Integral contravariant form of the Navier-Stokes equations

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Abstract: - An original integral formulation of the three-dimensional contravariant Navier-Stokes equations, devoid of the Christoffel symbols, in general time-dependent curvilinear coordinates is presented. The proposed integral form is obtained from the time derivative of the momentum of a material fluid volume and from the Leibniz rule of integration applied to a control volume that moves with a velocity which is different from the fluid velocity. The proposed integral formulation has general validity and makes it possible to obtain, with simple passages, the complete differential form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system. The integral form, devoid of the Christoffel symbols, proposed in this work is used in order to realise a three-dimensional non-hydrostatic numerical model for free surface flows, which is able to simulate the discontinuities in the solution related to the wave breaking on domains that reproduce the complex geometries of the coastal regions. The proposed model is validated by reproducing experimental test cases on time dependent curvilinear grids.

Key-Words: - Three-dimensional, Navier-Stokes equations, contravariant, integral form, time-dependent curvilinear coordinate system, shock-capturing scheme.

1 Introduction

The study of the fluid motion in three-dimensional form on domains characterised by complex geometries can be carried out by using boundary conforming curvilinear coordinate systems and by expressing the governing equations in contravariant formulation. In literature, several authors have used the contravariant formulation of the Navier-Stokes equations in fixed curvilinear coordinates in order to study flows in complex geometries [1,2]. When dealing with complex geometries which vary in time, some authors have used systems of moving curvilinear coordinates (which are time dependent): in this approach, the irregular time varying physical domain, whose boundaries are represented by time dependent curvilinear surfaces, is transformed into a uniform fixed computational domain [3,4]. The simulation of three-dimensional flow fields and free surface elevation in coastal regions characterized by complex morphology requires numerical models that make use of unstructured grid [5-6]. Rosenfeld and Kwak [7] used a contravariant form of the Navier-Stokes equations in moving curvilinear coordinates for the simulation of flows in cavities with variable geometry. A complete differential contravariant formulation of the Navier-Stokes equations in time dependent curvilinear coordinates was obtained by Luo and Bewley [8], who used a tensorial approach. The latter have obtained the differential form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system, starting from the intrinsic derivative of contravariant vectors in a moving frame. Recently, several authors have used the tensorial approach of Luo and Bewley [8] in order to express the differential motion equations in covariant form and contravariant form in time dependent curvilinear coordinate systems. The differential form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system obtained by Luo and Bewley [8] includes the covariant derivatives of contravariant vectors. Such covariant derivatives imply the presence of the Christoffel symbols. These terms are extra source terms that prevent the convective terms of the motion equations from being expressed in conservative form. In order to obtain a numerical model for the solution of conservation laws which is able to converge to the weak solution, it is necessary to express the convective terms of the differential motion equations in conservative form or express the motion equations directly in integral form [9].

In this work we propose an alternative approach to that proposed by Luo and Bewley [8], whereby it is possible to express the momentum equation in an integral contravariant form, in which the Christoffel terms are absent, in a time dependent curvilinear coordinate system [10]. This approach is based on the definition of the momentum time derivative of a fluid material volume and on the Leibniz rule of integration for a volume which moves with a velocity that is different from the fluid velocity. The resulting equation represents the general integral contravariant formulation of the momentum equation in a time dependent curvilinear coordinate system. Indeed, taking the limit as the volume approaches zero, with simple passages we obtain the complete differential formulation of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system, which is the same as the one obtained by Luo and Bewley [8].

The proposed integral contravariant momentum equation, devoid of the Christoffel symbols, is used to realise a three-dimensional non-hydrostatic numerical model for free surface flows, which is able to simulate the wave motion and the discontinuities in the solution, related to the wave breaking, on domains that reproduce the complex geometries of the coastal regions. The physical domain, which reproduce the geometry of the coastal region and the free surface variations along the vertical direction, is described by curvilinear boundary conforming time dependent coordinates.

In the proposed model, the integral contravariant form of the momentum and continuity equations are solved by a finite volume shock capturing scheme, which uses an HLL approximate Riemann solver [11] which is proven to be effective to simulate shocks both in depth-averaged [12-17] and in fully three-dimensional free surface flows [10,18-21].

The paper is organised as follows. In Section 2, the integral and contravariant formulation of the motion equations in a system of time varying curvilinear coordinates, devoid of the Christoffel symbols is presented. In Section 3 the procedure is shown by which, starting from the proposed integral formulation, the differential form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system is achieved.

In Section 4, we propose an original contravariant integral formulation of the three-dimensional motion equations for non-hydrostatic free surface flows in a time dependent curvilinear coordinate system. In Section 5 the results are shown and discussed. Conclusions are drawn in Section 6.

2 Derivation of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system

We consider a time-dependent transformation, $x^i = x^i(\xi^1, \xi^2, \xi^3, \tau)$, $t = \tau$, from the Cartesian coordinate system (x^1, x^2, x^3, t) to the curvilinear coordinate system, $(\xi^1, \xi^2, \xi^3, \tau)$, and the inverse transformation, $\xi^i = \xi^i(x^1, x^2, x^3, t)$, $\tau = t$. Let $\vec{g}_{(l)} = \partial\vec{x}/\partial\xi^l$ be the covariant base vectors and $\vec{g}^{(l)} = \partial\xi^l/\partial\vec{x}$ the contravariant base vectors and let us indicate by the point "·" the scalar product between vectors defined in the Cartesian coordinate system. The metric tensor and its inverse are defined, respectively, by $g_{lm} = \vec{g}_{(l)} \cdot \vec{g}_{(m)}$ and $g^{lm} = \vec{g}^{(l)} \cdot \vec{g}^{(m)}$ $(l, m = 1{,}3)$. The Jacobian of the transformation is given by $\sqrt{g} = \sqrt{|g_{lm}|}$. The transformation relationships between the components of the generic vector $\vec{b}$ in the Cartesian coordinate system and its contravariant and covariant components, b^l and b_l, in the curvilinear coordinate system are given by

$$b^l = \vec{g}^{(l)} \cdot \vec{b} \quad ; \quad \vec{b} = b^l \vec{g}_{(l)}$$
$$b_l = \vec{g}_{(l)} \cdot \vec{b} \quad ; \quad \vec{b} = b_l \vec{g}^{(l)} \tag{1}$$

In order to express the integral formulation of the contravariant momentum equation in a time dependent coordinate system, let us start from the contravariant expression of the three-dimensional Leibniz integral rule. Let ρ and u^l be, respectively, the density and the l^{th} $(l = 1{,}3)$ contravariant component of the fluid velocity vector. Let $\Delta V_1(\tau)$ be a time-varying control volume bounded by a surface, of area $\Delta A_1(\tau)$, every point of which moves with a velocity that is different from the fluid velocity. By using the three-dimensional Leibniz integral rule, the time derivative of the integral of ρu^l over the volume $\Delta V_1(\tau)$, in contravariant form, can be expressed as

$$\frac{d}{d\tau}\int_{\Delta V_1(\tau)} \rho u^l dV_1 = \int_{\Delta V_1(\tau)} \frac{\partial \rho u^l}{\partial \tau} dV_1 + \int_{\Delta A_1(\tau)} \rho u^l v^m \tilde{n}_m \, dA_1 \tag{2}$$

where $\tilde{n}_m (m = 1,3)$ is the outward unit vector normal to the surface of area $\Delta A_1(\tau)$ and v^m is the m^{th} $(m = 1,3)$ contravariant component of the velocity vector with which the points belonging to the surface of area $\Delta A_1(\tau)$ move.

Let us consider a fluid material volume, i.e. a time-varying volume which moves with the fluid and always encloses the same fluid particles. Let u^l $(l = 1,3)$ be the contravariant components of the velocity vector with which the above particles move. Let $\Delta V(\tau)$ be a time-varying control volume that at instant τ coincides with the above material volume and that is delimited by a surface of area $\Delta A(\tau)$ every point of which moves with the same velocity of the fluid. It is known that the time derivative of the integral of ρu^l over the above fluid material volume (material derivative), $\frac{D}{D\tau}\int_{\Delta V(\tau)} \rho u^l dV$, in contravariant form, is expressed as

$$\frac{D}{D\tau}\int_{\Delta V(\tau)} \rho u^l dV = \int_{\Delta V(\tau)} \frac{\partial \rho u^l}{\partial \tau} dV + \int_{\Delta A(\tau)} \rho u^l u^m n_m \, dA \tag{3}$$

where $n_m (m = 1,3)$ is the outward unit vector normal to the surface of area $\Delta A(\tau)$; u^m is the m^{th} $(m = 1,3)$ contravariant component of the velocity vector with which the points belonging to the surface of area $\Delta A(\tau)$ move and that coincides with the contravariant component of the fluid velocity vector. It is assumed that at instant τ, $\Delta V_1(\tau) = \Delta V(\tau)$. By replacing the first term on the right-hand side of Eq. 3 by the term $\int_{\Delta V_1(\tau)} \frac{\partial \rho u^l}{\partial \tau} dV_1$ extracted from the right-hand side of Eq. 2, Eq. 3 becomes

$$\frac{D}{D\tau}\int_{\Delta V(\tau)} \rho u^l dV = \frac{d}{d\tau}\int_{\Delta V(\tau)} \rho u^l dV + \int_{\Delta A(\tau)} \rho u^l (u^m - v^m) n_m dA \tag{4}$$

The right-hand side of Eq. 4 represents, in contravariant form, the expression of the time derivative of the integral of ρu^l over a material volume (material derivative), which is valid in the case of a control volume whose boundary surface points move with a velocity, v^m, that is different from the fluid velocity, u^m. By adopting the same control volume, $\Delta V(\tau)$, the expression, in contravariant form, of the time derivative of the integral of ρ over the fluid material volume reads

$$\frac{D}{D\tau}\int_{\Delta V(\tau)} \rho dV = \frac{d}{d\tau}\int_{\Delta V(\tau)} \rho dV + \int_{\Delta A(\tau)} \rho (u^m - v^m) n_m dA \tag{5}$$

In this work, Eqs. 4 and 5 are used to deduce the integral form of the contravariant Navier-Stokes equations in a time dependent coordinate system. By equating to zero the right-hand side of Eq. 5, the following integral contravariant form of the continuity equation is obtained

$$\frac{d}{d\tau}\int_{\Delta V(\tau)} \rho dV + \int_{\Delta A(\tau)} \rho (u^m - v^m) n_m dA = 0 \tag{6}$$

From a general point of view, in order to express the momentum conservation law in integral form, the rate of change of the momentum of a material volume and the total net force must be projected in a physical direction. The direction in space of a given curvilinear coordinate line changes, in contrast with the Cartesian case. Thus, the volume integral of the projection of the momentum equation onto a curvilinear coordinate line has no physical meaning, since it does not represent the volume integral of the projection of the aforementioned equation in a physical direction. We take a constant parallel vector field λ_l and equate the rate of change of the momentum of a material volume, expressed by the right-hand side of Eq. 4, to the total net force in this direction

$$\frac{d}{d\tau}\int_{\Delta V(\tau)} \rho u^l \lambda_l dV + \int_{\Delta A(\tau)} \rho u^l (u^m - v^m) \lambda_l n_m dA = \int_{\Delta V(\tau)} \rho f^l \lambda_l dV + \int_{\Delta A(\tau)} T^{lm} \lambda_l n_m dA \tag{7}$$

where f^l $(l = 1,3)$ represents the external body forces per unit mass vector and T^{lm} is the stress tensor. As a parallel vector field, we choose the one which is normal to the coordinate line on which the ξ^l coordinate is constant at point $P_0 \in \Delta V$. We indicate by ξ_0^1, ξ_0^2 and ξ_0^3 the coordinates of P_0. The contravariant base vector at point P_0, indicated by $\vec{g}^{(l)}(\xi_0^1, \xi_0^2, \xi_0^3)$, is, by definition, normal to the coordinate line on which ξ^l is constant and is used in this work to identify the constant parallel vector field. Let $\lambda_k(\xi^1, \xi^2, \xi^3)$ be the covariant component of $\vec{g}^{(l)}(\xi_0^1, \xi_0^2, \xi_0^3)$, given by

$$\lambda_k(\xi^1, \xi^2, \xi^3) = \vec{g}^{(l)}(\xi_0^1, \xi_0^2, \xi_0^3) \cdot \vec{g}_{(k)}(\xi^1, \xi^2, \xi^3) \tag{8}$$

For the sake of brevity, we indicate $\vec{\tilde{g}}^{(l)} = \vec{g}^{(l)}(\xi_0^1, \xi_0^2, \xi_0^3)$ and $\vec{g}_{(k)} = \vec{g}_{(k)}(\xi^1, \xi^2, \xi^3)$. By introducing Eq. 8 into Eq. 7 we obtain

$$\frac{d}{d\tau}\int_{\Delta V(\tau)} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k dV + \int_{\Delta A(\tau)} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k (u^m - v^m) n_m dA = \int_{\Delta V(\tau)} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho f^k dV + \int_{\Delta A(\tau)} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} T^{km} n_m dA \quad (9)$$

Let us introduce a restrictive condition on the control volume $\Delta V(\tau)$: in the following, $\Delta V(\tau)$ must be considered as the volume of a physical space that is bounded by surfaces lying on the curvilinear coordinate surfaces. In the curvilinear coordinate system, the aforementioned volume is, $\Delta V(\tau) = \int_{\Delta V_0} \sqrt{g} d\xi^1 d\xi^2 d\xi^3$, where ΔV_0 indicates the corresponding volume in the transformed space, which is defined as $\Delta V_0 = \Delta\xi^1 \Delta\xi^2 \Delta\xi^3$.

Analogously, in the curvilinear coordinate system, the area of a surface of the physical space that lies on the coordinate surface in which ξ^α is constant is, $\Delta A^\alpha(\tau) = \int_{\Delta A_0^\alpha} \left|\vec{g}_{(\beta)} \wedge \vec{g}_{(\gamma)}\right| d\xi^\beta d\xi^\gamma$, where ΔA_0^α indicates the corresponding area in the transformed space which is defined as, $\Delta A_0^\alpha = \Delta\xi^\beta \Delta\xi^\gamma$. It must be noted that the volume $\Delta V(\tau)$ and the surfaces $\Delta A^\alpha(\tau)$ are functions of time, because they are expressed as functions of the base vectors, $\vec{g}_{(l)}$, and the Jacobian of the transformation, $\sqrt{g}$, whose values change over time as the curvilinear coordinates follow the displacements of the free surface. Conversely, the volume ΔV_0 and the areas ΔA_0^α are not time dependent. By adopting the volume $\Delta V(\tau)$ (defined above) as control volume, in the transformed space, the integral Eq. 9 reads

$$\frac{d}{d\tau}\int_{\Delta V_0} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k \sqrt{g}\right) d\xi^1 d\xi^2 d\xi^3 + \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A_0^{\alpha+}} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k (u^\alpha - v^\alpha)\sqrt{g}\right) d\xi^\beta d\xi^\gamma - \int_{\Delta A_0^{\alpha-}} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k (u^\alpha - v^\alpha)\sqrt{g}\right) d\xi^\beta d\xi^\gamma \right\} = \int_{\Delta V_0} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho f^k \sqrt{g}\right) d\xi^1 d\xi^2 d\xi^3 + \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A_0^{\alpha+}} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} T^{k\alpha} \sqrt{g}\right) d\xi^\beta d\xi^\gamma - \int_{\Delta A_0^{\alpha-}} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} T^{k\alpha} \sqrt{g}\right) d\xi^\beta d\xi^\gamma \right\} \quad (10)$$

where $\Delta A_0^{\alpha+}$ and $\Delta A_0^{\alpha-}$ indicate the contour surfaces of the volume ΔV_0 on which ξ^α is constant and which are located at the larger and at the smaller value of ξ^α respectively. Here the indexes α, β and γ are cyclic. By adopting the same control volume, $\Delta V(\tau)$, the integral contravariant continuity Eq. 6 reads

$$\frac{d}{d\tau}\int_{\Delta V_0} \left(\rho\sqrt{g}\right) d\xi^1 d\xi^2 d\xi^3 + \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A_0^{\alpha+}} \left(\rho(u^\alpha - v^\alpha)\sqrt{g}\right) d\xi^\beta d\xi^\gamma - \int_{\Delta A_0^{\alpha-}} \left(\rho(u^\alpha - v^\alpha)\sqrt{g}\right) d\xi^\beta d\xi^\gamma \right\} = 0 \quad (11)$$

Eqs. 10 and 11 represent an integral form of the contravariant Navier-Stokes equations in a time-dependent curvilinear coordinate system in which the Christoffel symbols are absent.

3 Derivation of the differential form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system

The equation system 10 and 11 represents the general integral form of the Navier-Stokes equations expressed in a time dependent curvilinear coordinate system. Indeed, in this Section it is shown that, by simple passages, from the integral Eqs. 10 and 11 can be directly deduced the differential form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system, that is equal to the one obtained by Luo and Bewley [8].

The derivative with respect to time in the first term of the left-hand side of Eq. 10 can be carried under the integral sign (since, in the transformed space described by the curvilinear coordinates, the volume ΔV_0 over which the integral is calculated is not dependent on time) and the first term of Eq. 10 can be written as

$$\frac{d}{d\tau}\int_{\Delta V_0} \left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k \sqrt{g}\right) d\xi^1 d\xi^2 d\xi^3 = \int_{\Delta V_0} \frac{\partial\left(\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \rho u^k \sqrt{g}\right)}{\partial\tau} d\xi^1 d\xi^2 d\xi^3 \quad (12)$$

By introducing Eq. 12 into Eq. 10, by dividing both sides of Eq. 10 by the volume $\Delta V(\tau)$ and taking the limit as the volume $\Delta V(\tau)$ approaches zero, we obtain the following differential formulation of the contravariant momentum balance equation in a time-dependent curvilinear coordinate system

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^k\sqrt{g}\right)}{\partial\tau}+\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^k(u^\alpha-v^\alpha)\sqrt{g}\right)}{\partial\xi^\alpha}=\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}f^k+\frac{1}{\sqrt{g}}\frac{1}{\rho}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\vec{g}_{(k)}T^{k\alpha}\sqrt{g}\right)}{\partial\xi^\alpha} \tag{13}$$

It must be underlined that Eq. 13 is written in a differential conservative form in which the Christoffel symbols are not present. This differential formulation is general and can be further developed to derive the differential form obtained by Luo and Bewley [8]. By expanding the time derivative, the left-hand side of Eq. 13 reads

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^k\sqrt{g}\right)}{\partial\tau}=\rho u^k\vec{\tilde{g}}^{(l)}\cdot\frac{\partial\,\vec{g}_{(k)}}{\partial\tau}+\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\tau}\rho u^k+\frac{\partial\rho u^k}{\partial\tau}\right) \tag{14}$$

By using the definition of the covariant base vector, $\vec{g}_{(k)}=\partial\vec{x}/\partial\xi^k$, and the properties of the partial derivatives, the term $\frac{\partial\,\vec{g}_{(k)}}{\partial\tau}$ on the right-hand side of Eq. 14 becomes

$$\frac{\partial\,\vec{g}_{(k)}}{\partial\tau}=\frac{\partial}{\partial\tau}\frac{\partial\vec{x}}{\partial\xi^k}=\frac{\partial}{\partial\xi^k}\frac{\partial\vec{x}}{\partial\tau}=\frac{\partial\vec{v}_G}{\partial\xi^k}=\frac{\partial v^j\,\vec{g}_{(j)}}{\partial\xi^k}=\frac{\partial v^j}{\partial\xi^k}\,\vec{g}_{(j)}+v^j\frac{\partial\vec{g}_{(j)}}{\partial\xi^k} \tag{15}$$

In Eq. 20 the definition of the velocity vector of the moving curvilinear coordinates, $\frac{\partial\vec{x}}{\partial\tau}=\vec{v}_G$, and its expression in contravariant components, $\vec{v}_G=v^j\,\vec{g}_{(j)}$, has been used. By recalling that the derivative of the covariant base vector on the right-hand side of Eq. 15 involves the Christoffel symbols, $\frac{\partial\vec{g}_{(j)}}{\partial\xi^k}=\Gamma^r_{jk}\,\vec{g}_{(r)}$ and by using the expression $u^j_{,k}=\frac{\partial u^j}{\partial\xi^k}+u^r\Gamma^j_{rk}$, the right-hand side of Eq. 15 can be written in the form

$$\frac{\partial v^j}{\partial\xi^k}\,\vec{g}_{(j)}+v^j\frac{\partial\vec{g}_{(j)}}{\partial\xi^k}=\frac{\partial v^j}{\partial\xi^k}\,\vec{g}_{(j)}+v^j\Gamma^r_{jk}\,\vec{g}_{(r)}=\left(\frac{\partial v^j}{\partial\xi^k}+v^r\Gamma^j_{rk}\right)\vec{g}_{(j)}=v^j_{,k}\,\vec{g}_{(j)} \tag{16}$$

By using Eqs. 15 and 16, and changing the dummy indexes, the first term on the left-hand side of Eq. 13 reads

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^k\sqrt{g}\right)}{\partial\tau}=\vec{\tilde{g}}^{(l)}\,\vec{g}_{(k)}\rho u^\alpha v^k_{,\alpha}+\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\left(\rho u^k\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\tau}+\frac{\partial\rho u^k}{\partial\tau}\right)=\lambda_k\left(\rho u^\alpha v^k_{,\alpha}+\rho u^k\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\tau}+\frac{\partial\rho u^k}{\partial\tau}\right) \tag{17}$$

The further development of this term can be made by using the geometric identity [8] that imposes the conservation of a generic volume whose boundary surfaces move with velocity $\vec{v}_G$,

$$\frac{d}{d\tau}\int_{\Delta V(\tau)}dV=\int_{\Delta A(\tau)}\vec{v}_G\cdot\vec{n}dA \tag{18}$$

In fact, in the time dependent curvilinear coordinate system, the integral Eq. 18 reads

$$\frac{d}{d\tau}\int_{\Delta V_0}\sqrt{g}d\xi^1d\xi^2\,d\xi^3=\sum_{\alpha=1}^{3}\left\{\int_{\Delta A_0^{\alpha+}}v^\alpha\sqrt{g}d\xi^\beta\,d\xi^\gamma-\int_{\Delta A_0^{\alpha-}}v^\alpha\sqrt{g}d\xi^\beta\,d\xi^\gamma\right\} \tag{19}$$

By carrying the temporal derivative on the left hand side of Eq. 19 under the integral over the volume ΔV_0 (that is independent of time), by dividing both sides of Eq. 18 by the volume $\Delta V(\tau)$ and taking the limit as $\Delta V(\tau)$ approaches zero, we obtain the differential expression of the metric identity that holds in the time dependent curvilinear coordinate systems

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\tau}=\frac{1}{\sqrt{g}}\frac{\partial v^\alpha\sqrt{g}}{\partial\xi^\alpha} \tag{20}$$

By expanding the derivative on the right-hand side of Eq. 20, $\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\xi^\alpha}=\Gamma^i_{i\alpha}$, and by using the definition of covariant derivative, Eq. 20 can be written in the form

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\tau}=\frac{1}{\sqrt{g}}\frac{\partial v^\alpha\sqrt{g}}{\partial\xi^\alpha}=\frac{\partial v^\alpha}{\partial\xi^\alpha}+v^\alpha\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\xi^\alpha}=\frac{\partial v^\alpha}{\partial\xi^\alpha}+v^\alpha\Gamma^i_{i\alpha}=v^\alpha_{,\alpha} \tag{21}$$

By introducing Eq. 21 into Eq. 17, the left-hand side of Eq. 13 can be thus written in the form

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^{k}\sqrt{g}\right)}{\partial\tau}=$$

$$\lambda_k\left(\frac{\partial\rho u^k}{\partial\tau}+\rho u^\alpha v^k_{,\alpha}+\rho u^k v^\alpha_{,\alpha}\right) \quad (22)$$

In order to complete the derivation, it is sufficient to express the second term on the left-hand side of Eq. 13 and the last term on the right-hand side of Eq. 13 in non-conservative form. The second term on the left-hand side of Eq. 13 is thus rewritten by expanding the derivative with respect to the curvilinear coordinates and by using the definition of covariant derivative recalled above

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^{k}(u^\alpha-v^\alpha)\sqrt{g}\right)}{\partial\xi^\alpha}=\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}$$

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\rho u^{k}(u^\alpha-v^\alpha)\sqrt{g}\right)}{\partial\xi^\alpha}+$$

$$\rho u^k(u^\alpha-v^\alpha)\vec{\tilde{g}}^{(l)}\cdot\frac{\partial\vec{g}_{(k)}}{\partial\xi^\alpha}=$$

$$\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\frac{1}{\sqrt{g}}\frac{\partial\left(\rho u^{k}(u^\alpha-v^\alpha)\sqrt{g}\right)}{\partial\xi^\alpha}+$$

$$\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^r(u^\alpha-v^\alpha)\Gamma^k_{r\alpha}=$$

$$\lambda_k\left[\rho u^k(u^\alpha-v^\alpha)\right]_{,\alpha}=$$

$$\lambda_k\left((\rho u)^k_{,\alpha}(u^\alpha-v^\alpha)+\rho u^k u^\alpha_{,\alpha}-\rho u^k v^\alpha_{,\alpha}\right) \quad (23)$$

By using Eqs. 22 and 23 the left-hand side of Eq. 18 becomes

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^{k}\sqrt{g}\right)}{\partial\tau}+$$
$$\frac{1}{\sqrt{g}}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\rho u^{k}(u^\alpha-v^\alpha)\sqrt{g}\right)}{\partial\xi^\alpha}=$$
$$\lambda_k\left(\frac{\partial\rho u^k}{\partial\tau}+\rho u^\alpha v^k_{,\alpha}+\rho u^k v^\alpha_{,\alpha}+\right.$$
$$\left.(\rho u)^k_{,\alpha}(u^\alpha-v^\alpha)+\rho u^k u^\alpha_{,\alpha}-\rho u^k v^\alpha_{,\alpha}\right)=$$
$$\lambda_k\left(\frac{\partial\rho u^k}{\partial\tau}+\rho u^\alpha v^k_{,\alpha}+\rho u^k u^\alpha_{,\alpha}+\right.$$
$$\left.(\rho u)^k_{,\alpha}(u^\alpha-v^\alpha)\right) \quad (24)$$

Analogously, the last term on the right-hand side of Eq. 13 is rewritten by expanding the derivative and using the definition of covariant derivative

$$\frac{1}{\sqrt{g}}\frac{1}{\rho}\frac{\partial\left(\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}T^{k\alpha}\sqrt{g}\right)}{\partial\xi^\alpha}=$$
$$\frac{1}{\sqrt{g}}\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\frac{1}{\rho}\frac{\partial\left(T^{k\alpha}\sqrt{g}\right)}{\partial\xi^\alpha}+$$

$$\frac{1}{\rho}T^{k\alpha}\vec{\tilde{g}}^{(l)}\cdot\frac{\partial\left(\vec{g}_{(k)}\right)}{\partial\xi^\alpha}=$$

$$\vec{\tilde{g}}^{(l)}\cdot\vec{g}_{(k)}\frac{1}{\sqrt{g}}\frac{1}{\rho}\frac{\partial\left(T^{k\alpha}\sqrt{g}\right)}{\partial\xi^\alpha}+\vec{\tilde{g}}^{(l)}\cdot$$
$$\vec{g}_{(k)}\frac{1}{\rho}T^{r\alpha}\Gamma^k_{r\alpha}=\lambda_k\frac{1}{\rho}T^{k\alpha}_{,\alpha} \quad (25)$$

By replacing Eqs. 24 and 25, respectively, on the left hand and right-hand side of Eq. 13, and by dividing by λ_k, the differential contravariant momentum equation expressed by Eq. 13 can be written in the form

$$\frac{\partial\rho u^k}{\partial\tau}+\rho u^\alpha v^k_{,\alpha}+\rho u^k u^\alpha_{,\alpha}+$$
$$(\rho u)^k_{,\alpha}(u^\alpha-v^\alpha)=\rho f^k+T^{k\alpha}_{,\alpha} \quad (26)$$

By the same procedure, the integral continuity Eq. 11 can be expressed in the following differential form

$$\frac{d}{d\tau}\int_{\Delta V_0}\left(\rho\sqrt{g}\right)d\xi^1 d\xi^2 d\xi^3+$$

$$\sum_{\alpha=1}^{3}\left\{\int_{\Delta A_0^{\alpha+}}\left(\rho(u^\alpha-v^\alpha)\sqrt{g}\right)d\xi^\beta d\xi^\gamma-\right.$$

$$\left.\int_{\Delta A_0^{\alpha-}}\left(\rho(u^\alpha-v^\alpha)\sqrt{g}\right)d\xi^\beta d\xi^\gamma\right\}=$$

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\rho\sqrt{g}\right)}{\partial\tau}+\frac{1}{\sqrt{g}}\frac{\partial\left(\rho(u^\alpha-v^\alpha)\sqrt{g}\right)}{\partial\xi^\alpha}=$$

$$\frac{\partial\rho}{\partial\tau}+\rho v^\alpha_{,\alpha}+\left[\rho(u^\alpha-v^\alpha)\right]_{,\alpha}=0 \quad (27)$$

which, by expanding the covariant derivative, becomes

$$\frac{\partial\rho}{\partial\tau}-\frac{\partial\rho}{\partial\xi^\alpha}v^\alpha+(\rho u^\alpha)_{,\alpha}=0 \quad (28)$$

Eq. 28 is equal to the differential contravariant continuity equation obtained by Luo and Bewley [8] and can be used to further simplify the above momentum balance Eq. 26. In fact, by expanding the derivative on the left-hand side of Eq. 26 and by using Eq. 28, the momentum balance equation becomes

$$\frac{\partial u^k}{\partial \tau} + u^\alpha v^k_{,\alpha} + u^k_{,\alpha}(u^\alpha - v^\alpha) = f^k + \frac{1}{\rho} T^{k\alpha}_{,\alpha} \quad (29)$$

Eqs. 28 and 29 are equal, respectively, to the continuity and momentum balance equations obtained by Luo and Bewley [8] and represent the differential non-conservative form of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system.

4 Integral contravariant motion equations for three-dimensional non-hydrostatic free surface flows in a time dependent curvilinear coordinate system

In the differential Eqs. 27 and 28 the Christoffel symbols are present $u^k_{,\alpha}$, $v^k_{,\alpha}$ and $T^{k\alpha}_{,\alpha}$. The presence of the Christoffel symbols does not allow the numerical scheme to converge to the weak solution. In this work, we propose an original integral contravariant formulation of the three-dimensional motion equations, devoid of the Christoffel symbols, in order to produce a finite volume shock-capturing scheme. In order to simulate the fully dispersive wave processes Eq. 10 can be transformed in the following way.

Let $H(x^1,x^2,t) = h(x^1,x^2,t) + \eta(x^1,x^2,t)$ be the water depth, where h is the undisturbed water depth and η is the free surface elevation with respect to the undisturbed water level. The gravity acceleration is represented by G, the pressure p is divided into a hydrostatic part, $\rho G(\eta - x^3)$, and a dynamic one, q. We consider the following transformation from the Cartesian system of coordinates, (x^1,x^2,x^3,t), to the curvilinear one, (ξ^1,ξ^2,ξ^3,τ), in order to accurately represent the bottom and surface geometry

$$\xi^1 = \xi^1(x^1,x^2) \quad ; \quad \xi^2 = \xi^2(x^1,x^2)$$

$$\xi^3 = \frac{x^3 + h(x^1,x^2)}{H(x^1,x^2,t)} \quad ; \quad \tau = t \quad (30)$$

in which the horizontal curvilinear coordinates ξ^1 and ξ^2 conform to the horizontal boundaries of the physical domain and the vertical coordinate ξ^3 varies in time in order to adjust to the free surface movements. The contravariant components of the velocity vector of the moving coordinates are

$$v^1 = 0 \quad ; \quad v^2 = 0 \; ;$$
$$v^3 = \frac{\xi^3}{H} \left. \frac{\partial H(x^1,x^2,t)}{\partial t} \right|_{\vec{x}=cost} \quad (31)$$

The proposed coordinate transformation basically maps the irregular, varying domain in the physical space to a regular, fixed domain in the transformed space, where ξ^3 spans from 0 to 1. Let $\sqrt{g_0} = \vec{k} \cdot \left| \vec{g}_{(1)} \wedge \vec{g}_{(2)} \right|$, where $\wedge$ indicates the vector product. The Jacobian of the transformation becomes $\sqrt{g} = H\sqrt{g_0}$. Let us define the conserved variables that are given by the cell averaged product between the water depth H and the three contravariant components of the punctual velocity u^l with $l = 1,3$

$$\overline{H} = \frac{1}{\Delta A^3_0 \sqrt{g_0}} \int_{\Delta A^3_o} H\sqrt{g_0}\, d\xi^1 d\xi^2$$
$$\overline{Hu^l} = \frac{1}{\Delta V_0 \sqrt{g_0}} \int_{\Delta V_0} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} u^k H\sqrt{g_0}\, d\xi^1 d\xi^2 d\xi^3 \quad (32)$$

With simple passages it is possible to demonstrate that for the control volume $\Delta V(\tau)$, by using the definition of cell averaged given by Eq. (32), in the transformed space, the integral Eq. (10) reads

$$\frac{\partial \overline{Hu^l}}{\partial \tau} = -\frac{1}{\Delta V_0 \sqrt{g_0}} \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A^{\alpha+}_o} \left[\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} H u^k (u^\alpha - v^\alpha) + \vec{\tilde{g}}^{(l)} \cdot \vec{g}^{(\alpha)} G H^2 \right] \sqrt{g_0}\, d\xi^\beta d\xi^\gamma - \int_{\Delta A^{\alpha-}_o} \left[\vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} H u^k (u^\alpha - v^\alpha) + \vec{\tilde{g}}^{(l)} \cdot \vec{g}^{(\alpha)} G H^2 \right] \sqrt{g_0}\, d\xi^\beta d\xi^\gamma \right\} +$$
$$\frac{1}{\Delta V_0 \sqrt{g_0}} \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A^{\alpha+}_o} \vec{\tilde{g}}^{(l)} \cdot \vec{g}^{(\alpha)} G h H \sqrt{g_0}\, d\xi^\beta d\xi^\gamma - \int_{\Delta A^{\alpha-}_o} \vec{\tilde{g}}^{(l)} \cdot \vec{g}^{(\alpha)} G h H \sqrt{g_0}\, d\xi^\beta d\xi^\gamma \right\} +$$
$$\frac{1}{\Delta V_0 \sqrt{g_0}} \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A^{\alpha+}_o} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \frac{T^{k\alpha}}{\rho} H \sqrt{g_0}\, d\xi^\beta d\xi^\gamma - \int_{\Delta A^{\alpha-}_o} \vec{\tilde{g}}^{(l)} \cdot \vec{g}_{(k)} \frac{T^{k\alpha}}{\rho} H \sqrt{g_0}\, d\xi^\beta d\xi^\gamma \right\} -$$

$$\frac{1}{\Delta V_0\sqrt{g_0}}\int_{\Delta V_0}\vec{\tilde{g}}^{(l)}\cdot \vec{g}^{(m)}\frac{\partial q}{\partial \xi^m}H\sqrt{g_0}d\xi^1 d\xi^2 d\xi^3 \qquad (33)$$

in which $\overline{H}$ represents a two-dimensional quantity given by the averaged value of the water depth H on the base area of a water column, $\Delta A^3 = \Delta A_0^3\sqrt{g_0}$. $T^{k\alpha}$ is now the stress tensor in which the pressure is omitted, the gradient of the hydrostatic pressure is split into two parts by using $\eta = H - h$ and the last integral on the right-hand side of Eq. 33 is related to the gradient of the dynamic pressure, q. It is also possible to demonstrate that, by integrating the continuity Eq. 11 over a vertical water column (between the bottom and the free surface) which is bounded by coordinate surfaces, we obtained the governing equation for the free surface movement

$$\frac{\partial \overline{H}}{\partial \tau} = \frac{1}{\Delta A_o^3\sqrt{g_0}}\sum_{\alpha=1}^{2}\left[\int_0^1\int_{\Delta\xi_o^{\alpha+}}u^\alpha H\sqrt{g_0}d\xi^\beta d\xi^3 - \int_0^1\int_{\Delta\xi_o^{\alpha-}}u^\alpha H\sqrt{g_0}d\xi^\beta d\xi^3\right] \qquad (34)$$

Eqs. 33 and 34 represent the expressions of the three-dimensional motion equations as a function of the new conserved variables, given by the two-dimensional cell averaged values of the water depth, $\overline{H}$, and by the three-dimensional cell averaged values, $\overline{Hu^l}$, in the time dependent coordinate system, $(\xi^1, \xi^2, \xi^3, \tau)$.

5 Results

5.1 Propagation of monochromatic waves on a varying depth

In order to check the ability of the proposed model to simulate the shoaling and breaking wave processes, the experimental test performed by Steve [22] is here numerically reproduced. In this test, an incoming wave train of height 0.156m and period 1.79s is simulated, which propagates in a 55m long wave flume characterised by an initial constant depth of 0.85m, followed by a plane sloping beach of 1:40 (Figure 1).

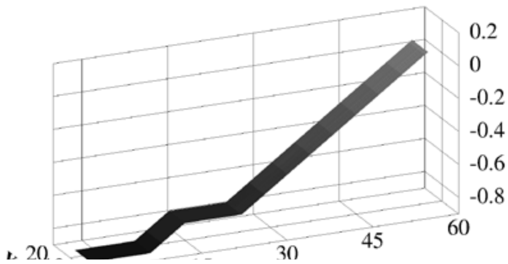


Figure 1 Topography of the bottom.

Aiming to demonstrate the independency of the results from the grid distortion, the numerical simulation is conducted both by using a Cartesian computational grid and a highly distorted curvilinear computational grid.

Figures 2 show a plan view of the distorted grid (Fig.2a) and a detailed view of the computational domain (Fig.2b). In both figures are represent one coordinate line out of every two.

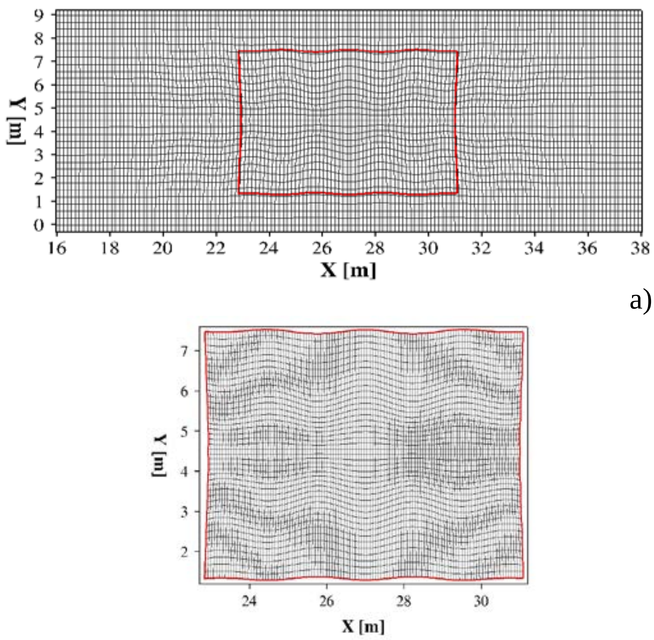


Figure 2 a) Plane view and b) detailed plane view of the highly distorted curvilinear computational grid. Only one coordinate line out of every two is shown.

The root mean square error, σ_{v_y}, of the difference between the numerical values and the expected values of the v_y velocity component (which are null since the direction of motion is parallel to the x-axis) is used as comparison parameter for the results obtained with the two different grids (see Table 1).

As it can be deduced from Table 1, the σ_{v_y} error calculated for the computational highly distorted grid case differs by less than 1 per cent from the corresponding Cartesian grid error.

In Table 1, the root mean square error, σ_{wh}, of the difference between the time averaged numerically computed wave height and the corresponding experimental data, and the root mean square error, σ_{mwl}, of the difference between the numerically computed mean water level and the experimental data are shown.

RMS error	σ_{vy}	σ_{wh}	σ_{mwl}
Cartesian Grid	2.95E-05	0.1198	0.1206
Distorted Grid	2.978E-05	0.1209	0.1218

Table 1 Numerical root mean square error of the velocity component (σ_{v_y}), of average wave height (σ_{wh}) and of still water level (σ_{mwl})

In Figure 3 an instantaneous wave field obtained with the curvilinear highly distorted computational grid is shown. Figure 4 shows a plane view detail of such instantaneous field in the area where the grid distortion is maximum. By observing these figures, it is possible to deduce that, despite the grid distortion, the wave train maintains even wave fronts as the wave propagates from deep water up to the shoreline and does not show spurious oscillations. Thus, it can be concluded that the grid distortion does not affect the ability of the proposed model to simulate the shoaling and breaking wave processes.

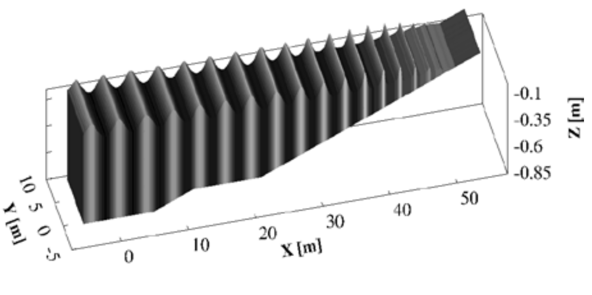


Figure 3 Instantaneous wave field obtained with the highly distorted curvilinear computational grid.

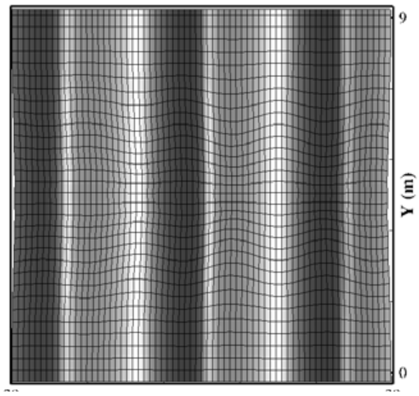


Figure 4 Detailed plane view of the instantaneous wave field obtained with the highly distorted curvilinear computational grid. Only one coordinate line out of every two is shown.

In Figures 5 comparison between the experimental data and the numerical results obtained with the distorted curvilinear grid is shown in terms of time averaged wave height (Fig. 5a) and mean water level (Fig. 5b).

5.2 Rip current test in a curved shaped coastal area

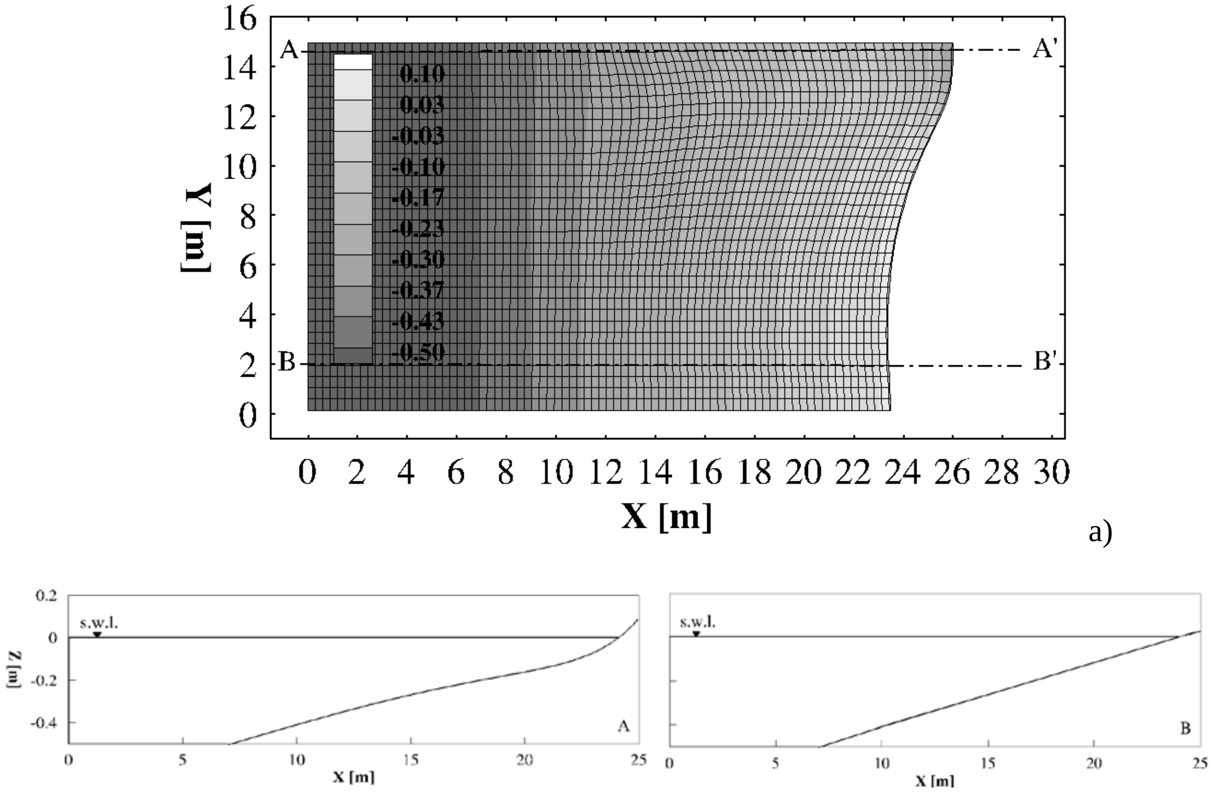
In this section, we verify the ability of the proposed model to numerically reproduce wave propagation, wave breaking and induced nearshore circulation due to the variable bathymetry in a curved shaped coastal area. To this end we reproduce a laboratory experiment carried out by Hamm [23]. These experiments were conducted in a 30x30 m wave tank. The geometry of the bottom consisted of a horizontal region of water depth 0.5m followed by a planar slope of 1:30 with a rip channel excavated along the centreline (see Fig. 6). The bottom variation is given by

$$z(x,y) = 0.5 \qquad x \le 7$$

$$z(x,y) = 0.1 - \frac{18-x}{30}\left[1 + 3exp\left(-\frac{18-x}{30}\right)cos^{10}\left(\frac{\pi(15-y)}{30}\right)\right] \qquad 7 < x < 25$$

$$z(x,y) = 0.1 + \frac{18-x}{30} \qquad x \le 25 \tag{41}$$

Because of the presence of an axis of symmetry perpendicular to the wave propagation direction, only half of the experimental domain has been reproduced. A to this end, we use a curvilinear boundary conforming grid which, in the horizontal directions reproduce the curved shaped coastline. Figure 6(a) shows a plan view of the curvilinear computational grid and bottom variation, in which only one out of every five coordinate lines is shown. Figures 6(b) and 6(c) show the beach profile at two significant cross sections, one inside the rip channel - yA =14.9625m - and one at the plane beach - yB =1.9875m - where the experimental data reported by Hamm [23] are available. The experiments considered a number of different incident wave conditions. Here the monochromatic, regular, incident waves are considered with a period of T = 1.25s and wave height of H = 0.07m.



a)

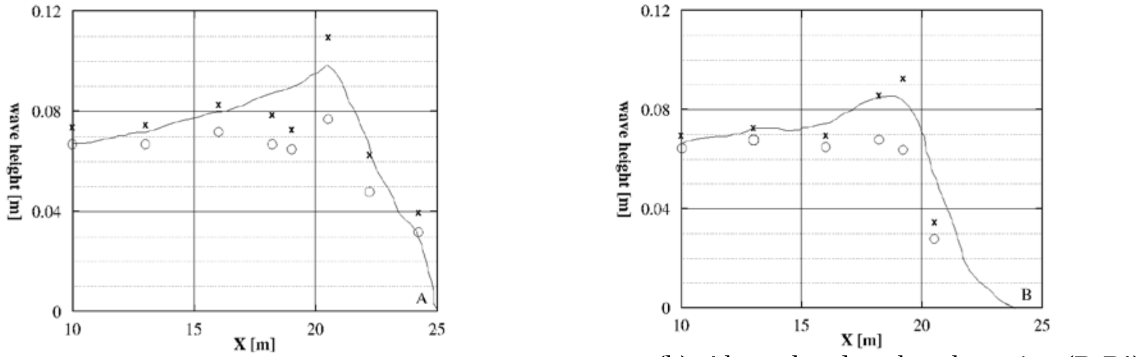
b) c)

Section A-A': bottom profile along the rip channel Section B-B': bottom profile along the plane beach

Figure 6 Bathymetry (Only one out of every five coordinate lines is shown). (b-c): bottom profiles in section A-A' and B-B'.

In Figure 7, the wave heights computed with the proposed model are compared with the wave heights measured by Hamm [23] along the two above mentioned cross sections and reported by Sørensen et al. [24].

It can be noticed that the numerical results in terms of wave height are in good agreement with the laboratory measurement. In particular, the wave height evolution and breaking point are well predicted in the rip channel section and in the plane beach section.



(a) Along the rip channel section (A-A’)

(b) Along the plane beach section (B-B’)

Figure 7 Comparison between the computed (solid line) and measured cross-shore variation of the wave height (crosses and circle).

Figure 8 shows a plane view detail of the time averaged velocity field near the bottom (in which only one out of every four vectors are shown). As it can be seen in Figure 8, the differences in the wave elevation between the plane beach and rip channel drives an alongshore current that turns offshore producing the rip current at the rip channel position. From this Figure it is easy to deduce that this circulation pattern represents an erosive condition.

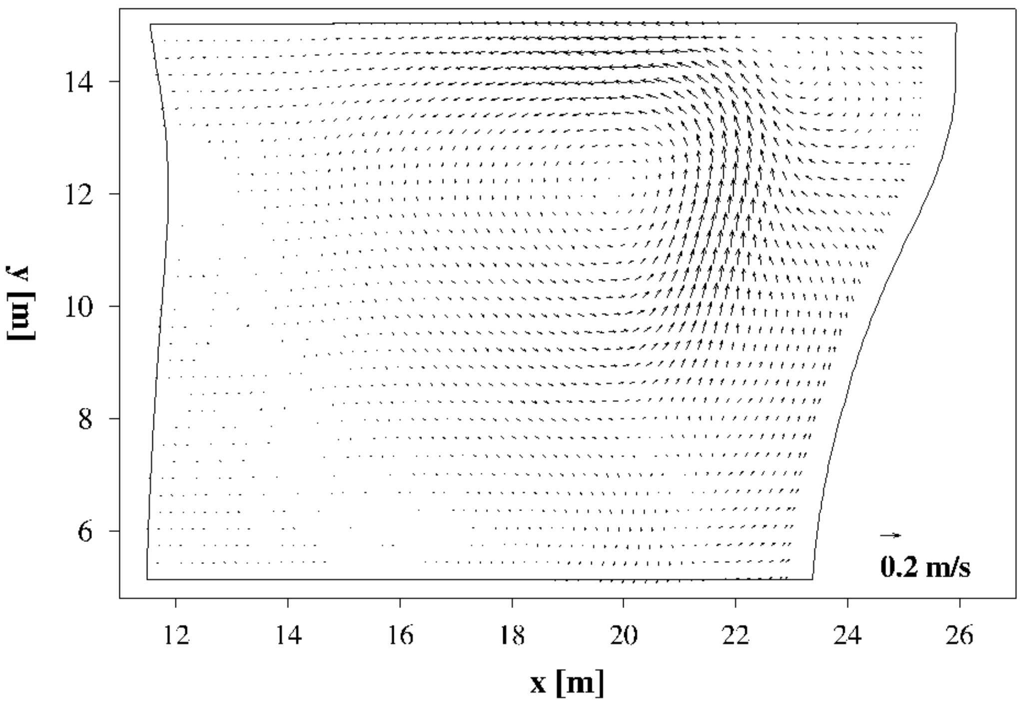


Figure 8 Plane view detail of the time averaged velocity field. Only one out of every four vectors are shown

5 Conclusion

In this work, an original integral formulation of the three-dimensional contravariant Navier-Stokes equations, devoid of the Christoffel symbols, in general time-dependent curvilinear coordinates has been presented. The proposed integral formulation has been obtained from the time derivative of the momentum of a fluid material volume and from the Leibniz rule of integration applied to a control volume that moves with a velocity which is different from the fluid velocity. In order to avoid the presence of the Christoffel symbols, the integral contravariant formulation of the momentum equation is solved in the direction identified by a constant parallel vector field. It has been demonstrated that, starting from the proposed integral formulation, with simple passages the complete differential form of the contravariant

Navier-Stokes equations in a time dependent curvilinear coordinate system can be deduced. The proposed integral formulation, devoid of the Christoffel symbols, is used in order to realise a three-dimensional non-hydrostatic numerical model for free surface flows, which is able to simulate the wave motion and the discontinuities in the solution related to the wave breaking on domains that reproduce the complex geometries of the coastal regions.

In the proposed model, the integral contravariant form of the momentum and continuity equations are solved by a finite volume shock capturing scheme, which uses an HLL approximate Riemann solver. The proposed model has been validated by reproducing experimental test cases on time dependent curvilinear grids.

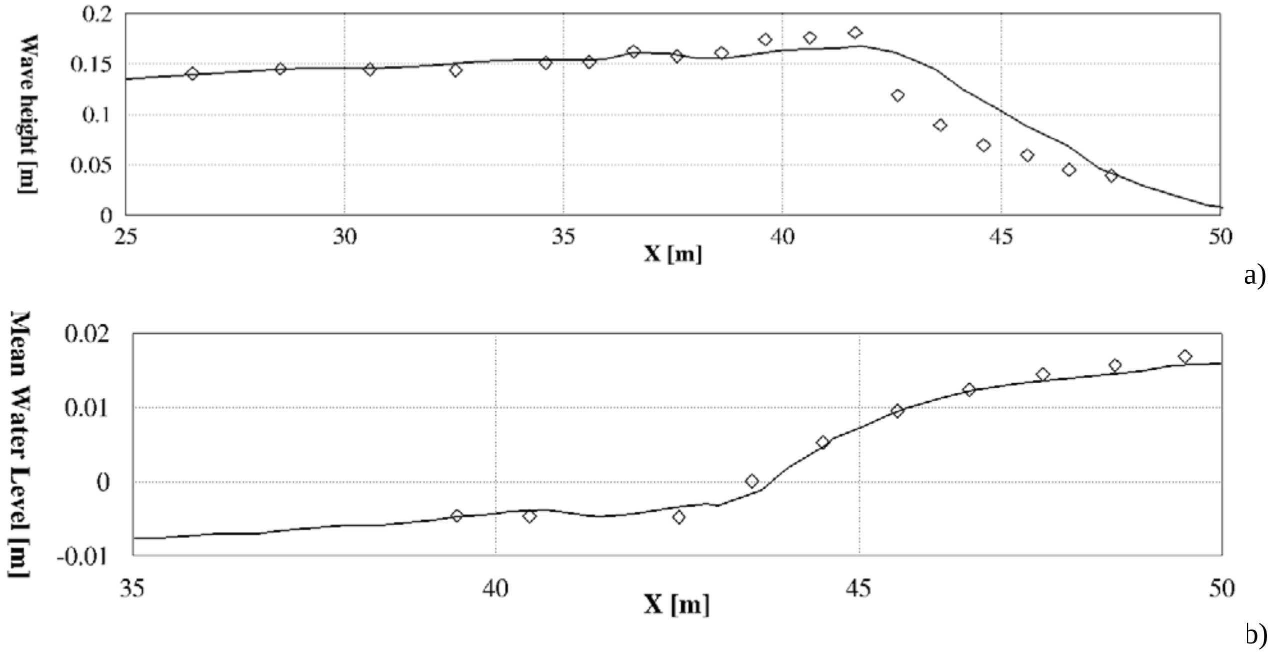


Figure 5 Comparison between the experimental data (diamond) and the numerical results (solid line) obtained with the highly distorted curvilinear computational grid in terms of time averaged wave height and mean water level.

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