Numerical and approximate solutions of boundary layer development due to a moving extensible surface

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Abstract: In this study, several numerical analysis methods were performed for solving boundary layer flow model development due to a moving surface ('sheet'). This model obeys to general stretching law and was presented by Kuiken in 1981. The numerical simulation methods which were used are the shooting method and finite difference method (FDM). Creating the final simulation model involved a calibration step. It was found that the shooting method does not describe properly the fluid physics as compared to finite difference method (FDM). Additionally, comparison between numerical and suggested approximate solutions was done while qualitative compatibility was found between solutions. Kuiken solution branch was found to be fully coincided with current FDM solution for $\kappa = 1/3$. Finally, comparison between ADM (Adomian Decomposition Method) and FDM has been done, while appropriate match was found between solutions. Quantitatively, all presented solutions have the same order of magnitude; nevertheless, inaccuracy between all kinds of solutions does exist.

Key-words: Shooting method, FDM, Two-dimensional flow, Numerical solution, Approximate solution, Analytical solution, Literature solution, ADM

1 Introduction

Boundary layer problems were studied and developed by different researchers for many decades. In 1965, Goldstein [1] has proved that outer fringes in which are described by potential flow (harmonic functions) of the boundary layer cannot be algebraic but should rather be exponential. According to Kuiken [2], Brown & Stewartson [3] have showed that algebraic behavior can be obtained at singular points in special cases (sink, source or point of separation). Moreover, algebraic behavior was examined in 1969 by Van Dyke [4] in the case of axisymmetric vertical needle motion with free convection.

During the last years, fresh studies on boundary layer flow have been published. In 2008, Ishak *et al*. [5] have published their study on incompressible viscous and electrically conducting fluid medium due to a moving extensible sheet that obeys a more general stretching law. According to Kuiken [2], the sheet suddenly (or "somehow") disappears in the origin while the flow medium has not been influenced by the sink presence. They found that dual solutions exist near $x = 0$, where the velocity profiles show a reversed flow. Similar studies have been done in 2011 and 2012 by Ibrahim & Shanker [6] and Soid *et al*. [7]. However, Ibrahim & Shanker [6] have been focused on the influence of the heat transfer due to a heat source by quasi-linearization technique. Similar study in a porous medium on mixed convection flow was performed by Imran *et al*. [8].

The present study concentrates on several numerical methods for solving boundary layer development equations due to moving extensible surface as originally brought by Kuiken [2] in 1981. The numerical methods which will be dealt here are the shooting method and finite difference method (FDM). Current study finds the suggested numerical solutions limitations by making comparison between those solutions and other studies.

Moreover, the author suggests four approximate analytic solutions which are based on boundary conditions only. The idea behind these solutions is to have a relatively quick solution type evaluation in the quantitative and qualitative aspects before finding an accurate solution for the non-linear differential equation. The analytic approximate solutions are inspired by the author mathematical approach on the study of laminar boundary layer [9]. Comparison between approximate and numerical solution is performed including other literature approximations and analytical solutions (like ADM - Adomian Decomposition Method and Kuiken analytical solution).

The shooting method appears broadly in Stoer & Bulirsch book [10] and by Press et al [11]. This method is used for solving boundary value problem (B.V.P) by reducing it to the solution of an initial value problem. Here, the method will be used for solving non-linear third order differential equation. Second method of solution is called finite difference method (FDM). This method represents a group of numerical methods which are based on finite derivatives approximations [12-13] and will be elaborated in the context of this essay.

2 Flow Field Equations

Consider a moving extensible sheet in Cartesian flow field coordinates (x, y) as appear in Fig. 1. While the sheet occupies the negative x-axis as shown below in Fig. 1. While the sheet occupies the negative x-axis as shown below in Fig. 1.

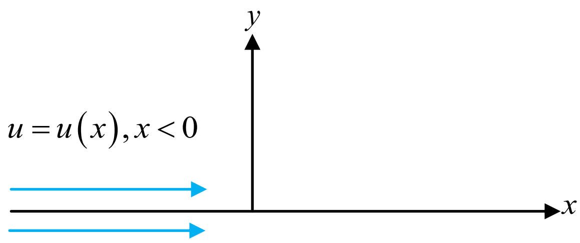

Fig. 1. 2D extensible sheet model.

The sheet moves continually in the positive x-direction at a velocity [2]:

$$u_s = u_0 \left(\frac{x_0}{|x|} \right)^{\kappa}, \kappa > 0, \quad (1)$$

while x_0 and u_0 are the characteristic length and velocity, respectively. According to Kuiken [2], the sheet suddenly (or "somehow") disappears in the origin while the flow medium has not been influenced by the sink presence. Suppose, Reynolds number is large enough, so "backward" boundary layer equations are given by [2]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \; u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2}, \quad (2)$$

where u, v are the velocity components in the (x, y) directions, respectively. while μ represents constant viscosity. Also, the boundary conditions are given by [2]:

$$\begin{aligned} &y = 0 : v = 0, \; u = u_s \\ &y \to \infty : u \to 0 \end{aligned}, \quad (3)$$

For parabolic equation, one should select specific value of $u(x,0)$. Hence, we will choose:

$$u(x \to -\infty, y) = 0. \quad (4)$$

Using similarity transformation for solving Eq. (2) with B.C. (3-4), as follows:

$$\psi = f(\xi)\sqrt{2\mu u_x |x|}, \; \xi = y\sqrt{\frac{u_s}{2\mu|x|}}, \quad (5)$$

while ψ represents stream function which is defined by the following derivatives:

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}. \quad (6)$$

Simplifying system Eqs. (1-3) into one non-linear differential equation using relations (5) and (6) with the appropriate boundary conditions is prescribed by [2]:

$$f''' + (\kappa - 1) f f'' - 2\kappa (f')^2 = 0, \quad (7)$$

With the following appropriate B.C.:

$$f(0)=0, f'(0)=1, f'(\infty)=0, \quad (8)$$

where differentiation is defined by the prime argument notation $(\ )'$.

3 Numerical methods formulation

Numerical solution procedure for solving parabolic equation has been studied by [14-17]. In order to simplify the problem, system order will be reduced using vector-matrix notation. Hence, $f(\xi)$ function derivatives would be defined by:

$$\begin{aligned} f'(\xi) &= F(\xi) \\ f''(\xi) &= F'(\xi) = G(\xi) \end{aligned}. \quad (9)$$

Substituting Eq. (9) into Eqs. (7-8) leads to vectors notation such that $U=[f\,F\,G]^T$ and $\underline{U'}=V'$:

$$\begin{bmatrix} f' \\ F' \\ G' \end{bmatrix} = \begin{bmatrix} F \\ G \\ 2\kappa F^2+(1-\kappa)fG \end{bmatrix}, \quad (10)$$

While

$$f(0)=0, F(0)=1, F(\infty)=0. \quad (11)$$

In this problem the following parameters will be used:

- $\kappa=\frac{1}{3},\frac{1}{5},\frac{2}{5},\frac{4}{5},1,4,10$. While elementary numerical method analysis will be concentrated on $\kappa=\frac{1}{5},\frac{2}{5},\frac{4}{5}$. Moreover, further examination and comparisons for analytical and approximate solutions will be included values of $\kappa=\frac{1}{3},1,4,10$ (see Sec. 6).
- The range of independent parameter ξ values is $0 \le \xi \le 30$ such as $\xi=30$ is considered "far enough" (as will be discussed continually) to represent infinity $\xi \to \infty$.

Two main numerical methods for solving this problem are:

- The shooting method.
- Finite Difference Method (FDM).

Shooting Method

The shooting method requires a definition of an initial condition. Guessing $f(0)$ involved with numerical integration from $\xi=0$ to $\xi=1$. Next step is to guess the function value at $\xi=30$ and then making an equality check by comparing it to $f(0)$. Thus the error will defined by:

$$E(f^{(n)})=f(\xi=30, f^{(n)})-f(\xi=30), \quad (12)$$

where $f^{(n)}$ is a series of $f(\xi=30)$ guesses and $f(\xi=30, f^{(n)})$ is the appropriate guess value. The shooting method can be solved iteratively (implicit methods) or alternatively, by using explicit methods as shown below in Fig. 2 diagram.

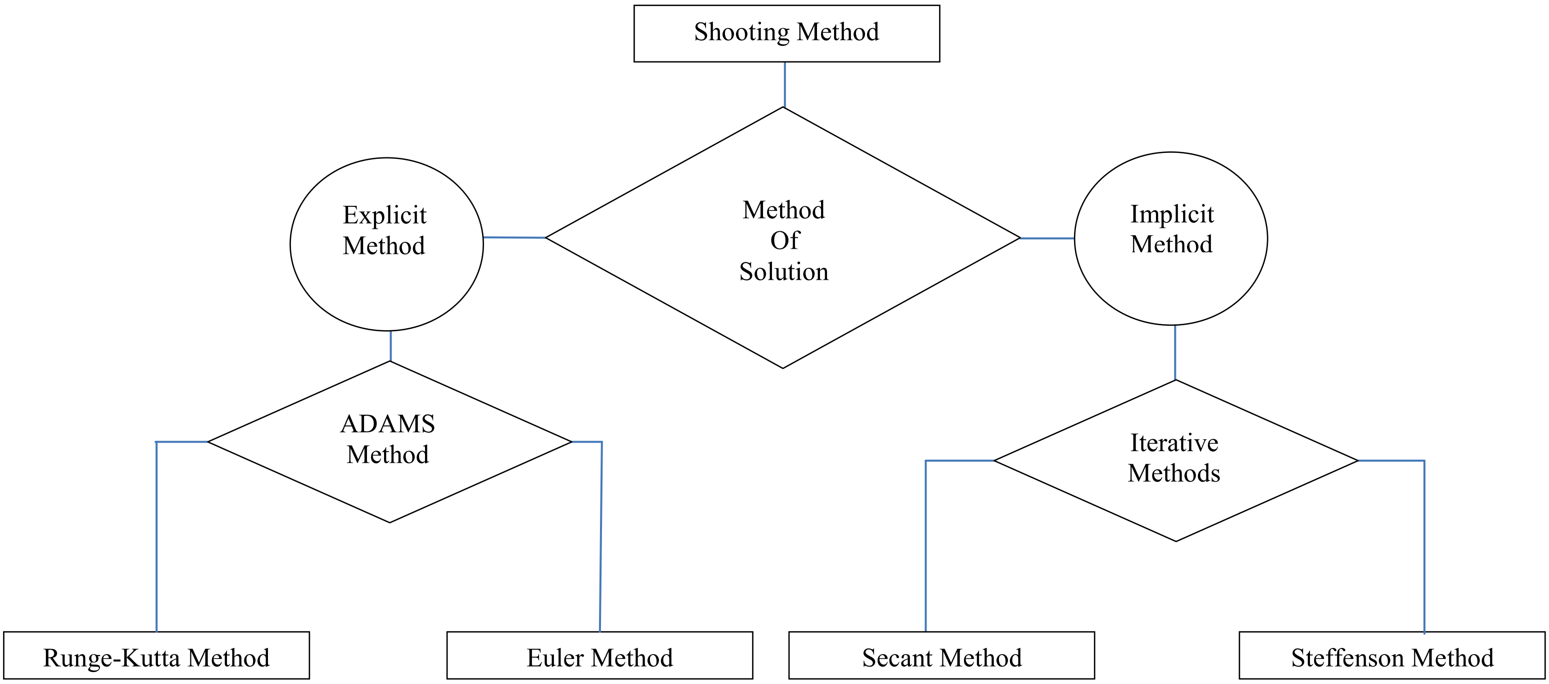

Fig. 2. Schematic diagram of the shooting method of solution.

ADAMS Method

Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution. Single-step methods (such as Euler's method) refer to only one previous point and its derivative to determine the current value. Methods such as Runge-Kutta take some intermediate steps to obtain a higher order method, but then discard all previous information before taking a second step. This method has been discussed by Butcher [14] and Süli & Mayers [15]. Mathematically formulation of this method is obtained by:

$$f_{n+1} = f_n + h_1 F_n + \frac{h_1}{2}[F_n - F_{n-1}] + O\left(h_1^{\ 3}\right)$$
$$F_{n+1} = F_n + h_2 G_n + \frac{h_2}{2}[G_n - G_{n-1}] + O\left(h_2^{\ 3}\right)$$
$$G_{n+1} = G_n + h_3 Q_n + \frac{h_3}{2}[Q_n - Q_{n-1}] + O\left(h_3^{\ 3}\right)$$
$$Q_n = 2\kappa F^2 + (1-\kappa) fG \tag{13}$$

In order to find the value at the initial point, one should use one of the following methods:

- Euler Method.
- Runge-Kutta Method.

While Euler method is based on error of the second order $O\left(h^2\right)$ and is written by:

$$f_2 = f_1 + h_1 f_1 + O(h_1^{\ 2})$$
$$F_2 = F_1 + h_2 F_1 + O(h_2^{\ 2}) \ , \tag{14}$$
$$G_2 = G_1 + h_3 G_1 + O(h_3^{\ 2})$$

where f_1, F_1, G_1 are obtained by initial guesses.

Alternatively, Runge-Kutta method which is one-step method with an error of the third order $O\left(h^3\right)$ can also be applied for initiation. This method can be implemented by:

$$f_2 = f_1 + \lambda_1 h_1 f_1 + \lambda_2 h_1 f_1(\eta + \mu h_1) + O(h_1^{\ 3})$$
$$F_2 = F_1 + \lambda_1 h_2 F_1 + \lambda_2 h_2 F_1(\eta + \mu h_2) + O(h_2^{\ 3}) \ ,$$
$$G_2 = G_1 + \lambda_1 h_3 G_1 + \lambda_2 h_2 G_1(\eta + \mu h_3) + O(h_3^{\ 3}) \tag{15}$$

while $\lambda_1 = \lambda_2 = \frac{1}{2}, \mu_1 = \mu_2 = 1$.

Until now, explicit solutions to algebraic equations have been demonstrated. From this point, iterative solution methods will be discussed. Two main methods that may be suitable for solving Eq. (12) are:

- Secant Method.
- Steffenson Method.

Secant Method

In numerical analysis, the Secant method is a root-finding algorithm that uses a roots succession of secant lines to better approximate a root of specific function. The secant method can be thought of as a finite difference approximation of Newton's method. However, the method was developed independently of Newton's method as reported by Papakonstantinou [16]. This method is being considered as a multi-step method and data of two points is required. The secant method can be applied here by:

$$u^{(n+1)} = u^{(n)} - \frac{f\left[u^{(n)}\right]\left(u^{(n)} - u^{(n-1)}\right)}{f\left[u^{(n)}\right] - f\left[u^{(n-1)}\right]}. \quad (16)$$

Another iterative method that should be considered is called Steffenson method and will be elaborated here.

Steffenson Method

This method is actually a modification of Newton-Raphson method which is also being called "variable derivative" method. This method is categorized to be one step method which means that knowledge of only one point initiation data is required. However, the mathematical formulation is:

$$u^{(n+1)} = u^{(n)} - \frac{\left\{f\left[u^{(n)}\right]\right\}^2}{f\left[u^{(n)} + f\left[u^{(n)}\right]\right] - f\left[u^{(n)}\right]}. \quad (17)$$

Other numerical solution which will be implemented and dealt here is the finite difference method (FDM).

FDM Method

FDM method has been studied by Morton and Mayers [12]. Accordingly, it can be solved both iteratively (implicit methods) or alternatively, by using explicit methods as shown below in Fig. 3 diagram.

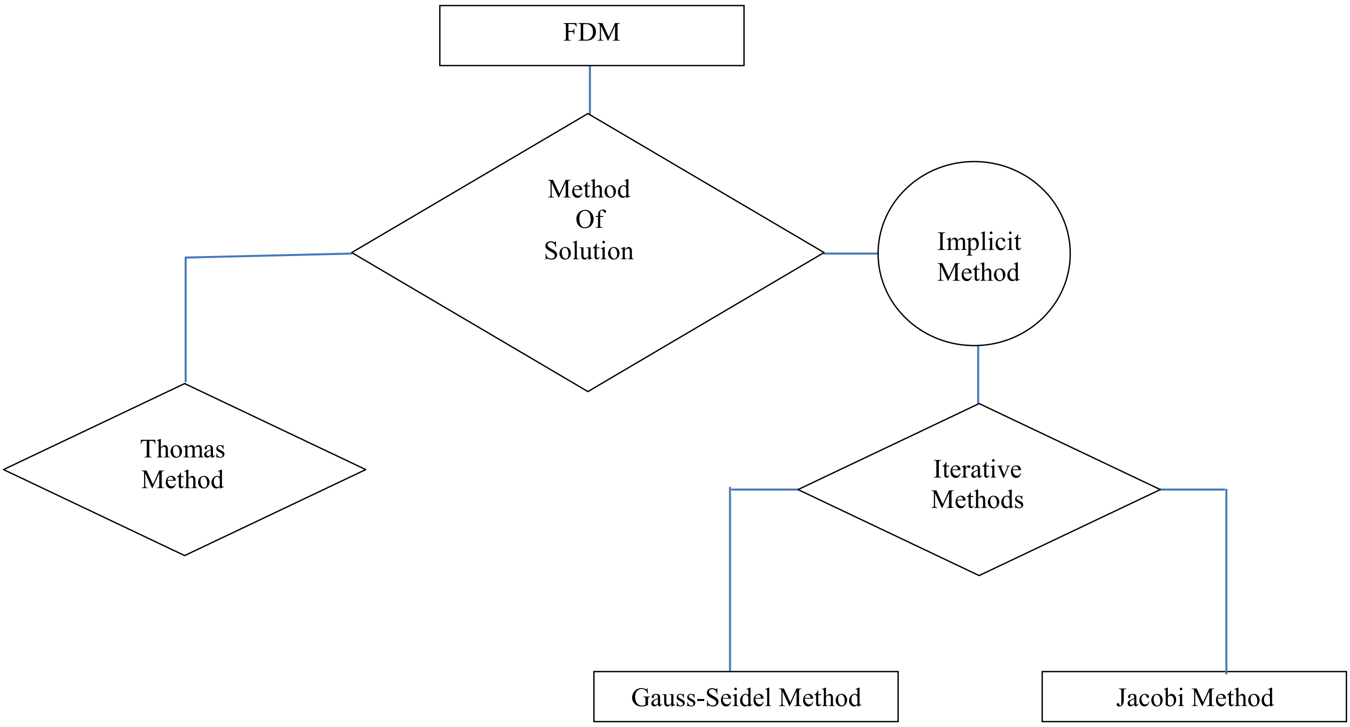

Fig. 3. Schematic diagram of the FDM method of solution.

Finite-difference methods (FDM) are numerical methods which are based on approximate derivatives. For better explanation, general problem will be defined in the shape of:

$$\begin{cases} y'' + p(x)\,y' + g(x)\,y^2 = q(x), & 0 \le x \le 1 \\ y(0) = \alpha,\ y(1) = \beta \end{cases} \tag{18}$$

While dividing x range into N segments. Taylor central operator relations will be used for numerical purpose, by:

$$y'_n = \frac{y_{n+1} - y_{n-1}}{2h} + O(h^2), \quad y''_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + O(h^2) \tag{19}$$

Substituting relations (19) into Eq. (18) yields:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + \frac{y_{n+1} - y_{n-1}}{2h} p(x) + g(x) y_n{}^2 = q(x) \tag{20}$$

After applying discretization on Eq. (20) we have:

$$y_{i-1}\left[1 - \frac{hp_i}{2}\right] + y_i\left[h^2 g_i - 2\right] + y_{i+1}\left[1 + \frac{hp_i}{2}\right]. = h^2 q_i,\ i = 1, 2, .., N \tag{21}$$

While this problem should be solved according to schematic diagram as shown in Fig. 3. Parenthetically, G-S (Gauss-Seidel) method implementation for solving Eq. (21) is given by:

$$y_i{}^{n+1} = \frac{y_{i-1}^{n+1}\left(1 - \frac{hp_i}{2}\right) + y_{i+1}^{n}\left(1 + \frac{hp_i}{2}\right) - h^2 g_i}{2 - h^2 g_i y_i{}^n}, \quad n = 0, 1, 2.. \tag{22}$$

Under the following relaxation convergence criterion:

$$y_{SUR}^{n+1} = y^n + \Omega\left(y^{n+1} - y^n\right), 0 \le \Omega \le 1. \tag{23}$$

These method prominent advantages are easy programming and faster convergence than Jacobi and SOR (successive over-relaxation) methods. The next paragraph will discuss Thomas algorithm for solving Eq. (21).

Thomas algorithm

In this paragraph general formulation for solving numerical algebraic Eq. (21) using Thomas algorithm will be introduced. Thomas tri-diagonal matrix is written by [11 ,17]:

$$\begin{bmatrix} b_2 & c_2 & & & \\ a_3 & b_3 & c_3 & & \\ & a_4 & b_4 & c_4 & \\ & & \ddots & & \\ & & & \ddots & \\ & & & a_{N-1} & b_{N-1} \end{bmatrix} \begin{bmatrix} C_{i,2} \\ C_{i,3} \\ . \\ . \\ . \\ C_{i,N} \end{bmatrix} = \begin{bmatrix} d_2 \\ d_3 \\ . \\ . \\ . \\ d_{N-1} \end{bmatrix} \tag{24}$$

Similar to system (24), the tri-diagonal matrix in our problem is:

$$\begin{bmatrix} -2-2\kappa h^2 F_1 & 1+(\kappa-1)\frac{hf_1}{2} & 0 & \cdots & 0 \\ 1-(\kappa-1)\frac{hf_2}{2} & -2-2\kappa h^2 F_2 & 1+(\kappa-1)\frac{hf_2}{2} & & \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & & & \\ 0 & \cdots & & 1-(\kappa-1)\frac{hf_2}{2} & -2-2\kappa h^2 F_N \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix} = \begin{bmatrix} -\alpha\left[1-(\kappa-1)\frac{hf_1}{2}\right] \\ 0 \\ \vdots \\ 0 \\ -\beta\left[1+(\kappa-1)\frac{hf_N}{2}\right] \end{bmatrix} \tag{25}$$

As appear in equations system (25), matrix coefficient vector is non-linear and fulfills:

$$\underline{A}\left[f^{(n)}\underline{y}^{(n)}; \underline{y}^{(n)}\right] \cdot \underline{y}^{(n+1)} = \underline{B} f^{(n)} \underline{y}^{(n)}. \tag{26}$$

While $\underline{A}\left(\underline{y}^{(n)}\right)$ and $\underline{B}$ are the coefficients and solutions vectors, respectively. System of Eqs. (26) will be solved by initial guess of F function. Convergence condition is fulfilled by:

$$\left|\underline{F}^{(n+1)} - \underline{F}^{(n)}\right| < \varepsilon, \tag{27}$$

while $\varepsilon = \text{constant} > 0$.

Additionally, f_i parameter is calculated by using Euler backwards differences:

$$f_i = f_{i-1} + hF_i + O_{(h^2)}. \tag{28}$$

4 Numerical Model Calibration

Calibration of numerical methods results will be examined here by comparing results of f, f' functions. Initialization process difference between Euler and Runga-Kutta methods is presented in Fig. 4. The difference value between these methods is very small and almost not exists for each κ value as shown in Fig. 4 .a – b for f, f', respectively. The maximum difference value is in order of magnitude of 10^{-3} for each value of $\kappa = 1/5, 2/5$ as presented in Fig. 4. It can be concluded that one can use Euler method for integration process instead of using Adams method and yet, achieve relatively accurate results.

Convergence processes for solution are obtained using Secant and Steffenson methods which are of the same order $O\left(h^2\right)$. The difference between these methods is illustrated in Fig. 5. It can be observed that there is a slight difference between solution profile functions f, f', especially for $\kappa = 2/5$. The reason for this slight difference is derived from the fact that Secant method is multi-step method which required two initial guesses. On the other hand, Steffenson method is one-step method which required only one initial guess. The maximum error between Secant and Steffenson profiles is about 14% for $\kappa = 1/5$. Thus Steffenson method will be used. Moreover, shooting method is too sensitive for initial guess choice and therefore unstable enough.

Examination of integration step (grid step) as illustrated in Fig. 6, reveals that there is no clear line ('thumb rule') between f, f' function values and integration step values. This incompatibility is derived due to instability of the shooting method, as was mentioned before. From here, $h = 0.01$ will be used for achieving minimum run-time. ε parameter value has no influence on convergence since edge conditions are not possibly to achieve.

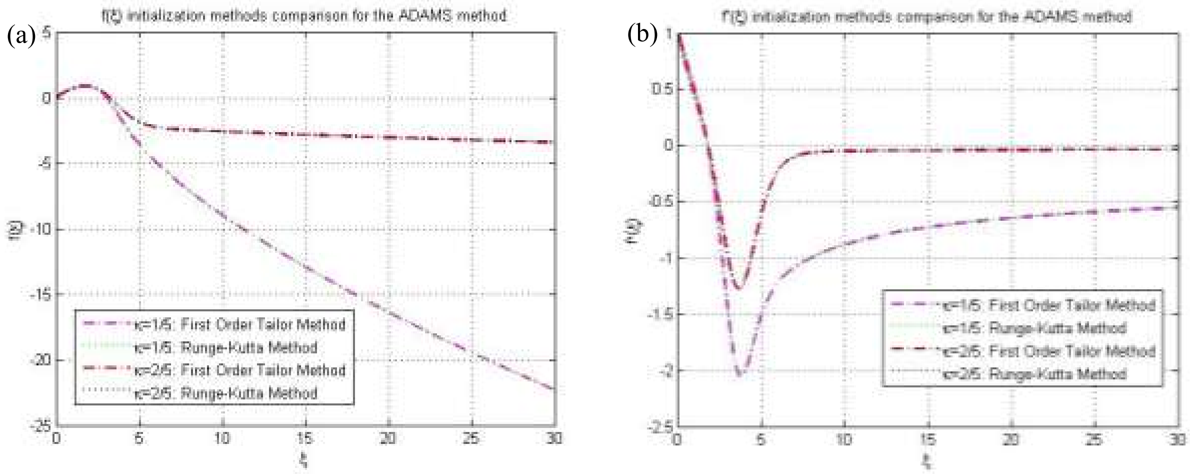

Fig. 4. Comparison between initialization methods for $\kappa = 1/5,\ 2/5$: a. f . b. f' .

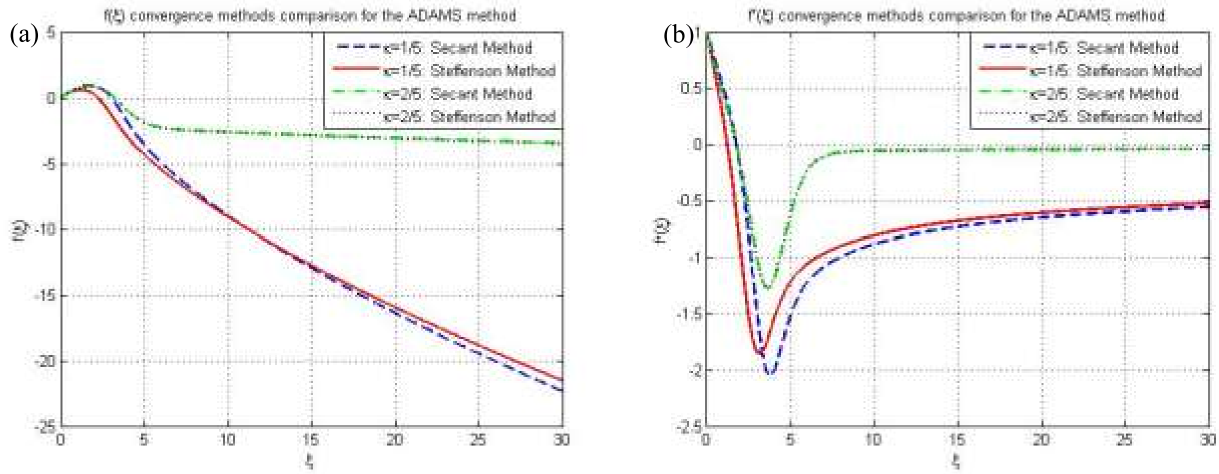

Fig. 5. Comparison between convergence methods for $\kappa = 1/5,\ 2/5$: a. f b. f'

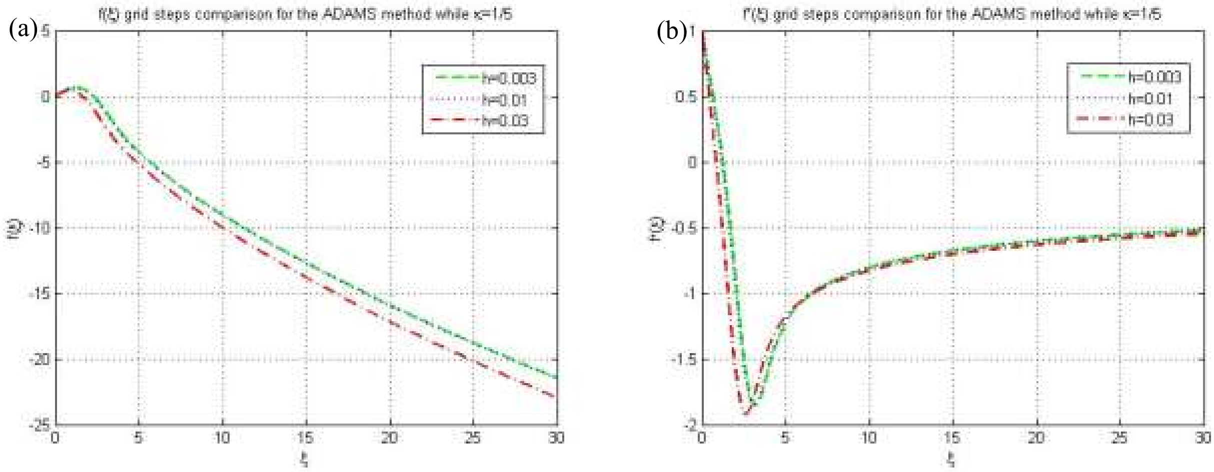

Fig. 6. Comparison between various grid steps for $\kappa = 1/5$: a. f b. f'

Now, examination of FDM analysis parameters will be done. FDM order of magnitude error is identical to Euler method $O(h^2)$. After assuming $\varepsilon = 0.01$, integration grid step comparison is demonstrated in Fig. 7. It can be easily inferred from Fig. 7 that solution accuracy convergence is achieved for smaller integration step values. Moreover, integration step has no influence on number of iterations required for convergence. Hence, we will use $h = 0.01$ as an integration step. Convergence condition comparison for various ε parameters is presented in Fig. 8. It can be easily concluded that for decreasing ε parameter, solution convergence improves accuracy. However, $\varepsilon = 0.01$ will be taken for further calculations since solution accuracy is appropriate enough for supplying convergence condition and solution run – time is relatively short.

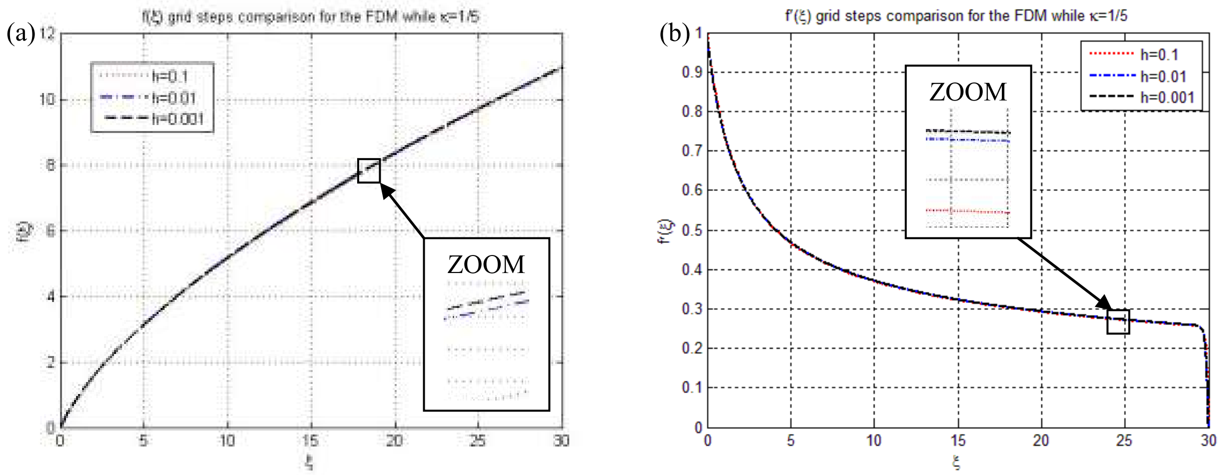

Fig. 7. FDM comparison between various grid steps for $\kappa = 1/5$: a. f . b. f' .

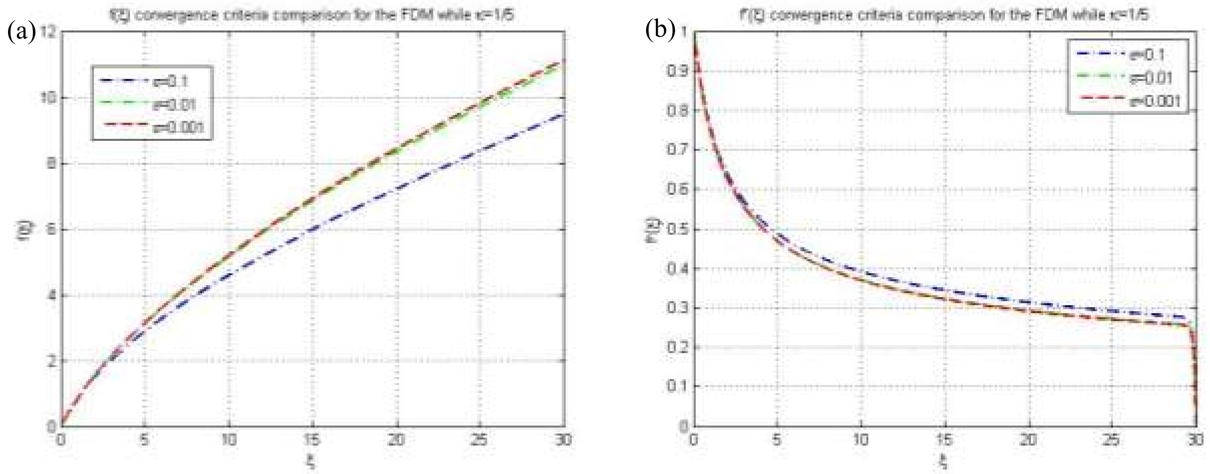

Fig. 8. FDM comparison between various convergence criteria for $\kappa = 1/5$: a. f. b. f'.

5 Numerical results

This section presents final numerical results for the following specific parameters:

- $h = 0.01$.
- $\varepsilon = 0.01$.
- $\kappa = 1/5, 2/5, 4/5$.

Since the shooting method is unstable and obtained results have no physical meaning since normalized function values are negative in large part of the range ($f, f' < 0$) as shown in Fig. 1-3, only final results for f and f' functions using FDM will be presented here. Examination of FDM final solution as shown in Fig. 9 indicates that solution functions (f, f') achieve their maximum value for minimum κ parameter value. The sharp kink behavior at $\xi = 30$ is caused by fulfilling B.C. $f'(\xi \to \infty) = 0$ which proves that FDM numerical solution is still insufficient for providing accurate solution. Moreover, f' is proportional to u velocity and represents velocity component qualitative behavior in the x direction ($f' \propto u$). In the next section, $\xi = 30$ will be proved to be considered as "far enough" to represent infinity ("∞").

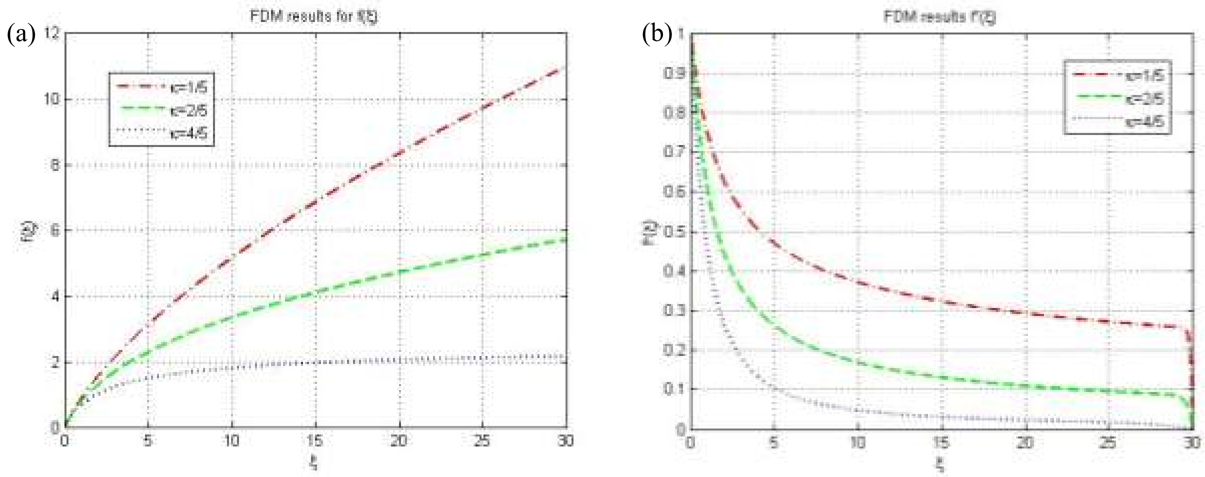

Fig. 9. Final results of the FDM for $\kappa = 1/5, 2/5, 4/5$: a. f. b. f'.

6 Comparisons & Discussion

The solution has been obtained using relevant mathematical knowledge from the author's study on laminar boundary layer [9]. The solutions are aimed to fulfill B.C. (8) only while the basic functions shapes (for instance: sin, ln, exponent or polynomial, etc.) have been chosen intuitively. These approximate functions do not fulfill numerical B.C. for $\xi = 30$ but only the original B.C. for $\xi \to \infty$. It should be emphasized that these solutions are aimed to model the original analytic solution of Eq. (7) and not the numerical problem solution.

On the one hand, qualitative compatibility was found between FDM and approximate solutions (29-32) as shown in Fig. 10. a-b. Also, quantitatively observation reveals compatibility in order of magnitude between both solutions. On the other hand, accuracy quantitative differences between solutions do exist due to the following reasons:

- Numerical solution convergence condition (11).
- Numerical solution dependency on different κ parameter values while approximate solution has no dependency on this parameter (only B.C.).
- Function shapes (29-32).
- Numerical solution discretization.

The significant advantages of approximate solutions are:

- Quick evaluation of solution.
- Good qualitative compatibility.
- Good quantitative compatibility (order of magnitude).

In continue to the former discussion, literature approximation and analytical solutions will be presented. Analogous behavior between Kuiken [2], Crane [20], Liao & Pop [21] and others [18-19,23-24] was found. Kuiken [2] has suggested analytic solution (f) for $\kappa = 1/3$ in the form:

$$f_5(\xi) = -3\left(c^2/9\right)^{1/6}\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)},\quad c \approx 0.56,\ z = \frac{1}{3}(c\eta - 1)\left(\frac{9}{c^2}\right)^{1/3} \tag{33}$$

Where $\mathrm{Ai}(z)$ is Airy function and z is the accommodate argument. Moreover, in 1970 Crane [20] and three decades later Liao & Pop [21] have solved similar problem on boundary layer development due to flow past a stretching plate in the form:

$$f'''(\xi) + f(\xi)f''(\xi) - \beta f'^2(\xi) = 0 \tag{34}$$

with identical B.C. (8). However, Eq. (34) is equivalent to Eq. (7) for $\beta = 4$ and $\kappa = 2$. Two analytical solutions have been proposed by Crane [20] for Eq. (34) with specific β values:

$$f_6(\xi) = 1 - e^{-\xi},\ \beta = 1,\ f''(0) = 1. \tag{35}$$

$$f_7(\xi) = \sqrt{2}\tanh\left(\xi/\sqrt{2}\right),\quad \beta = -1, f''(0) = 0 \tag{36}$$

while $f''(0)$ is the appropriate convergence condition. Even though solutions (35-36) are appointed to solve specific differential equation (34) with distinct coefficients compared to Eq. (7), nevertheless, these solutions are still valuable in there qualitative and quantitative aspects despite of its value inaccuracy. After intensive sorting and classification of approximate, FDM and literature solutions it was found that all kinds of solutions have similar qualitative behavior (see Fig. 10 - 11) but are different, quantitatively. However, numerical values have the same order of magnitude for all solutions. FDM solution and Kuiken solution are coincided at $\kappa = 1/3$ as shown in Fig. 11. Moreover, Crane [20] first (35) and second (36) analytical solutions have been coincided with first (31) and second (32) approximate solutions, respectively.

Reasons for differences between literature, approximate and numerical solutions are:

- Different differential equation coefficients lead to different equations which influences the function solution shape.
- Using $f''(0)$ convergence condition (35 - 36) instead of numerical convergence condition (12).
- Approximate solution is based on B.C. only.
- Numerical discretization accuracy.

Despite those differences, approximate and literature solutions are still have similar order of magnitude as compared to numerical solutions and also fits qualitatively.

Kechil et al. [22] have solved Kuiken boundary layer equation by suggesting simple and efficient approximate analytical technique which called ADM (Adomian Decomposition Method). This method is an iterative semi-analytical method for solving ordinary and partial nonlinear differential equations. The analytic solution has convergent infinite series form. In this part, comparison between ADM solution as brought by Kechil *et al.* [22] and current FDM solution will be presented and discussed. The comparison shown in Fig. 12 are for f, f' functions where $\kappa = 1/3, 1, 4, 10$. Note that symbols appear in this illustration are equivalent $m = \kappa, \eta = \xi$. In case where $\kappa = 1, 4, 10$ both ADM and FDM solutions yield identical results quantitatively and qualitatively. Moreover, FDM solution for $\kappa = 4$ seems to fulfill condition $f'(\xi = 5) = 0$ with relatively better accuracy than ADM solution as appear in Fig. 12 (c-d). In case where $\kappa = 1/3$ ADM solution is fully coincided with Kuiken [2] solution branch as shown in Fig. 12 (a-b). However, current solution is partly coincided with Kuiken [2]. At $\xi = 2.5$ current solution branch splits from Kuiken branch. There are two main reasons for this phenomenon: 1. FDM solution obligates to fulfill B.C. $f'(\xi = 5) = 0$, while Kuiken solution doesn't supply this condition. 2. FDM solution becomes accurate for large ξ values (like $\xi = 30$) than small ξ values (like $\xi = 5$), since otherwise sharp kink shape is likely to be created (compare with Fig. 11 (a-b)).

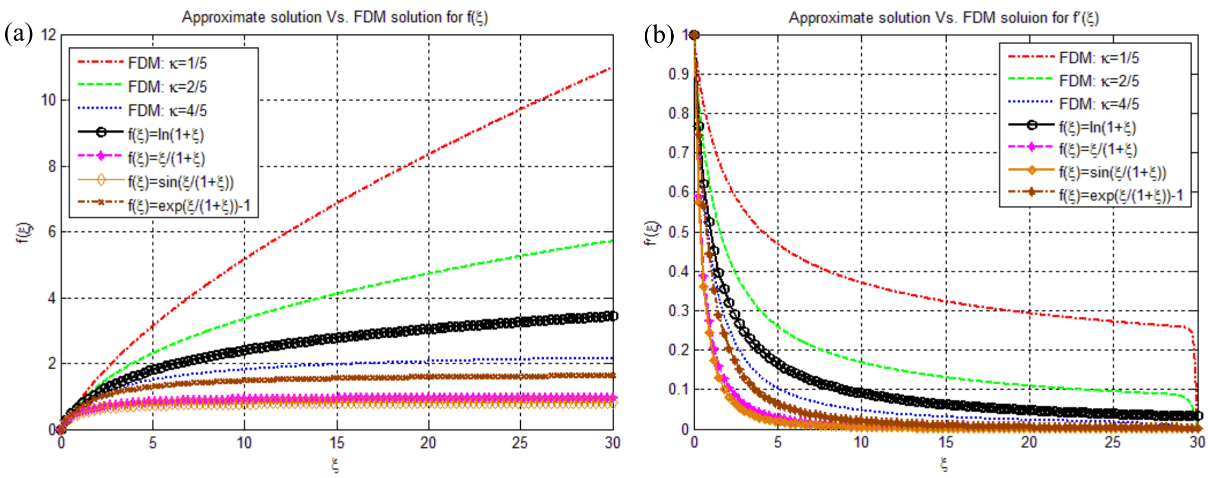

Fig. 10. Final results comparison between FDM and approximate solutions for: a. f. b. f'.

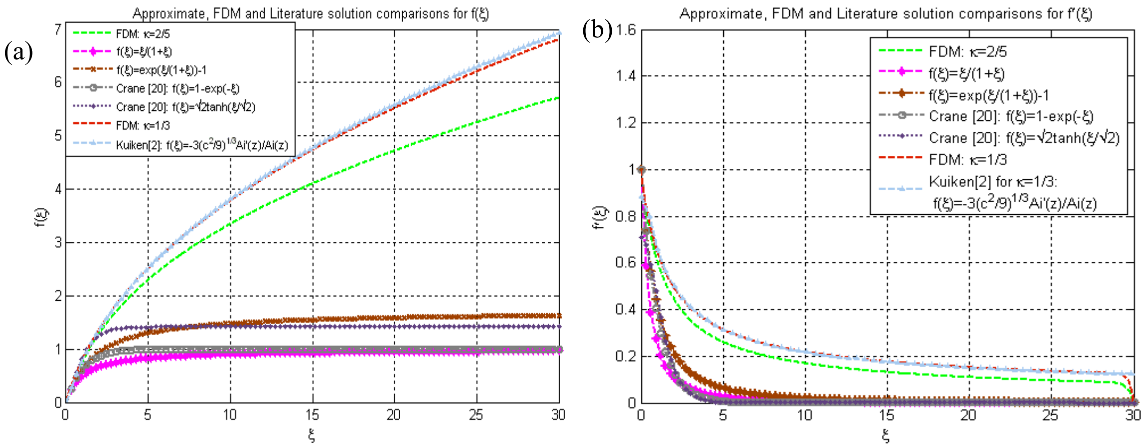

Fig. 11. Final results comparison between FDM, approximate and literature solutions for a. f . b. f' .

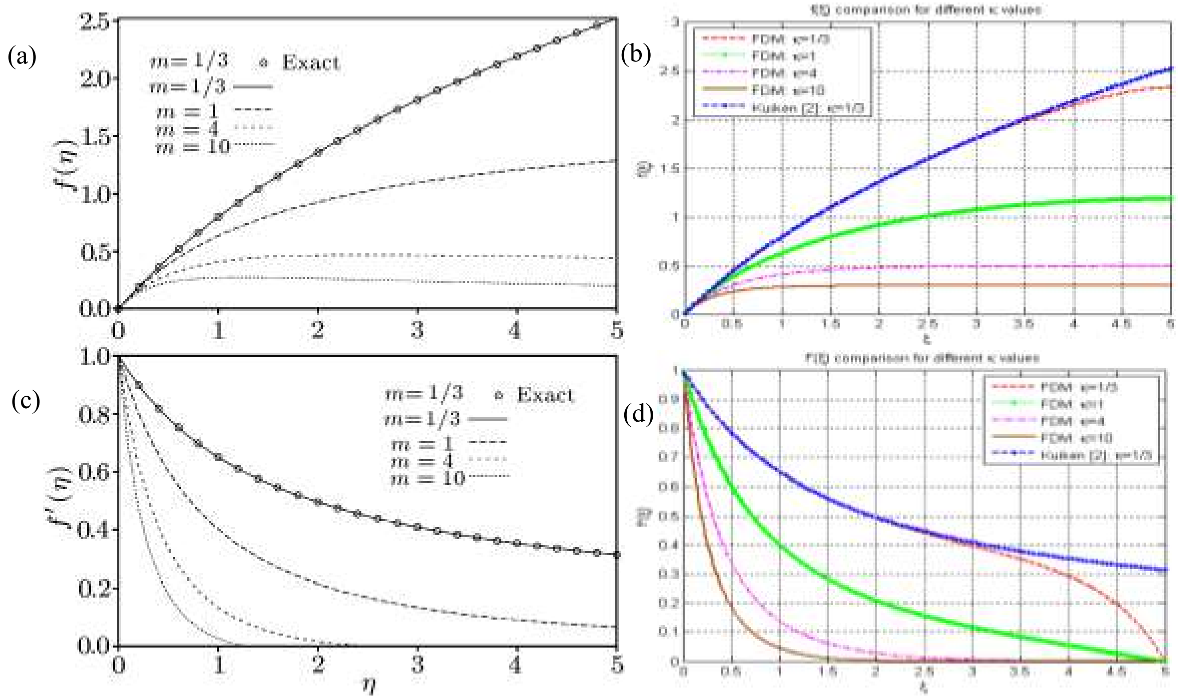

Fig. 12. a– c. Numerical and approximate results for f, f' in case of $\kappa = 1/3, 1, 4, 10$ by Kechil et al. [19]. b - d. f, f' current study results for $\kappa = 1/3, 1, 4, 10$.

7 Conclusion

This study presents several kinds of numerical and analytical approximate solutions for boundary layer development equations due to moving extensible surface as originally brought by Kuiken [2] in 1981. Two main numerical methods for solving this problem are the shooting method and Finite Difference Method (FDM). Main parameters for solving the numerical problem are: $\kappa = \frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, 1, 4, 10$ and independent parameter ξ fulfills $0 \le \xi \le 30$.

Examination of the shooting method shows:

- Initialization value difference due to initialization process between Euler and Runga-Kutta methods is very small and almost not exists for specific κ values parameters.
- Convergence processes for solution were obtained using Secant and Steffenson methods which are of the same order $O(h^2)$. Slight difference between solution profiles was found, especially for $\kappa = 2/5$.

- The shooting method is too sensitive for initial guess choice and therefore is unstable. Typical symptom of its instability is incompatibility between profile values and integration step values.
- ε parameter value has no influence on convergence since edge conditions are not possibly to achieve.

In conclusion, the shooting method is unstable enough to use including the fact that obtained results have no physical meaning ($f, f' < 0$).

Examination of the FDM shows:

- Solution accuracy convergence is achieved for smaller integration step values.
- Solution convergence improves accuracy for decreasing ε parameter.

Moreover, simple approximate solutions have been suggested by the author together with some literature solutions. After comparison between analytical and numerical solutions it was found that:

- Quick evaluation of solution.
- Good qualitative compatibility.
- Good quantitative compatibility (order of magnitude).
- Inaccuracy of quantitative results.

Comparison between ADM (Adomian Decomposition Method) and FDM solutions has raised that:

- In case where $\kappa = 1, 4, 10$ both ADM and FDM solutions yield identical results quantitatively and qualitatively.
- Moreover, FDM solution for $\kappa = 4$ seems to fulfill condition $f'(\xi = 5) = 0$ with relatively better accuracy than ADM solution.
- In case where $\kappa = 1/3$ ADM solution is fully coincided with Kuiken [2] solution branch.
- Current solution is partly coincided with Kuiken [2]. At $\xi = 2.5$ current solution branch splits from Kuiken branch. There are two main reasons for this phenomenon: 1. FDM solution obligates to fulfill B.C. $f'(\xi = 5) = 0$, while Kuiken solution doesn't supply this condition. 2. FDM solution becomes accurate for large ξ values (like $\xi = 30$) than small ξ values (like $\xi = 5$), since otherwise sharp kink shape is likely to be created.

Future studies on subject should be:

- Full mathematical examination of general equation for boundary layer development due to moving extensible surface, including influences, like: magnetic, heat transfer etc.
- Examination of B.C. influence on flow behavior under previous or similar conditions.
- Finding mathematical link between approximate solution that based only on B. C. and accurate numerical solution.
- Comparison with more advanced numerical approach, like Homotopy method.

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