A Theoretical Study of an Extended KDV Equation

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Abstract: Discovered experimentally by Russell and described theoretically by Korteweg and de Vries, KdV equation has been a nonlinear evolution equation describing the propagation of weakly dispersive and weakly nonlinear waves. This equation received a lot of attention from mathematical and physical communities as an integrable equation. The objectives of this paper are: first, providing a rigorous mathematical derivation of an extended KdV equations, one on the velocity, other on the surface elevation, next, solving explicitly the one on the velocity. In order to derive rigorously these equations, we will refer to the definition of consistency, and to find an explicit solution for this equation, we will use the sine-cosine method. As a result of this work, a rigorous justification of theextendedKdvequationoffifthorderwillbedone,andanexplicitsolutionofthisequationwillbederived.

key-words: KdV equation, rigorous derivation, equation on the velocity, equation on the surface elevation, sine-cosinemethod,explicitsolution.

1 Introduction

The topic of water waves is an old one, more than half a century has passed and the study of ocean surface waves has greatly advanced. Shallow water waves are studied because of their impacts on coasts, the economy, recreation and defense. Although there is still interest in shallow water waves as a source of pollution-free renewable energy. The water waves problem has attracted physicians and mathematicians because of its extremely rich structure. In order to give a sketch of the historical development of the modelisation of water waves problems, assume that the domain occupied by the fluid at time t is denoted by $\Omega_t = \{(x,z) \in \mathbb{R} \times \mathbb{R}; -h_0 + b < z < \zeta\}$ where the surface of the fluid is a graph parametrized by ζ and its bottom is parametrized by $-h_0 + b$, with h_0 is the reference depth. Under the following definitions of: a as the amplitude of the wave, λ the wavelength of the wave, let us introduce the following parameters: $\mu = \frac{h_0^2}{\lambda}$ and $\varepsilon = \frac{a}{h_0}$, the former parameter characterizes the shallowness of the wave, whereas the lat-

ter characterizes its nonlinearity. The first mathematical description of the motion of an homogeneous, inviscid, incompressible and irrotational fluid was provided by Euler model, where the driving force is due to gravity g, and the effect of the surface tension is neglected for the sake of simplicity:

(1)
$$\begin{cases} \partial_t V + (V \cdot \nabla_{x,z})V = -g\vec{e_z} - \nabla_{x,z}P & in \quad (x,z) \in \Omega_t, \\ divV = 0 & in \quad (x,z) \in \Omega_t, t \geq 0, \\ curlV = 0 & in \quad (x,z) \in \Omega_t, t \geq 0, \\ P = P_{atm} & at \quad z = \zeta(t,x), t \geq 0, \\ \partial_t \zeta - \sqrt{1+(\partial_x \zeta)^2}V.n_+ & at \quad z = \zeta(t,x), t \geq 0, \\ V.n_- = 0 & at \quad z = -h_0 + b(x), t \geq 0, \\ \lim_{|(x,z)|\to\infty} |\zeta(t,x)| + |V| = 0 & in \quad (x,z) \in \Omega_t, t \geq 0, \end{cases}$$

with V is the fluid velocity, P is the fluid pressure, $\vec{e_z}$ is a unit vector in vertical direction, n_- and n_+ are the outward and inward normal vectors respectively.

These fluid equations expressed the mass and momentum conservation, where every physical assumption was stated by an equation.

From the equations mentioned above, Bernoulli equations for traveling water waves was obtained by employing two key reductions: the traveling wave assumption and the introduction of a velocity potential. The assumption of irrotationality is equivalent to assuming thee existence of a velocity potential φ : $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\nabla \varphi$ represents the fluid velocity vector, thereby, another formulation of the free surface Euler equations:

(2)
$$\begin{cases} \partial_x^2 \varphi + \partial_z^2 \varphi = 0 & at \quad -h_0 + b(x) < z < \zeta \\ \partial_z \varphi - \partial_x b \cdot \partial_x \zeta = 0 & at \quad z = -h_0 + b(x) \\ \partial_t \zeta + \partial_x \varphi \cdot \partial_x \zeta - \partial_z \varphi = 0 & at \quad z = \zeta \\ \partial_t \varphi + \frac{1}{2}\left((\partial_x \varphi)^2 + (\partial_z \varphi)^2\right) = -g\zeta & at \quad z = \zeta \end{cases}$$

Toward the dimensionalization of Bernoulli's equations, use the following relations

$$x = \lambda x' \qquad z = h_0 z' \qquad \zeta = a\zeta'$$

$$\varphi = \frac{a}{h_0}\lambda\sqrt{gh_0}\varphi' \qquad b = b_0 b' \qquad t = \frac{\lambda}{\sqrt{gh_0}}t'$$

The governing equations of the water waves problem in terms of the dimensionless variables become

(3)
$$\begin{cases} \mu\partial_x^2 \varphi + \partial_z^2 \varphi = 0 & at \quad -1+\beta b < z < \varepsilon\zeta \\ \partial_z \varphi - \mu\beta\partial_x b \cdot \partial_x \zeta = 0 & at \quad z = -1+\beta b \\ \partial_t \zeta + \partial_x \varphi \cdot \partial_x \zeta - \partial_z \varphi = 0 & at \quad z = \varepsilon\zeta \\ \partial_t \varphi + \frac{1}{2}\left((\partial_x \varphi)^2 + (\partial_z \varphi)^2\right) + \zeta = 0 & at \quad z = \varepsilon\zeta \end{cases}$$

where $\beta = \frac{b_0}{h_0}$ represents the topography parameter. A popular form of the water waves system is given by the Zakharov/Craig-Sulem formulation. This is an elegant formulation of the water waves equations where all the unknowns are evaluated at the free surface only. Following Zakharov, introduce the trace of the potential on the free surface $\psi(t,x) = \varphi(t,x,\varepsilon\zeta) = \varphi|_{z=\varepsilon\zeta}$. Craig and Sulem observed that one can form a system of two evolution equations for ζ and ψ. To do so, they introduced the Dirichlet-Neumann operator G_μ that relates ψ to the normal derivative $\partial_n \varphi$ of the potential by

$$G_\mu[\varepsilon\zeta, \beta b]\psi = -\mu\varepsilon\partial_x\zeta \cdot \partial_x\varphi|_{z=\varepsilon\zeta} + \partial_z\varphi|_{z=\varepsilon\zeta}$$

$$= \sqrt{1+\mu\varepsilon^2\partial_x\zeta^2}\partial_n\varphi|_{z=\varepsilon\zeta}$$

with φ solving the boundary value problem:

(4)
$$\begin{cases} \mu\partial_x^2 \varphi + \partial_z^2 \varphi = 0 & at \quad -1+\beta b < z < \varepsilon\zeta \\ \partial_n\varphi_{z=-1+\beta b} = 0 \\ \varphi|_{z=\varepsilon\zeta} = \psi \end{cases}$$

Benefiting from the definition of G_μ and (4) and doing some calculations we get:

(5)
$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2}(\partial_x\psi)^2 - \mu\varepsilon\frac{(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\partial_x\zeta \cdot \partial_x\psi)^2}{2(1+\varepsilon^2\mu(\partial_x\zeta)^2)} = 0 \end{cases}$$

Considering long waves propagation in shallow water but without assuming small amplitudes $(\mu << 1, \varepsilon = O(1))$, Green and Naghdi derived a new improved model of equations describing significant dispersive effects of water waves phenomena. To derive these equations, let us inroduce the depth averaged horizontal velocity

$$u(t,x) = \frac{1}{h}\int_{-1+\beta b(x)}^{\varepsilon\zeta(t,x)} \partial_x\varphi(t,x,z)dz$$

Also, we are going to use the Taylor expansion of φ:

$$\varphi_{app} = \sum_{j=0}^{N} \mu^j \varphi_j$$

Since we are seeking to find these equations up to $O(\mu^3)$, we need to find φ_0, φ_1, and φ_2.

After a long calculations and brainwork we get the one dimensional Green-Naghdi system of equations, for uneven bottom:

(6)
$$\begin{cases} \partial_t \zeta + \partial_x(hu) = 0, \\ (h + \mu \mathcal{T}[h, \beta b] + \mu^2 \mathcal{I}[h, \beta b])\partial_t u + h\partial_x \zeta + \varepsilon huu_x \\ + \mu\varepsilon Q_1[U]u + \mu\varepsilon\beta B_1[U]u + \mu\varepsilon\beta^2 B_2[U]u \\ + \mu^2\varepsilon Q_2[U]u + \mu^2\varepsilon\beta B_3[U]u + \mu^2\beta^2\varepsilon B_4[U]u = O(\mu^3), \end{cases}$$

where $U = (\zeta, u)^T$ and denoting by $h = h(t, x) = 1 + \varepsilon\zeta(t, x) - \beta b(x)$ the total non-dimensional height of the liquid, with

$$\mathcal{T}[h, \beta b]u = -\frac{1}{3}\partial_x(h^3\partial_x u) + \frac{\beta}{2}[\partial_x(h^2 b_x u) - h^2 b_x u_x] \\ + \beta^2 hb_x^2 u,$$

$$\mathcal{I}[h, \beta b]u = -\frac{1}{45}\partial_x^2(h^5\partial_x^2 u) \\ + \frac{\beta}{24}[\partial_x(\partial_x(h^4 b_x)\partial_x u) + \partial_x^2(h^4\partial_x(b_x u)) - b_x\partial_x(h^4\partial_x^2 u)] \\ + \frac{\beta^2}{12}[2\partial_x(h^3 b_x^2\partial_x u) + \partial_x(h^3 b_x b_{xx} u) + 2b_x\partial_x(h^3 b_x\partial_x u)] \\ + \frac{\beta^2}{12}b_x\partial_x(h^3 b_{xx}u),$$

where the non-topographical terms are represented by

$$Q_1[U]u = -\frac{1}{3}\partial_x\left(h^3(uu_{xx} - u_x^2)\right),$$

$$Q_2[U]u = -\frac{1}{45}\partial_x\left(\partial_x(h^5(uu_{xxx} - 5u_x u_{xx})) - 3h^5(u_{xx})^2\right)$$

while the purely-topographical terms are introduced by

$$B_1[U]u = \frac{1}{2}[\partial_x(h^2 b_{xx}u^2) + \partial_x(h^2 b_x uu_x)] \\ - \frac{1}{2}h^2(uu_{xx} - u_x^2)b_x,$$

$$B_2[U]u = h(b_{xx}u^2 + b_x uu_x)b_x,$$

$$B_3[U]u = \frac{1}{24}\partial_x^2\left(h^4(b_{xxx}u^2 - b_{xx}uu_x - 8b_x u_x^2)\right) \\ - \frac{1}{4}h^4 b_{xx}u_x u_{xx} + \frac{1}{24}\partial_x(h^4 b_x uu_{xx}) \\ - \frac{5}{24}\partial_x(h^4 b_{xx}uu_{xx}) - \frac{1}{24}b_x\partial_x\left(h^4(uu_{xxx} + u_x u_{xx})\right),$$

$$B_4[U]u = \frac{1}{12}\partial_x\left(h^3(b_{xxx}b_x u^2 + 2b_x^2 uu_{xx} + 10b_x b_{xx}uu_x)\right) \\ + \frac{1}{12}\partial_x\left(h^3(2b_x^2 u_x^2 + 3b_{xx}^2 u^2)\right) \\ + \frac{1}{12}h^3(b_{xx}u + 2b_x u_x)b_{xx}u_x \\ + \frac{1}{12}b_x\partial_x\left(h^3(b_{xxx}u^2 + 2b_x uu_{xx} - 6b_x u_x^2)\right).$$

In as mush as this water waves problem is a difficult nonlinear problem to solve, approximate theories have been developed. Under the KdV regime $\varepsilon = O(\mu)$, Boussinsq model of equations for flat bottom ($b = 0$) can be derived:

(7)
$$\begin{cases} \partial_t \zeta + [(1 + \varepsilon\zeta)u]_x = 0 \\ u_t + \zeta_x + \varepsilon uu_x = \frac{\mu}{3}u_{xxt} - \mu\varepsilon\left[\frac{1}{2}u_x^2 + \frac{1}{3}uu_{xx}\right]_x \\ \qquad + \frac{\mu^2}{45}\partial_x^2(u_{xxt}) \end{cases}$$

A large body of literature has been dedicated to the development of efficient techniques to solve these equations. An innovative technique was the derivation of consistent asymptotic models with the main models, like Camassa-Holm and KdV equations.

The most renowned KdV equation is [5]:

$$u_t + u_x + \frac{3}{2}\varepsilon uu_x + \frac{\mu}{6}u_{xxx} = 0$$

that was originally derived for flat bottom. What attracted the focus of scientists in this equation, was its integrability property, and thus its solitons (solitary waves) solutions. Since the derivation of the equation mentioned before, several methods have been used to derive new extended KdV equations, with different ocean conditions and properties. General derivations of this equation were justified with bottom, and with non constant coefficients, with topography (called KdV-top equation) [5, 6]. A formal derivation of KdV equation was provided using Whitham method, in the presence of surface tension [2], all these previous works were done up to $O(\mu^2)$. Also, using some physical principles, an extended KdV equation was

formally derived up to $O(\mu^3)$ [8]. Characterized with its integrable property, KdV equation catch the attention of scientists, since it was solved using some successive integrations by parts. With the extension of this equation, more nonlinear terms appeared, causing difficulties in finding its explicit solution. Therefore, scientists tried to use new techniques for solving explicitly these extended equations. In [9], an explicit solution for a generalized KdV equation of third order was derived using sine-cosine method.

In the paper at hand, we deal with an irrotational, incompressible, inviscid fluid with a free surface, and constant density, acted on only by gravity. Knowing that a is the amplitude of the wave, λ is the wavelength of the wave, h_0 is the reference depth, denote by $\Omega_t = \{(x,z) \in \mathbb{R} \times \mathbb{R}; -h_0 + b < z < \zeta\}$ the domain of the fluid for each time t where the surface of the fluid is a graph parametrized by ζ and its bottom is parametrized by $-h_0 + b$.

In this paper we consider the extended Boussinesq system of equations 7, describing the motion of an incompressible, irrotational, inviscid fluid with free surface, under the influence of gravity. Recall that the KdV scaling is $\varepsilon = O(\mu)$, with $0 < \varepsilon \leq 1$ and $0 < \mu \ll 1$. The organization of this paper is as follows: in the secind section, a derivation of the extended KdV equation will be done, in the first subsection 2.1, we will derive rigorously an extended KdV equation on the velocity u. In the second subsection 2.2, a rigorous mathematical derivation of extended KdV equation on the surface elevation ζ will be provided, and hence a rigorous verification of this imposed equation in [8]. In the third section, we will use the sine-cosine method to find an explicit solution for the derived extended KdV equation on the velocity. The aim of this paper is to give a rigorous mathematical derivation of the extended KdV equations, up to μ^3, and solve it explicitly. Concerning the methodolgy, after the examination of some previous works, we will proceed as in [1], so we will use the definition of consistency to provide these rigorous derivations, which guarantee the relevance of these equations with

(7), and serve in the construction of approximate solutions of Boussinesq equations. Also, we will apply a new technique called sine-cosine method to solve explicitly the one on the velocity.

2 Derivation of the new extended KDV Equations

The main goal of this section is to find extended KdV equations on velocity and on surface elevation. The new derived KdV equation on the velocity will be:

(8)
$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \frac{\mu}{6} u_{xxx}$$
$$+ \mu\varepsilon \left[\frac{5}{12} u u_{xxx} + \frac{21}{24} u_x u_{xx} \right] + \frac{11}{360} \mu^2 \partial_x^2(u_{xxx}) = O(\mu^3)$$

And one on the surface elevation will be:

(9)
$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon \zeta \zeta_x - \frac{3}{8}\varepsilon^2 \zeta^2 \zeta_x + \frac{\mu}{6}\zeta_{xxt}$$
$$+ \mu\varepsilon \left[\frac{5}{12}\zeta \zeta_{xx} + \frac{23}{24}\zeta_x^2 \right] + \frac{19}{360}\mu^2 \partial_x^2(\zeta_{xxx}) = O(\mu^3)$$

2.1 Equation on the velocity.

In order to get (8), we will introduce the following equation on u

(10)
$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu\alpha u_{xxx}$$
$$= \mu\varepsilon \left[\beta u u_{xxx} + \gamma u_x u_{xx} \right] + \mu^2 \delta \partial_x^2(u_{xxx})$$

where α, β, γ, δ are parameters in $\mathbb{R}$.
Next, we need to find the values of parameters mentioned above, that will be done in three steps.

- **Step1:**
  From (10) we get:

  $$u_x = -u_t - \frac{3}{2}\varepsilon u u_x - \mu\alpha u_{xxx} + O(\varepsilon, \mu)$$

  $$u_{xxx} = -u_{xxt} - \frac{3}{2}\varepsilon \partial_x^2(u u_x) - \mu\alpha \partial_x^2(u_{xxx}) + O(\varepsilon, \mu)$$

Substitute the expression of u_{xxx} in (10) to get:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x - \mu\alpha u_{xxt} - \mu\varepsilon \left[\frac{3\alpha}{2}u_x^2 + \frac{3\alpha}{2}u u_{xx}\right]_x$$
$$- \mu^2 \partial_x^2(u_{xxx}) = \mu\varepsilon[\beta u u_{xxx} + \gamma u_x u_{xx}]$$
$$+ \mu^2 \delta \partial_x^2(u_{xxx}) + O(\mu^3)$$

But

$$\beta u u_{xxx} + \gamma u_x u_{xx}$$
$$= \beta u u_{xxx} + \beta u_x u_{xx} - \beta u_x u_{xx} + \gamma u_x u_{xx}$$
$$= \beta(u u_{xx})_x - \frac{\beta}{2}(u_x^2)_x + \frac{\gamma}{2}(u_x^2)_x$$
$$= \left[\beta u_x u_{xx} + \frac{\gamma - \beta}{2}u_x^2\right]_x$$

Then

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x - \mu\alpha u_{xxt}$$
$$= \mu\varepsilon\left[\left(\frac{3\alpha}{2} + \beta\right)u u_{xx} + \left(\frac{3\alpha + \gamma - \beta}{2}\right)u_x^2\right]_x$$
$$+ \mu^2\left(\alpha^2 + \delta\right)\partial_x^2(u_{xxx}) + O(\mu^3)$$

One can get

(11)
$$u_t + u_x + \frac{3}{2}\varepsilon u u_x - \mu\alpha u_{xxt} = \mu\varepsilon\left[a u u_{xx} + b u_x^2\right]_x$$
$$+ \mu^2 c \partial_x^2(u_{xxx}) + O(\mu^3)$$

where

$$a = \frac{3\alpha}{2} + \beta; \quad b = \frac{3\alpha + \gamma - \beta}{2}; \quad c = \delta + \alpha^2.$$

To find another equation on a, b, c and α we need to use the equations of (7).
In the next step, the second equation of (7) we will be used.

- **Step2:**
  Let v be a smooth enough function such that $\zeta = u + \varepsilon v$. Then $(7)_2$ becomes

$$u_t + u_x + (\varepsilon v)_x + \varepsilon u u_x = \frac{\mu}{3}u_{xxt} - \mu\varepsilon\left[\frac{1}{2}u_x^2 + \frac{1}{3}u u_{xx}\right]_x$$
$$+ \frac{\mu^2}{45}\partial_x^2(u_{xxt})$$

We know from (11) that

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x - \mu\alpha u_{xxt} - \mu\varepsilon\left[a u u_{xx} + b u_x^2\right]_x$$
$$- \mu^2 c \partial_x^2(u_{xxx}) = O(\mu^3)$$

Then

$$(\varepsilon v)_x + u_t + u_x + \frac{3}{2}\varepsilon u u_x - \mu\alpha u_{xxt} - \mu\varepsilon\left[a u u_{xx} + b u_x^2\right]_x$$
$$- \mu^2 c \partial_x^2(u_{xxx})$$
$$= u_t + u_x + (\varepsilon v)_x + \varepsilon u u_x + \frac{\varepsilon}{2}u u_x - \mu\alpha u_{xxt}$$
$$- \mu\varepsilon\left[a u u_{xx} + b u_x^2\right]_x - \mu^2 c \partial_x^2(u_{xxx})$$

So we can deduce up to $O(\mu^3)$

$$(\varepsilon v)_x = \frac{\varepsilon}{2}u u_x + \mu\left(\frac{1}{3} - \alpha\right)u_{xxt}$$
$$- \mu\varepsilon\left[\left(a + \frac{1}{3}\right)u u_{xx} + \left(b + \frac{1}{2}\right)u_x^2\right]_x$$
$$+ \mu^2\left(\frac{1}{45} - c\right)\partial_x^2(u_{xxx}) + O(\mu^3)$$

Hence

(12)
$$\varepsilon v = \frac{\varepsilon}{4}u^2 + \mu\left(\frac{1}{3} - \alpha\right)u_{xt} - \mu\varepsilon\left[\left(a + \frac{1}{3}\right)u u_{xx} + \left(b + \frac{1}{2}\right)u_x^2\right]$$
$$+ \mu^2\left(\frac{1}{45} - c\right)\partial_x(u_{xxx})$$

Now, we will use the first equation of (7).

- **Step3:**
  Put $\zeta = u + \varepsilon v$ in $(7)_1$ that is

$$u_t + u_x + 2\varepsilon u u_x + (\varepsilon v)_t + \varepsilon^2(uv)_x = 0$$

From (11) we have

(13)
$$u_t = -u_x + O(\varepsilon, \mu)$$
$$u_{xt} = -u_{xx} + O(\varepsilon, \mu)$$

Multiply (12) by εu, derive with respect to x and use (13) to get

$$\varepsilon^2 uv = \frac{\varepsilon^2}{4}u^3 + \mu\varepsilon\left(\frac{1}{3} - \alpha\right)u u_{xt} + O(\mu^3)$$

$$\varepsilon^2(uv)_x = \frac{3}{4}\varepsilon^2 u^2 u_x + \mu\varepsilon\left(\frac{1}{3} - \alpha\right)(u u_{xt})_x + O(\mu^3)$$
$$= \frac{3}{4}\varepsilon^2 u^2 u_x - \mu\varepsilon\left(\frac{1}{3} - \alpha\right)(u u_{xx})_x + O(\mu^3)$$

Next, deriving (12) with repect to t one can get

$$(\varepsilon v)_t = \frac{\varepsilon}{2}uu_t + \mu\left(\frac{1}{3}-\alpha\right)u_{xtt}$$
$$-\mu\varepsilon\left[\left(a+\frac{1}{3}\right)uu_{xx} + \left(b+\frac{1}{2}\right)u_x^2\right]_t$$
$$+\mu^2\left(\frac{1}{45}-c\right)\partial_{xt}^2(u_{xxx}) + O(\mu^3)$$

From (11) we get

$$u_t = -u_x - \frac{3}{2}\varepsilon uu_x + \mu\alpha u_{xxt} + O(\varepsilon,\mu)$$

$$u_{xtt} = -u_{xxt} - \frac{3}{2}\varepsilon\partial_{xt}(uu_x) + \mu\alpha\partial_{xt}(u_{xxt}) + O(\varepsilon,\mu)$$

$$u_{xxt} = -u_{xxx} - \frac{3}{2}\varepsilon\partial_{xx}(uu_x) + \mu\alpha\partial_x^2(u_{xxt}) + O(\varepsilon,\mu)$$

Remark that

$$\partial_{xt}(u_{xxt}) = \partial_x^2(u_{xxx}) + O(\varepsilon,\mu)$$
$$\partial_{xt}(u_{xxx}) = -\partial_x^2(u_{xxx}) + O(\varepsilon,\mu)$$

Also we have

$$\partial_{xt}^2(uu_x) = -\partial_x(u_tu_x + uu_{xt})$$
$$= -\partial_x(u_x^2 + uu_{xx}) = -\partial_x^2(uu_x) + O(\varepsilon,\mu)$$

Then

$$u_{xtt} = -u_{xxt} + \frac{3}{2}\varepsilon\left[u_x^2 + uu_{xx}\right]_x$$
$$+ \mu\alpha\partial_x^2(u_{xxx}) + O(\varepsilon,\mu)$$

Gathering all the computations done above one can get

$$(\varepsilon v)_t = -\frac{\varepsilon}{2}uu_x - \frac{3}{4}\varepsilon^2u^2u_x + \mu\varepsilon\frac{\alpha}{2}uu_{xxt}$$
$$-\mu\left(\frac{1}{3}-\alpha\right)u_{xxt}$$
$$+\mu\varepsilon\frac{3}{2}\left(\frac{1}{3}-\alpha\right)\left[u_x^2 + uu_{xx}\right]_x$$
$$+\mu^2\alpha\left(\frac{1}{3}-\alpha\right)\partial_x^2(u_{xxx})$$
$$+\mu\varepsilon\left[\left(a+\frac{1}{3}\right)uu_{xx} + \left(b+\frac{1}{2}\right)u_x^2\right]_x$$
$$-\mu^2\left(\frac{1}{45}-c\right)\partial_x^2(u_{xxx}) + O(\mu^3)$$

But

$$\mu\varepsilon\frac{\alpha}{2}uu_{xxt} = -\mu\varepsilon\frac{\alpha}{2}uu_{xxx} + O(\mu^3)$$
$$= -\mu\varepsilon\frac{\alpha}{2}\left[uu_{xx} - \frac{1}{2}u_x^2\right]_x + O(\mu^3)$$

So

$$(\varepsilon v)_t = -\frac{\varepsilon}{2}uu_x - \frac{3}{4}\varepsilon^2u^2u_x - \mu\left(\frac{1}{3}-\alpha\right)u_{xxt}$$
$$+\mu\varepsilon\left[\left(a-2\alpha+\frac{5}{6}\right)uu_{xx} + \left(1+b-\frac{5\alpha}{4}\right)u_x^2\right]_x$$
$$+\mu^2\left(\alpha\left(\frac{1}{3}-\alpha\right)+c-\frac{1}{45}\right)\partial_x^2(u_{xxx}) + O(\mu^3)$$

Substitute $\varepsilon^2(uv)_x$ and $(\varepsilon v)_t$ in $(7)_1$ to get

(14)

$$u_t + u_x + \frac{3}{2}\varepsilon uu_x + \mu\left(\alpha-\frac{1}{3}\right)u_{xxt}$$
$$= \mu\varepsilon\left[\left(\alpha-a-\frac{1}{2}\right)uu_{xx} + \left(-1-b+\frac{5\alpha}{4}\right)u_x^2\right]_x$$
$$+\mu^2\left(-\alpha\left(\frac{1}{3}-\alpha\right)-c+\frac{1}{45}\right)\partial_x^2(u_{xxx}) + O(\mu^3)$$

Compare the equations (11) and (14) to deduce

$$\alpha = \frac{1}{6}; a = -\frac{1}{6}; b = -\frac{19}{48}; c = -\frac{1}{360}.$$

Doing some calculations one can get

$$\beta = -\frac{5}{12}; \gamma = -\frac{21}{24}; \delta = -\frac{11}{360}.$$

2.2 Equation on the surface elevation.

In this subsection, we are going to derive the extended KdV equation on the surface elevation ζ as a rigorous verification of a previous work [8], where it was imposed formally.
For this purpose, let us introduce the following equation

(15)

$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \mu\alpha\zeta_{xxx}$$
$$= \mu\varepsilon[\beta\zeta\zeta_{xxx} + \gamma\zeta_x\zeta_{xx}] + \mu^2\delta\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

where α, β, γ, δ are parameters in $\mathbb{R}$, and will be determined next.

- **Step1:**
  From (15) one can get

  $$\zeta_x = -\zeta_t - \frac{3}{2}\varepsilon\zeta\zeta_x + \frac{3}{8}\varepsilon^2\zeta^2\zeta_x - \mu\alpha\zeta_{xxx} + O(\varepsilon,\mu)$$

  $$\zeta_{xxx} = -\zeta_{xxt} - \frac{3}{2}\varepsilon\partial_x^2(\zeta\zeta_x) + \frac{3}{8}\varepsilon^2\partial_x^2(\zeta^2\zeta_x)$$
  $$- \mu\alpha\partial_x^2(\zeta_{xxx}) + O(\varepsilon,\mu)$$

  Substitute ζ_{xxx} in (15) to get

  $$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x - \mu\alpha\zeta_{xxt}$$
  $$- \frac{3}{2}\alpha\mu\varepsilon\partial_x^2(\zeta\zeta_x) - \mu^2\alpha^2\partial_x^2(\zeta_{xxx})$$
  $$= \mu\varepsilon[\beta\zeta\zeta_{xxx} + \gamma\zeta_x\zeta_{xx}] + \mu^2\delta\partial_x^2(\zeta_{xxx})$$
  $$+ O(\mu^3)$$

  Use the fact that

  $$\beta\zeta\zeta_{xxx} + \gamma\zeta_x\zeta_{xx} = \left[\beta\zeta\zeta_{xxx} + \frac{\gamma-\beta}{2}\zeta_x^2\right]_x$$

  To get the following equation

  (16)
  $$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x - \mu\alpha\zeta_{xxt}$$
  $$= \mu\varepsilon[a\zeta\zeta_{xx} + b\zeta_x^2]_x + \mu^2c\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

  where
  $$a = \frac{3\alpha}{2} + \beta; \quad b = \frac{3\alpha+\gamma-\beta}{2}; \quad c = \delta + \alpha^2.$$

- **Step2:**
  Let w be a smooth enough function such that
  $u = \zeta + \varepsilon w$.
  Compute

  $$(1+\varepsilon\zeta)u = (1+\varepsilon\zeta)(\zeta+\varepsilon w) = \zeta + \varepsilon\zeta^2 + (1+\varepsilon\zeta)(\varepsilon w)$$

  Then the first equation of (7) gives

  $$\zeta_t + \zeta_x + 2\varepsilon\zeta\zeta_x + [(1+\varepsilon\zeta)(\varepsilon w)]_x = 0$$

  Use (16) to compute

  $$[(1+\varepsilon\zeta)(\varepsilon w)]_x + \zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x$$
  $$- \mu\alpha\zeta_{xxt} - \mu\varepsilon[a\zeta\zeta_{xx} + b\zeta_x^2]_x - \mu^2c\partial_x^2(\zeta_{xxx})$$
  $$= \zeta_t + \zeta_x + 2\varepsilon\zeta\zeta_x + [(1+\varepsilon\zeta)(\varepsilon w)]_x - \frac{\varepsilon}{2}\zeta\zeta_x$$
  $$- \frac{3}{8}\varepsilon^2\zeta^2\zeta_x - \mu\alpha\zeta_{xxt} - \mu\varepsilon[a\zeta\zeta_{xx} + b\zeta_x^2]_x - \mu^2c\partial_x^2(\zeta_{xxx})$$
  $$+ O(\mu^3)$$

Then

$$[(1+\varepsilon\zeta)(\varepsilon w)]_x = -\frac{\varepsilon}{2}\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x - \mu\alpha\zeta_{xxt}$$
$$- \mu\varepsilon[a\zeta\zeta_{xx} + b\zeta_x^2]_x - \mu^2c\partial_x^2(\zeta_{xxx})$$
$$+ O(\mu^3)$$

Hence

(17)
$$(1+\varepsilon\zeta)(\varepsilon w) = -\frac{\varepsilon}{4}\zeta^2 - \frac{1}{8}\varepsilon^2\zeta^3 - \mu\alpha\zeta_{xt}$$
$$- \mu\varepsilon[a\zeta\zeta_{xx} + b\zeta_x^2] - \mu^2c\partial_x(\zeta_{xxx})$$

- **Step3:**
  Here we will use the second equation of (7).
  Since we need to keep the terms ζ_t and ζ_x, and
  since the latter derived term is $(1+\varepsilon\zeta)(\varepsilon w)$ and
  not εw, we will multiply the second equation of
  (7) by $(1+\varepsilon\zeta)$. Hence, one can get

  (18)
  $$(1+\varepsilon\zeta)u_t + \zeta_x + \varepsilon\zeta\zeta_x + (1+\varepsilon\zeta)\varepsilon u u_x$$
  $$= \frac{\mu}{3}\zeta_{xxt} + \frac{\mu}{3}(\varepsilon w)_{xxt} + \frac{\mu\varepsilon}{3}\zeta\zeta_{xxt}$$
  $$- \mu\varepsilon\left[\frac{1}{2}u_x^2 + \frac{1}{3}uu_{xx}\right]_x + \frac{\mu^2}{45}\partial_x^2(\zeta_{xxt}) + O(\mu^3)$$

For the right hand side of (18):

- $$\frac{\mu}{3}(\varepsilon w) = -\frac{\mu\varepsilon}{12}\zeta^2 - \mu^2\frac{\alpha}{3}\zeta_{xt} + O(\mu^3)$$
  $$\frac{\mu}{3}(\varepsilon w)_{xxt} = -\frac{\mu\varepsilon}{12}\partial_{xxt}(\zeta^2) - \mu^2\frac{\alpha}{3}\partial_{xxt}(\zeta_{xt}) + O(\mu^3)$$

Recall that (16) gives:

$$\zeta_t = -\zeta_x + O(\varepsilon,\mu)$$
$$\zeta_{xtt} = -\zeta_{xxt} + O(\varepsilon,\mu)$$

Also

$$\partial_{xxt}(\zeta_{xt}) = \partial_x^2(\zeta_{xxx}) + O(\varepsilon,\mu)$$

Therefore

$$\frac{\mu}{3}(\varepsilon w)_{xxt} = \frac{\mu\varepsilon}{6}[\zeta_x^2 + \zeta\zeta_{xx}]_x - \mu^2\frac{\alpha}{3}\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

Now, use

$$\zeta_{xxt} = -\zeta_{xxx} + O(\varepsilon, \mu)$$

to get

- $$\frac{\mu\varepsilon}{3}\zeta\zeta_{xxt} = -\frac{\mu\varepsilon}{3}\zeta\zeta_{xxx} + O(\mu^3)$$
$$= -\frac{\mu\varepsilon}{3}\left[\zeta\zeta_{xx} - \frac{1}{2}\zeta_x^2\right]_x + O(\mu^3)$$

Obviously, one gets

$$\mu\varepsilon\left[\frac{1}{2}u_x^2 + \frac{1}{3}uu_{xx}\right]_x = \mu\varepsilon\left[\frac{1}{2}\zeta_x^2 + \frac{1}{3}\zeta\zeta_{xx}\right]_x + O(\mu^3)$$

Using the identity

$$\partial_x^2(\zeta_{xxt}) = -\partial_x^2(\zeta_{xxx}) + O(\varepsilon, \mu)$$

one can deduces

$$\frac{\mu^2}{45}\partial_x^2(\zeta_{xxt}) = -\frac{\mu^2}{45}\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

Gathering all the informations found above in the right hand side of (18) we get

(19)
$$\frac{\mu}{3}\zeta_{xxt} + \frac{\mu}{3}(\varepsilon w)_{xxt} + \frac{\mu\varepsilon}{3}\zeta\zeta_{xxt} - \mu\varepsilon\left[\frac{1}{2}u_x^2 + \frac{1}{3}uu_{xx}\right]_x$$
$$+ \frac{\mu^2}{45}\partial_x^2(\zeta_{xxt}) = \frac{\mu}{3}\zeta_{xxt} - \mu\varepsilon\left[\frac{1}{2}\zeta\zeta_{xx} + \frac{1}{6}\zeta_x^2\right]_x$$
$$- \mu^2\left(\frac{\alpha}{3} + \frac{1}{45}\right)\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

For the left hand side of (18):

- $$\varepsilon uu_x = \varepsilon\zeta\zeta_x + [(\varepsilon w)(\varepsilon\zeta)]_x + O(\mu^3)$$
$$(1 + \varepsilon\zeta)(\varepsilon uu_x) = \varepsilon\zeta\zeta_x + [(\varepsilon w)(\varepsilon\zeta)]_x + \varepsilon^2\zeta^2\zeta_x + O(\mu^3)$$

We have

$$(\varepsilon w)(\varepsilon\zeta) = -\frac{\varepsilon^2}{4}\zeta^3 - \mu\varepsilon\alpha\zeta\zeta_{xt} + O(\mu^3)$$

$$[(\varepsilon w)(\varepsilon\zeta)]_x = -\frac{3\varepsilon^2}{4}\zeta^2\zeta_x - \mu\varepsilon\alpha(\zeta\zeta_{xt})_x + O(\mu^3)$$
$$= -\frac{3\varepsilon^2}{4}\zeta^2\zeta_x + \mu\varepsilon\alpha(\zeta\zeta_{xx})_x + O(\mu^3)$$

$$(1 + \varepsilon\zeta)(\varepsilon uu_x) = \varepsilon\zeta\zeta_x + \frac{\varepsilon^2}{4}\zeta^2\zeta_x + \mu\varepsilon\alpha(\zeta\zeta_{xx})_x + O(\mu^3)$$

Next,

- $$(1 + \varepsilon\zeta)u_t = (1 + \varepsilon\zeta)(\zeta_t + (\varepsilon w)_t)$$
$$= \zeta_t + (\varepsilon w)_t + \varepsilon\zeta\zeta_t + \varepsilon^2\zeta w_t$$
$$= \zeta_t + \varepsilon\zeta\zeta_t + [(1 + \varepsilon\zeta)(\varepsilon w)]_t - \varepsilon^2\zeta_t w$$
$$= \zeta_t + \varepsilon\zeta\zeta_t + [(1 + \varepsilon\zeta)(\varepsilon w)]_t + \varepsilon^2\zeta_x w$$
$$+ O(\mu^3)$$

Multiply (17) by $\varepsilon\zeta_x$ to get

$$\varepsilon^2\zeta_x w = -\frac{\varepsilon^2}{4}\zeta^2\zeta_x - \mu\varepsilon\alpha(\zeta_x\zeta_{xt}) + O(\mu^3)$$
$$= -\frac{\varepsilon^2}{4}\zeta^2\zeta_x + \mu\varepsilon\frac{\alpha}{2}\left[\zeta_x^2\right]_x + O(\mu^3)$$

Also

$$\varepsilon\zeta\zeta_t + [(1 + \varepsilon\zeta)(\varepsilon w)]_t = \frac{\varepsilon}{2}\zeta\zeta_t - \frac{3}{8}\varepsilon^2\zeta^2\zeta_t$$
$$- \mu\alpha\zeta_{xtt} - \mu\varepsilon\left[a\zeta\zeta_{xx} + b\zeta_x^2\right]_t - \mu^2 c\partial_{xt}(\zeta_{xxx})$$
$$+ O(\mu^3)$$

Recall that

$$\zeta_t = -\zeta_x - \frac{3}{2}\varepsilon\zeta\zeta_x + \mu\alpha\zeta_{xxt} + O(\varepsilon, \mu)$$

To compute

$$\partial_{xt}(\zeta_{xxx}) = -\partial_x^2(\zeta_{xxx}) + O(\varepsilon, \mu)$$
$$\partial_{xt}(\zeta_{xxt}) = \partial_x^2(\zeta_{xxx}) + O(\varepsilon, \mu)$$
$$\partial_{xt}(\zeta\zeta_x) = \partial_t[\zeta_x^2 + \zeta\zeta_{xx}] = -\left[\zeta_x^2 + \zeta\zeta_{xx}\right]_x + O(\varepsilon, \mu)$$

And

$$\zeta_{xtt} = -\zeta_{xxt} - \frac{3}{2}\varepsilon\partial_{xt}(\zeta\zeta_x) + \mu\alpha\partial_{xt}(\zeta_{xxt}) + O(\varepsilon, \mu)$$
$$= -\zeta_{xxt} + \frac{3}{2}\varepsilon\left[\zeta_x^2 + \zeta\zeta_{xx}\right]_x + \mu\alpha\partial_x^2(\zeta_{xxx}) + O(\varepsilon, \mu)$$

Therefore

$$\varepsilon\zeta\zeta_t + [(1 + \varepsilon\zeta)(\varepsilon w)]_t = -\frac{\varepsilon}{2}\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \mu\alpha\zeta_{xxt}$$
$$+ \mu\varepsilon\frac{\alpha}{2}\zeta\zeta_{xxt} - \mu\varepsilon\left[\frac{3\alpha}{2}\zeta_x^2 + \frac{3\alpha}{2}\zeta\zeta_{xx}\right]_x$$
$$+ \mu\varepsilon\left[a\zeta\zeta_{xx} + b\zeta_x^2\right]_x$$
$$+ \mu^2(c - \alpha^2)\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

Hence

$$(1 + \varepsilon\zeta)u_t + \zeta_x + \varepsilon\zeta\zeta_x + (1 + \varepsilon\zeta)\varepsilon u u_x$$

$$(20) \quad \begin{aligned} &= \zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \mu\alpha\zeta_{xxt} \\ &\quad + \mu\varepsilon\left[(a-\alpha)\zeta\zeta_{xx} + \left(b - \frac{3\alpha}{4}\right)\zeta_x^2\right]_x \\ &\quad + \mu^2\left(c - \alpha^2\right)\partial_x^2(\zeta_{xxx}) + O(\mu^3) \end{aligned}$$

Eventually, gathering (20) and (19) we get the equation

(21)

$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \mu\left(\alpha - \frac{1}{3}\right)\zeta_{xxt}$$

$$= \mu\varepsilon\left[\left(\alpha - a - \frac{1}{2}\right)\zeta\zeta_{xx} + \left(\frac{3\alpha}{4} - b - \frac{1}{6}\right)\zeta_x^2\right]_x$$

$$+ \mu^2\left(\alpha^2 - c - \frac{\alpha}{3} - \frac{1}{45}\right)\partial_x^2(\zeta_{xxx}) + O(\mu^3)$$

Comparing (16) and (21) one gets

$$\alpha = \frac{1}{6}; \quad a = -\frac{1}{6}; \quad b = -\frac{1}{48}; \quad c = -\frac{1}{40}.$$

Do some calculations to deduce

$$\beta = -\frac{5}{12}; \quad \gamma = -\frac{23}{24}; \quad \delta = -\frac{19}{360}.$$

3 An explicit solution of the extended KdV equation

We will use the sine-cosine method to solve the latter equation explicitly. It will need 3 steps.

- **Step1:** Use the traveling wave transformation $\theta = x - ct$. Then

$$u_t = -cu'$$
$$u_x = u'$$
$$u_{xx} = u''$$

Then we get

$$(1-c)u' + \frac{3}{2}\varepsilon u u' + \frac{\mu}{6}u''' + \mu\varepsilon\left[\frac{5}{12}uu''' + \frac{21}{24}u'u''\right]$$

$$+ \frac{11}{360}\mu^2 u^{(5)} = O(\mu^3)$$

Integrate with respect to θ to get

(22)

$$(1-c)u + \frac{3}{4}\varepsilon u^2 + \frac{\mu}{6}u'' + \mu\varepsilon\left[\frac{5}{12}uu'' + \frac{11}{48}(u')^2\right]$$

$$+ \frac{11}{360}\mu^2 u^{(4)} = O(\mu^3)$$

- **Step2:** Suppose that the solution of the equation is:

$$u(\theta) = p\cos(q\theta)^s$$

where $p \neq 0$, $q \neq 0$, $s \neq 0$, and $|\theta| \leq \frac{\pi}{2q}$.
Differentiate with respect to θ

$$u'(\theta) = -pqs\sin(q\theta)\cos(q\theta)^{s-1}$$
$$u''(\theta) = pq^2s(s-1)\cos(q\theta)^{s-2} - pq^2s^2\cos(q\theta)^s$$
$$u'''(\theta) = -pq^3s(s-1)(s-2)\sin(q\theta)\cos(q\theta)^{s-3}$$
$$\qquad + pq^3s^3\sin(q\theta)\cos(q\theta)^{s-1}$$
$$u^{(4)}(\theta) = pq^4s^4\cos(q\theta)^s$$
$$\qquad + pq^4s(s-1)(s-2)(s-3)\cos(q\theta)^{s-4}$$
$$\qquad - 2pq^4s(s-1)(s^2-2s+2)\cos(q\theta)^{s-2}$$

- Step3: Put $u, u', u'', u^{(4)}$ in (22) to get

$$(1-c)p\cos(q\theta)^s + \frac{3}{4}\varepsilon p^2\cos(q\theta)^{2s}$$

$$+ \frac{\mu}{6}pq^2s(s-1)\cos(q\theta)^{s-2} - \frac{\mu}{6}pq^2s^2\cos(q\theta)^s$$

$$+ \frac{5}{12}\mu\varepsilon p\cos(q\theta)^s\left[pq^2s(s-1)\cos(q\theta)^{s-2} - pq^2s^2\cos(q\theta)^s\right]$$

$$+ \frac{11}{48}\mu\varepsilon p^2q^2s^2\sin(q\theta)^2\cos(q\theta)^{2s-2} + \frac{11}{360}\mu^2pq^4s^4\cos(q\theta)^s$$

$$+ \frac{11}{360}\mu^2pq^4s(s-1)(s-2)(s-3)\cos(q\theta)^{s-4}$$

$$- \frac{11}{180}pq^4s(s-1)(s^2-2s+2)\cos(q\theta)^{s-2}$$

$$= O(\mu^3)$$

Use $\sin(q\theta)^2 = 1 - \cos(q\theta)^2$ to get

$(1-c)p\cos(q\theta)^s + \dfrac{3}{4}\varepsilon p^2\cos(q\theta)^{2s}$

$+ \dfrac{\mu}{6}pq^2 s(s-1)\cos(q\theta)^{s-2} - \dfrac{\mu}{6}pq^2 s^2\cos(q\theta)^s$

$+ \dfrac{5}{12}\mu\varepsilon p^2 q^2 s(s-1)\cos(q\theta)^{2s-2}$

$- \dfrac{5}{12}\mu\varepsilon p^2 q^2 s^2\cos(q\theta)^{2s}$

$+ \dfrac{11}{48}\mu\varepsilon p^2 q^2 s^2\cos(q\theta)^{2s-2}$

$- \dfrac{11}{48}\mu\varepsilon p^2 q^2 s^2\sin(q\theta)^2\cos(q\theta)^{2s}$

$+ \dfrac{11}{360}\mu^2 pq^4 s^4\cos(q\theta)^s$

$+ \dfrac{11}{360}\mu^2 pq^4 s(s-1)(s-2)(s-3)\cos(q\theta)^{s-4}$

$- \dfrac{11}{180}pq^4 s(s-1)(s^2-2s+2)\cos(q\theta)^{s-2} = O(\mu^3)$

1. Higher order of power = Lower order of power of cos:

$$2s = s - 4 \implies s = -4$$

2. Terms of $\cos(q\theta)^{s-2}$:

$$\dfrac{\mu}{6}pq^2 s(s-1) = \dfrac{11}{180}\mu^2 pq^4 s(s-1)(s^2-2s+2)$$

Use the fact $s = -4$ and $p \neq 0$ and $q \neq 0$ to get

$$q^2 = \dfrac{15}{143\mu}$$

3. Terms of $\cos(q\theta)^s$:

$$(1-c)p = \dfrac{\mu}{6}pq^2 s^2 - \dfrac{11}{360}\mu^2 pq^4 s^4$$

After some calculations, using $s = -4$ and $p \neq 0$ and the term of q^2 we get

$$c = \dfrac{1499}{1859}$$

4. Terms of $\cos(q\theta)^{2s}$: Since $2s = s - 4$ we deduce that the $\cos(q\theta)^{s-4}$-terms are equal to $\cos(q\theta)^{2s}$ ones. So we deduce

$$\dfrac{3}{4}\varepsilon p^2 + \dfrac{11}{360}\mu^2 pq^4 s(s-1)(s-2)(s-3)$$
$$= \dfrac{5}{12}\mu\varepsilon p^2 q^2 s^2 + \dfrac{11}{48}\mu\varepsilon p^2 q^2 s^2$$

Do some calculations to get

$$p = \dfrac{2100}{2483\varepsilon}$$

Hence the solution of (8) is

$$u(x,t) = \dfrac{2100}{2483\varepsilon}\sec^4\left(\sqrt{\dfrac{15}{143\mu}}\left(x - \dfrac{1499t}{1859}\right)\right)$$

or

$$u(x,t) = \dfrac{2100}{2483\varepsilon}\sec^4\left(-\sqrt{\dfrac{15}{143\mu}}\left(x - \dfrac{1499t}{1859}\right)\right)$$

4   Conclusion

The ocean energy sector is all about innovation and has been evidence of some notable progress. Many studies in this domain have been done, and many questions have given insight into new studies. After providing a rigorous derivation of KdV equation on u, and a rigorous verification of one on ζ, for flat bottom, new lights have been casted on some future researches, the extended KdV equation on the velocity could be derived with the presence of surface tension effect, this study could be done using the pseudo-differential operator theory. Also, a numerical framework could be done using the finite element method. Finally, we would study its well-posedness employing the modified-energy method.

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