes, assuming that the perturbation quantities are of the form

$$[w, h_z, \theta, \zeta, \xi] = [W(z), K(z), \Theta(z), Z(z), X(z)] \exp(ik_x x + ik_y y + nt), \tag{14}$$

where k_x, k_y are the wave numbers along the $x-$ and $y-$ directions respectively, $k = \sqrt{(k_x^2 + k_y^2)}$ is the resultant wave number and n is the growth rate which is, in general, a complex constant. $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and $\xi = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$ stand for the z components of vorticity and current density, respectively.

Using expression (14), equations (9)-(13) in non-dimensional form transform to

$$(D^2 - a^2)\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l} - \frac{F}{P_1}(D^2 - a^2)\right] W + \frac{g\alpha d^2}{\nu} a^2 \Theta + \frac{2\Omega d^3}{\nu\varepsilon} DZ - \frac{\mu_e H d}{4\pi\rho_0 \nu}(D^2 - a^2) DK = 0, \tag{15}$$

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l} - \frac{F}{P_l}(D^2 - a^2)\right] Z = \left(\frac{\mu_e H d}{4\pi\rho_0 \nu}\right) DX + \left(\frac{2\Omega d}{\nu\varepsilon}\right) DW, \tag{16}$$

$$(D^2 - a^2 - p_2\sigma) K = -\left(\frac{Hd}{\varepsilon\eta}\right) DW, \tag{17}$$

$$(D^2 - a^2 - p_2\sigma) X = -\left(\frac{Hd}{\varepsilon\eta}\right) DZ, \tag{18}$$

$$(D^2 - a^2 - Ep_1\sigma)\Theta = -\left(\frac{\beta d^2}{\kappa}\right) W, \tag{19}$$

where we have expressed the coordinates x, y, z in the new unit of length d and put $a = kd, \sigma = \frac{nd^2}{\nu}$ and $D = \frac{d}{dz}, p_1 = \frac{\nu}{\kappa}$ is the Prandtl number, $P_l = \frac{k_1}{d^2}$ is the dimensionless medium permeability, $p_2 = \frac{\nu}{\eta}$ is the magnetic Prandtl number and $F = \frac{\mu'/(\rho_0 d^2)}{\nu}$ is the dimensionless couple-stress viscosity.

Consider the case where both the boundaries are free as well as perfect conductors of heat, while the adjoining medium is also perfectly conducting. The case of two free boundaries, though a little artificial (realistic in stellar atmospheres), enables us to find analytical solutions and to make some qualitative conclusions. The appropriate boundary conditions, with respect to which equations (15)-(19) must be solved are

$$W = D^2W = 0, DZ = 0, \Theta = 0, at\ z = 0\ and\ z = 1,$$

$DX = 0, K = 0$ on a perfectly conducting boundary. (20)

Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for $z = 0$ and $z = 1$ and hence the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{21}$$

where W_0 is a constant.

Eliminating Θ, X, Z, and K between equations (15)-(19) and substituting the proper solution (21) in the resultant equation, we obtain the dispersion relation

$$R_1 = \left(\frac{1+x}{x}\right)\left[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{\pi^2 F}{P}(1+x)\right][1 + x + iEp_1\sigma_1] + Q_1\left(\frac{1+x}{x}\right)\frac{(1+x+iEp_1\sigma_1)}{(1+x+ip_2\sigma_1)} + T_{A_1}\left[\frac{\frac{(1+x+iEp_1\sigma_1)}{(1++ip_2\sigma_1)}}{x\left\{\left(\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{\pi^2 F}{P}(1+x)\right)(1+x+ip_2\sigma_1) + Q_1\right\}}\right], \tag{22}$$

where

$$R_1 = \frac{g\alpha\beta d^4}{\nu\kappa\pi^4}, Q_1 = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon\pi^2}, T_{A_1} = \left(\frac{2\Omega d^2}{\varepsilon\nu\pi^2}\right)^2, x = \frac{a^2}{\pi^2}, i\sigma_1 = \frac{\sigma}{\pi^2}, and\ P = \pi^2 P_l.$$

Equation (22) is the required dispersion relation including the effects of magnetic field, rotation, couple-stress, and medium permeability on the couple-stress fluid heated from below in a porous medium in the presence of uniform vertical magnetic field and uniform vertical rotation. In the absence of rotation $(T_{A_1} = 0)$, the dispersion relation (22) reduces to that by, [22].

4 Important Theorems and Discussion

Theorem 1: For stationary convection case:

(i) Rotation postpones the onset of convection i.e. rotation has a stabilizing effect on the system.

(ii) In the absence of rotation, the magnetic field and couple-stress parameter postpone the onset of convection i.e. has a stabilizing effect and in the presence of rotation, the magnetic field and couple-stress parameter has both stabilizing and destabilizing effects on the system.

(iii) In the absence of rotation, the medium permeability hastens the onset of convection i.e. has a destabilizing effect and in the presence of rotation, the medium permeability has both stabilizing and destabilizing effects on the system.

Proof: When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (22) reduces to

$$R_1 = \left(\frac{1+x}{x}\right)\left[\left\{\frac{1+x}{P} + \frac{\pi^2 F}{P}(1+x)^2\right\} + Q_1\right] + T_{A_1}\frac{(1+x)^2}{x\left[\left\{\frac{1+x}{P} + \frac{\pi^2 F}{P}(1+x)^2\right\} + Q_1\right]}. \quad (23)$$

To investigate the effects of rotation, magnetic field, couple-stress parameter, and medium permeability, we examine the nature of $\frac{dR_1}{dT_{A_1}}, \frac{dR_1}{dQ_1}, \frac{dR_1}{dF}$ and $\frac{dR_1}{dP}$ analytically.

(i) Equation (23) yields

$$\frac{dR_1}{dT_{A_1}} = \frac{(1+x)^2 P}{x\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}}, \quad (24)$$

It is clear from equation (24) that for stationary convection, the rotation postpones the onset of convection in a couple-stress fluid heated from below in a porous medium in the presence of a magnetic field.

(ii) It is evident from equation (23) that

$$\frac{dR_1}{dQ_1} = \left(\frac{1+x}{x}\right)\left[1 - \frac{(1+x)P^2 T_{A_1}}{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}\right], \quad (25)$$

$$\frac{dR_1}{dF} = \frac{\pi^2(1+x)^3}{xP}\left[1 - \frac{(1+x)P^2 T_{A_1}}{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}\right], \quad (26)$$

It is evident from equation (25) and equation (26) that for stationary convection, the magnetic field and the couple-stress postpone the onset of convection in the absence of rotation and also postpone the onset of convection in the presence of rotation if

$$T_{A_1} < \frac{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}{(1+x)P^2}, \quad (27)$$

whereas the magnetic field and the couple stress hastens the onset of convection if

$$T_{A_1} > \frac{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}{(1+x)P^2}. \quad (28)$$

(iii) Equation (23) yields

$$\frac{dR_1}{dP} = -\frac{(1+x)^2\{1+\pi^2 F(1+x)\}}{xP^2}\left[1 - \frac{(1+x)P^2 T_{A_1}}{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}\right]. \quad (29)$$

It is evident from equation (29) that for stationary convection, the medium permeability hastens the onset of convection in the absence of rotation and also hastens the onset of convection in the presence of rotation if

$$T_{A_1} < \frac{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}{(1+x)P^2},$$

whereas the medium permeability postpones the onset of convection if

$$T_{A_1} > \frac{\{(1+x) + \pi^2 F(1+x)^2 + PQ_1\}^2}{(1+x)P^2}.$$

Theorem 2: The system is stable or unstable.

Proof: Multiplying equation (15) by W^*, the complex conjugate of W, integrating over the range of z, and making use of equations (16)-(19) together with the boundary conditions (20), we obtain

$$FI_1 + \left(1 + P_l\frac{\sigma}{\varepsilon}\right)I_2 - \left(\frac{g\alpha\kappa a^2}{\nu\beta}P_l\right)[I_3 + Ep_1\sigma^* I_4] + \frac{\mu_e \varepsilon\eta}{4\pi\rho_0\nu}P_l[I_5 + p_2\sigma^* I_6]$$
$$+ d^2\left[\left(1 + P_l\frac{\sigma^*}{\varepsilon}\right)I_8 + FI_7\right] + \frac{\mu_e\varepsilon\eta d^2}{4\pi\rho_0\nu}P_l[I_9 + p_2\sigma I_{10}]$$
$$= 0, \quad (30)$$

where

$$I_1 = \int_0^1 (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)dz,$$
$$I_2 = \int_0^1 (|DW|^2 + a^2|W|^2)dz,$$
$$I_3 = \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz,$$
$$I_4 = \int_0^1 (|\Theta|^2)dz,$$
$$I_5 = \int_0^1 (|D^2K|^2 + 2a^2|DK|^2 + a^4|K|^2)dz,$$
$$I_6 = \int_0^1 (|DK|^2 + a^2|K|^2)dz,$$
$$I_7 = \int_0^1 (|DZ|^2 + a^2|Z|^2)dz,$$
$$I_8 = \int_0^1 (|Z|^2)dz,$$
$$I_9 = \int_0^1 (|DX|^2 + a^2|X|^2)dz,$$
$$I_{10} = \int_0^1 (|X|^2)dz. \quad (31)$$

The integrals $I_1, \dots\dots\dots, I_{10}$ are all positive definite. Substituting $\sigma = \sigma_r + i\sigma_i$, where σ_r, σ_i are real and then equating the real and imaginary parts of equation (30), we obtain

$$\sigma_r\left[\frac{I_2}{\varepsilon}-\frac{g\alpha\kappa a^2}{\nu\beta}Ep_1I_4+\frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}p_2(I_6+d^2I_{10})+\frac{d^2}{\varepsilon}I_8\right]$$
$$=-\left[\frac{F}{P_l}I_1+\frac{1}{P_l}I_2-\frac{g\alpha\kappa a^2}{\nu\beta}I_3+\frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}(I_5+d^2I_9)+\frac{d^2}{P_l}(I_8+FI_7)\right], \qquad (32)$$

$$\sigma_i\left[\frac{I_2}{\varepsilon}+\frac{g\alpha\kappa a^2}{\nu\beta}Ep_1I_4-\frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}p_2(I_6-d^2I_{10})-\frac{d^2}{\varepsilon}I_8\right]=0. \qquad (33)$$

It is evident from equation (32) that σ_r is either positive or negative. The system is, therefore, either stable or unstable.

Theorem 3: The modes may be either oscillatory or non-oscillatory in contrast to the non-magneto-rotatory case.

Proof: Equation (33) yields that σ_i may be either zero or non-zero, which means that the modes may be either non-oscillatory or oscillatory. In the absence of rotation and magnetic field, equation (33) reduces to

$$\sigma_i\left[\frac{I_2}{\varepsilon}+\frac{g\alpha\kappa a^2}{\nu\beta}Ep_1I_4=0,\right]$$

and the terms in brackets are positive definite. Thus $\sigma_i=0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for the couple-stress fluid heated from below in a porous medium in the absence of rotation and magnetic field. It is clear from equation (33) that the oscillatory modes are introduced due to the presence of rotation and the magnetic field, which were non-existent in their absence.

Theorem 4: The system is stable for $\frac{g\alpha\kappa}{\nu\beta}\le\frac{27\pi^4}{4}$ and under the condition $\frac{g\alpha\kappa}{\nu\beta}>\frac{27\pi^4}{4}$, the system becomes unstable.

Proof: From equation (33), it is clear that σ_i is zero when the quantity is multiplied it is not zero and arbitrary when this quantity is zero.

If $\sigma_i\neq 0$, equation (32) upon utilizing equation (33) and the Rayleigh-Ritz inequality gives

$$\left[\frac{27\pi^4}{4}-\frac{g\alpha\kappa}{\nu\beta}\right]\int_0^1|W|^2dz+\frac{(\pi^2+a^2)}{a^2}\left\{\frac{2\sigma_r\mu_e\varepsilon\eta}{4\pi\rho_0\nu}p_2d^2I_{10}+\frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}(I_5+d^2I_9)+\frac{d^2}{P_l}(I_8+FI_7)+\left(\frac{2\sigma_r}{\varepsilon}+\frac{1}{P_l}\right)I_2+\frac{F}{P_l}I_1\right\}\le 0, \qquad (34)$$

since the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ with respect to a^2 is $\frac{27\pi^4}{4}$.

Now, let $\sigma_r\ge 0$, we necessarily have from inequality (34) that

$$\frac{g\alpha\kappa}{\nu\beta}>\frac{27\pi^4}{4}. \qquad (35)$$

Hence, if

$$\frac{g\alpha\kappa}{\nu\beta}\le\frac{27\pi^4}{4}, \qquad (36)$$

then $\sigma_r<0$. Therefore, the system is stable.

Thus, under condition (36), the system is stable and under condition (35) the system becomes unstable.

Theorem 5: The sufficient condition for the non-existence of overstability is

$$\kappa<Min\left[E\eta, E\left(\frac{\nu\varepsilon}{k_1}\right)^3\left(\frac{d}{2\Omega\pi}\right)^2\right].$$

Proof: Here we discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (22) will admit solutions with σ_1 real. Equating real and imaginary parts of equation (22) and eliminating R_1 between them, we obtain

$$A_3c_1^3+A_2c_1^2+A_1c_1+A_0=0, \qquad (37)$$

where we have put $c_1=\sigma_1^2$, $b=1+x$ and

$$A_0=\left[\frac{\pi^4F^2}{P^2}\left\{\frac{1}{\varepsilon}+\frac{Ep_1\pi^2F}{P}\right\}b^8+\frac{\pi^2F}{P^2}\left\{\frac{2}{\varepsilon}+\frac{3Ep_1\pi^2F}{P}\right\}b^7\right.$$
$$+\left\{\frac{\pi^4F^2Q_1}{P^2}(3Ep_1-p_2)+\frac{1}{P^2}\left(\frac{1}{\varepsilon}+\frac{3Ep_1\pi^2F}{P}\right)+\frac{2Q_1\pi^2F}{\varepsilon P}\right\}b^6$$
$$+\left\{\frac{2\pi^2FQ_1}{P^2}(3Ep_1-p_2)+\left(\frac{Ep_1}{P^3}-\frac{T_{A_1}}{\varepsilon}\right)+\frac{1}{P}\left(\frac{2Q_1}{\varepsilon}+Ep_1\pi^2FT_{A_1}\right)\right\}b^5$$
$$+\left\{\frac{\pi^2FQ_1^2}{P}(3Ep_1-2p_2)+\frac{Q_1}{P^2}(3Ep_1-p_2)+\frac{Q_1^2}{\varepsilon}+\frac{Ep_1T_{A_1}}{P}\right\}b^4$$
$$\left.+\left\{\frac{Q_1^2}{P}(3Ep_1-2p_2)+Q_1T_{A_1}(Ep_1+p_2)\right\}b^3+\{Q_1^3(Ep_1-p_2)\}b^2\right], \qquad (38)$$

$$A_3=\frac{p_2^4}{\varepsilon^2}\left[\left\{\frac{1}{\varepsilon}+\frac{Ep_1\pi^2F}{P}\right\}b^2+\left\{\frac{Ep_1}{P}\right\}b\right]. \qquad (39)$$

Since σ_1 is real for overstability, the values of $c_1(=\sigma_1^2)$ of equation (37) are positive. So the product of the roots of equation (37) is positive but this is impossible if $A_0>0$ (since the product of the roots of equation (37) is $-\frac{A_0}{A_3}$ and $A_3>0$). $A_0>0$

is, therefore, a sufficient condition for the non-existence of overstability.

It is clear from (38) that A_0 is always positive if

$$Ep_1 > p_2 \ and \ T_{A_1} < \frac{Ep_1\varepsilon}{P^3} \qquad (40)$$

i.e.

$$E\eta > \kappa \ and \ \kappa < E\left(\frac{\nu\varepsilon}{k_1}\right)^3\left(\frac{d}{2\Omega\pi}\right)^2, \qquad (41)$$

which imply that

$$\kappa < Min\left[E\eta, E\left(\frac{\nu\varepsilon}{k_1}\right)^3\left(\frac{d}{2\Omega\pi}\right)^2\right]. \qquad (42)$$

The condition (42) is, therefore, a sufficient condition for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

5 Conclusions

With the growing importance of non-Newtonian fluids in modern technology and industries, investigation of such fluids is desirable. The study, [14], formulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use in the study of mechanisms of lubrication of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing that has articular cartilage as the bearing and synovial fluid as the lubricant. The shoulder, ankle, knee, and hip joints are the loaded-bearing synovial joints of the human body. Since long-chain hyaluronic acid molecules are found as additives in synovial fluids, Walicki and Walicka modeled the synovial fluid as a couple-stress fluid. Therefore, an attempt has been made to investigate the combined effects of uniform vertical magnetic field and uniform vertical rotation on a layer of couple-stress fluid heated from below in a porous medium. The main conclusions from the analysis of this paper are as follows:

- In the case of stationary convection, the rotation postpones the onset of convection.
- It is also observed for the case of stationary convection that in the absence of rotation, the magnetic field and couple-stress parameter postpone the onset of convection i.e. has a stabilizing effect and in the presence of rotation, the magnetic field and couple-stress parameter has both stabilizing and destabilizing effects on the system. Also in the absence of rotation, the medium permeability hastens the onset of convection i.e. has a destabilizing effect and in the presence of rotation, the medium permeability has both stabilizing and destabilizing effects on the system.
- It is found that magnetic field and rotation introduce oscillatory modes in the system which were non-existent in their absence.
- It is observed that the system is stable for $\frac{g\alpha\kappa}{\nu\beta} \leq \frac{27\pi^4}{4}$ and under the condition $\frac{g\alpha\kappa}{\nu\beta} > \frac{27\pi^4}{4}$, the system becomes unstable.
- The case of overstability is also considered. The condition

$$\kappa < Min\left[E\eta, E\left(\frac{\nu\varepsilon}{k_1}\right)^3\left(\frac{d}{2\Omega\pi}\right)^2\right]$$

is the sufficient condition for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

Acknowledgements:

The author is grateful to all three learned referees for their useful technical comments and valuable suggestions, which led to a significant improvement of the paper.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.