The Effect of Non-Uniform Transverse Friction on the Linear Stability of Shallow Mixing Layers

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Abstract: - Linear stability analysis of mixing layers in shallow water is performed in the present paper under the following assumptions: (a) the fluid contains uniformly distributed heavy small particles, (b) the mixing layer is slightly curved in the longitudinal direction, (c) the friction coefficient changes in the transverse direction. Marginal stability curves are calculated for different base flow velocity profiles which represent both stably and unstably curved mixing layers. The effect of all parameters on the stability characteristics of the flow is investigated.

Key-Words: - Linear stability, shallow mixing layer, non-uniform friction, curvature

1 Introduction

Typical examples of shallow mixing layers are flows at river junctions or in compound and composite channels. Linear stability analyses of shallow mixing layers with constant bottom friction show that shallowness of the fluid layer plays an important role in preventing the development of three-dimensional instabilities [1]-[4]. The presence of a solid boundary is an additional factor that stabilizes the flow. Experimental data show that bottom friction also affects the growth of a mixing layer [5]-[8].

Shallow mixing layers in nature and engineering can be also slightly curved. The effect of small curvature on the stability of free shear layers is investigated in [9] where it is shown that curvature has a stabilizing effect for the case of stably curved mixing layer and destabilizes the flow for unstably curved layer.

The analysis in [1]-[4] is performed for the case where bottom friction is modeled by means of the Chezy or Manning formulas [10]. It is assumed in [1]-[4] that the friction coefficient is constant in the transverse direction. Recent experimental analyses [11]-[15] indicated that if fluid is in contact with porous layer then the friction force changes considerably in the transverse direction. From a practical point of view such a situation occurs in compound channels (or rivers) during floods. In this case friction in the floodplain is much higher than the friction in the main channel. It is shown in [11]-[15] that the characteristics of mass and momentum exchange in case of variable friction are different from the corresponding characteristics for the case of constant friction.

Many environmental flows (for example, flows in rivers or channels) contain particles [16]. The presence of heavy particles also can affect the dynamics of the flow and, in particular, modify linear stability characteristics of the flow. Spatial and temporal instability of slightly curved particle-laden shallow mixing layers for the case of constant friction is investigated in [17].

In the present paper we investigate the combined effect of small curvature, variable friction in the transverse direction and presence of small heavy particles on the stability characteristics of shallow mixing layers. Preliminary results of stability analysis in this case are reported in [18]. The corresponding linear stability problem is solved numerically for different values of the parameters of the problem. It is shown that increase of the particle concentration and small curvature, as well as bottom friction has stabilizing effect on the flow for the case of stably curved mixing layers. On the other hand, unstably curved mixing layer has a destabilizing influence on the base flow. Marginal stability curves and surfaces for different values of the parameters of the problem are plotted and analyzed.

2 Mathematical Formulation of the Problem

Consider the two-dimensional shallow water equations under the rigid-lid assumption [17]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f(y)}{2h}u\sqrt{u^2+v^2} = B(u^p-u), \qquad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f(y)}{2h}v\sqrt{u^2+v^2} +\frac{1}{R}u^2 = B(v^p-v), \qquad (3)$$

where p is the pressure, u and v are the velocity components of the fluid in the x and y-directions, respectively, u^p and v^p are the components of particle velocities, h is water depth, $c_f(y)$ is the non-constant friction coefficient, B is the particle loading parameter [16], [17], and R is the radius of curvature ($R>>1$).

Equations (1)-(3) are derived under the following simplifying assumptions: (a) water depth is constant (this assumption is usually referred to as the rigid-lid assumption); (b) bottom friction is modeled by the Chezy fomula [13] with non-constant friction coefficient; (c) curvature is assumed to be small $(1/R<<1)$; (d) particles are uniformly distributed in fluid; (e) no dynamic interaction between carrier fluid and particles is assumed. Assumption (a) is verified in [8] where it is shown that from a linear stability point of view rigid-lid assumption works well for small Froude numbers. Assumptions (d) and (e) are discussed in [16] where it is shown that these assumptions are reasonable for the case of large Stokes number of the flow.

Introducing the stream function by the relations

$$u=\frac{\partial \psi}{\partial y},\ v=-\frac{\partial \psi}{\partial x} \qquad (4)$$

and eliminating the pressure we obtain from (1)-(3) the following equation:

$$(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{2}{R}\psi_y\psi_{xy} +\frac{c_f(y)}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}+B\Delta\psi +\frac{c_f(y)}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}) +\frac{c_{fy}(y)}{2h}\psi_y\sqrt{\psi_x^2+\psi_y^2}=0. \qquad (5)$$

The function $c_f(y)$ is assumed to be of the form

$$c_f(y)=c_{f_0}\gamma(y), \qquad (6)$$

where c_{f0} is constant and $\gamma(y)$ is sufficiently smooth "shape" function.

In a classical hydrodynamic stability theory the flow and pressure field is represented as a sum of the base flow and small perturbations. The base flow $U(y)$ is obtained as a simple (usually one-dimensional) solution of the equations of motion. Such an approach cannot be used for shallow water equations in the form (1)-(3). The reason is the presence of empirical friction terms in (2)-(3). As a result, the base flow profiles for the system of shallow water equations are selected on the basis of the available experimental data and numerical simulations. In particular, hyperbolic tangent profile is usually used as a base flow for the case of mixing layers. In the present study the following three base flow profiles are used:

$$U(y)=(1+\tanh y)/2, \qquad (7)$$

$$U(y)=2+\tanh y, \qquad (8)$$

$$U(y)=2-\tanh y. \qquad (9)$$

Experimental data [19] show that for mildly curved mixing layers base flow profiles are similar to that of plane mixing layers. As a result, we adopted profiles (7)-(9) in the present study. Profiles (7) and (8) represent the case where the high-speed stream is on the outside of the low-speed stream. Such a case is referred to as a stably curved mixing layer. Similarly, (9) is known as an unstably curved mixing layer since the high-speed stream is on the inside of the low-speed stream. The difference between profiles (7) and (8) is that $U(y)\to 0$ as $y\to-\infty$ for the base flow (7) while $U(y)\to 1$ as $y\to-\infty$ for the base flow (8). Thus, profile (7) has a vanishing velocity as $y\to-\infty$.

In order to study linear stability of flows (7)-(9) we represent the stream function in the form

$$\psi(x,y,t)=\psi_0(y)+\varepsilon\psi_1(x,y,t)+... \qquad (10)$$

where $\psi_0(y)$ is the stream function of the base flow and $\psi_{0y}(y) = U(y)$. Substituting (10) into (5) and linearizing the resulting equation in the neighborhood of the base flow we obtain

$$\psi_{1xxt} + \psi_{1yyt} + \psi_{0y}(\psi_{1xxx} + \psi_{1yyx}) - \psi_{0yyy}\psi_{1x} + \\ + \frac{c_f(y)}{2h}(\psi_{0y}\psi_{1xx} + 2\psi_{0yy}\psi_{1y} + 2\psi_{0y}\psi_{1yy}) \\ + \frac{c_{fy}(y)}{h}\psi_{0y}\psi_{1x} + \frac{2}{R}\psi_{0y}\psi_{1xy} + B(\psi_{1xx} + \psi_{1yy}) = 0. \tag{11}$$

Using the method of normal modes [20] the perturbed stream function $\psi_1(x, y, t)$ is represented in the form

$$\psi_1(x, y, t) = \varphi(y)e^{i\alpha(x-ct)} \tag{12}$$

where $\varphi(y)$ is the amplitude of the normal perturbation, α is the wave number and $c = c_r + ic_i$ is the complex eigenvalue. The base flow $U(y)$ is said to be stable if all $c_i < 0$ and unstable if at least one $c_i > 0$. Substituting (12) into (11) we obtain

$$\varphi_{yy}[\alpha(U - c) - iSU\gamma - iB) - iS(\gamma U_y + \gamma_y U)\varphi_y \\ + \varphi[\alpha^3(c - U) - \alpha U_{yy} + i\alpha^2 US\gamma/2 + i\alpha^2 B) = 0. \tag{13}$$

The boundary conditions are

$$\varphi(\pm\infty) = 0. \tag{14}$$

Here $S = \dfrac{c_{f0}b}{2h}$ is the stability parameter and b is the half-width of the mixing layer.

Note that by adopting the base flow profiles (7)-(9) we are using parallel flow assumption (the base flow is independent on the longitudinal coordinate x). Such an approximation represents the leading order solution in a multiple scale expansion which takes into account slow flow variation in the longitudinal direction [21].

3 Numerical Method

Eigenvalue problem (13), (14) is solved numerically by means of the pseudospectal collocation method based on the Chebyshev polynomials [22]. The interval $-\infty < y < +\infty$ is mapped onto the interval $-1 \le \xi \le 1$ by means of the substitution $\xi = \dfrac{2}{\pi}\arctan y$. The solution to (13) is sought in the following form

$$\varphi(\xi) = \sum_{k=0}^{N-1} a_k(1 - \xi^2)T_k(\xi), \tag{15}$$

where $T_k(\xi) = \cos k \arccos \xi$ is the Chebyshev polynomial of the first kind of order k and a_k are unknown coefficients. The factor $(1 - \xi^2)$ in (15) is chosen in order to satisfy zero boundary conditions at $\xi = \pm 1$ automatically.

The collocation points are

$$\xi_m = \cos\frac{\pi m}{N}, \quad m = 1, 2, ..., N - 1. \tag{16}$$

Using (13)-(16) we obtain the generalized eigenvalue problem of the form

$$(A + cB)a = 0, \tag{17}$$

where A and B are complex-valued matrices and $a = (a_0 a_1 ... a_{N-1})^T$. Since $\varphi(\xi)$ is selected in the form (15), the matrix B in (17) is non-singular.

Numerical results are obtained for the following form of the "shape" function $\gamma(y)$:

$$\gamma(y) = \frac{\beta + 1}{2} + \frac{(\beta - 1)}{2}\tanh \lambda y, \tag{18}$$

where $\beta = \dfrac{c_{f_1}}{c_{f_0}} \ge 1$ is the ratio of the friction coefficients in the floodplain and main channel and λ is the parameter which characterizes sharpness of the variation of the friction coefficient in the transverse direction.

4 Numerical Results for the Base Flow Profile (7)

The marginal stability curves (the curves where $c_i = 0$) for the case of uniform friction ($\beta = 1$) and straight channel ($R = \infty$) are shown in Fig. 1. Three curves in Fig. 1 (from top to bottom) correspond to the following three values of the particle loading parameter B: 0, 0.02 and 0.04. Since the critical bed friction number S_{cr} decreases as the parameter B increases, we conclude that the particle loading parameter has a stabilizing effect on the flow.

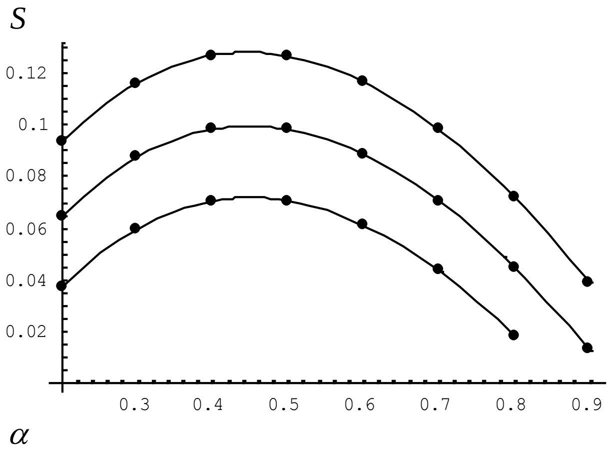

Fig. 1. Marginal stability curves for the case $R=\infty, \beta=1, \lambda=1$ and three values of $B: B=0, 0.02$ and $B=0.04$ (from top to bottom).

The effect of non-uniform friction on the marginal stability curves for the case $R=\infty, \beta=1.5$ is shown in Fig. 2. The three curves in Fig. 2 correspond to the same values of B as in Fig. 1. Comparing Figs. 1 and 2 we can see that non-uniform friction stabilizes the flow: the maxima of the marginal stability curves in Fig. 2 occur at lower values of S than in Fig. 1 (uniform friction).

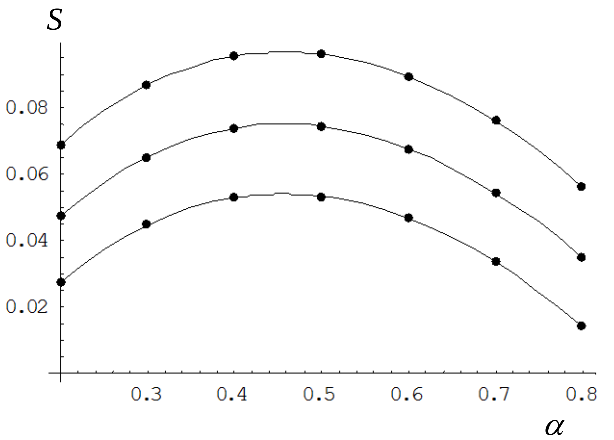

Fig. 2. Marginal stability curves for the case $R=\infty, \beta=1.5, \lambda=1$ and three values of $B: B=0, 0.02$ and $B=0.04$ (from top to bottom).

The combined effect of the three parameters R, β and B on the stability boundary can be analyzed if we compute the critical values of the bed friction number S, namely, $S_{cr}=\max_{\alpha} S(\alpha)$ for several values of the parameters. The marginal stability surfaces are shown in Figs. 3 – 5. Fig. 3 plots the critical values of S (on the vertical axis) for different values of β and $B: 1 \le \beta \le 3$ and $0 \le B \le 0.05$ for $R=\infty$.

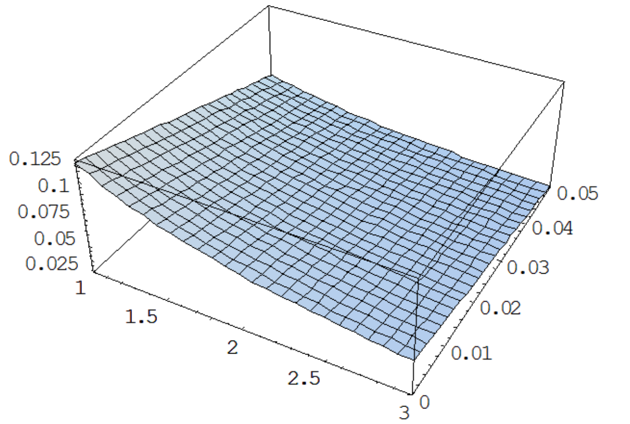

Fig. 3. Marginal stability surface for the case $R=\infty$ and different values of β and B.

Similar graphs are shown in Fig. 4 where the marginal stability surfaces are shown for the case $1/R=0.03$ (slightly curved mixing layer).

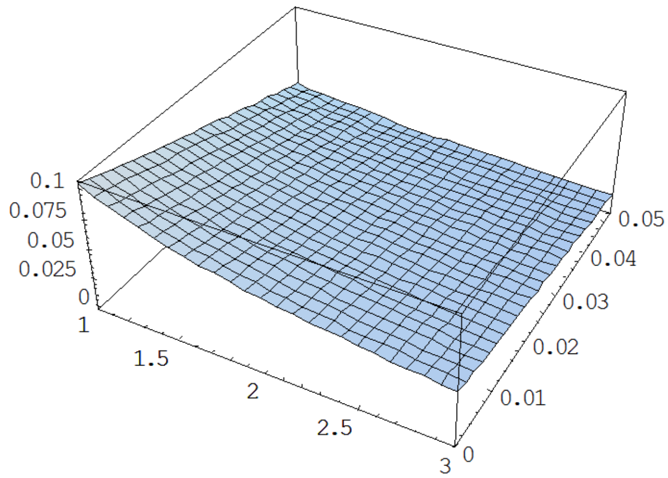

Fig. 4. Marginal stability surface for the case $1/R=0.03$ and different values of β and B.

The stabilizing effect of small curvature can be seen from the analysis of Figs. 3 and 4: critical bed friction numbers decrease as $1/R$ increases.
Larger value of the parameter $1/R$ is shown in Fig. 5 (the marginal stability surface is constructed for the same range of β and B values as in Figs. 3 and 4). Stabilization of the base flow is even more pronounced for the case $1/R=0.06$.

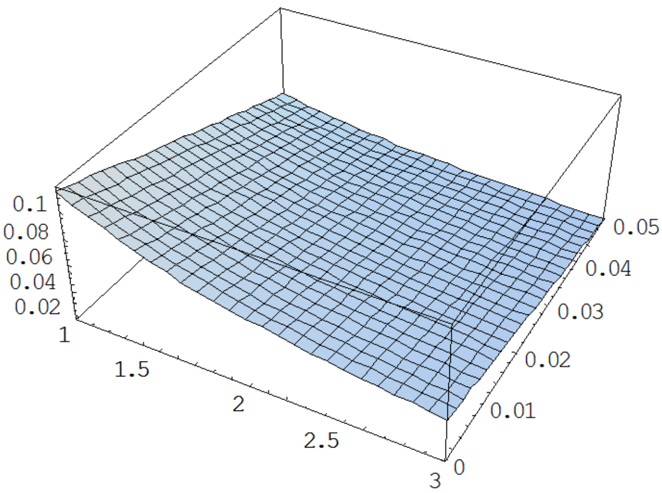

Fig. 5. Marginal stability surface for the case $1/R = 0.06$ and and different values of β and B.

Comparing Figs. 3 – 5 we see that all three parameters R, β and B have a stabilizing effect on the flow: the critical bed friction number decreases as all parameters increase.

5 Numerical Results for the Base Flow Profile (8)

In this section we present the results of numerical calculations (marginal stability curves and surfaces) for the base flow profile (8). In this case the base flow velocity approaches non-zero limits as $y \to +\infty$ and $y \to -\infty$.

Fig. 6 plots the marginal stability curves for plane mixing layer ($1/R = 0$) and uniform friction ($\beta = 1$). It can be seen from Fig. 6 that the particle loading parameter has a stabilizing effect on the flow.

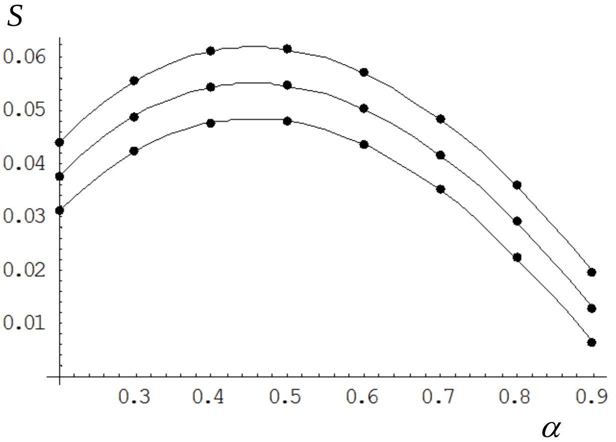

Fig. 6. Marginal stability curves for the case $R = \infty, \beta = 1, \lambda = 1$ and three values of $B: B = 0, 0.02$ and $B = 0.04$ (from top to bottom).

Comparing Fig. 1 and Fig. 5 we see that the velocity profile (8) is more stable from the linear stability point of view than the profile (7).

The marginal stability curves for a planar mixing layer and non-uniform friction ($\beta = 1.5$) are shown in Fig. 7 for the same values of the other parameters as in Fig. 6. Non-uniform friction stabilizes the flow as one can see from the comparison of Figs. 2 and 7.

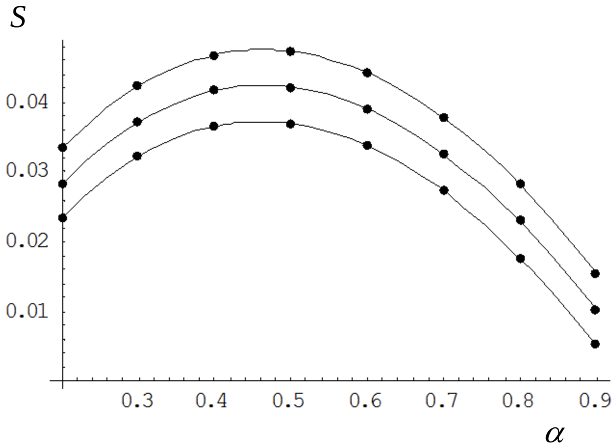

Fig. 7. Marginal stability curves for the case $R = \infty, \beta = 1.5, \lambda = 1$ and three values of $B: B = 0, 0.02$ and $B = 0.04$ (from top to bottom).

Fig. 8 plots the marginal stability surface in the range $1 \le \beta \le 3$ and $0 \le B \le 0.05$ for $R = \infty$.

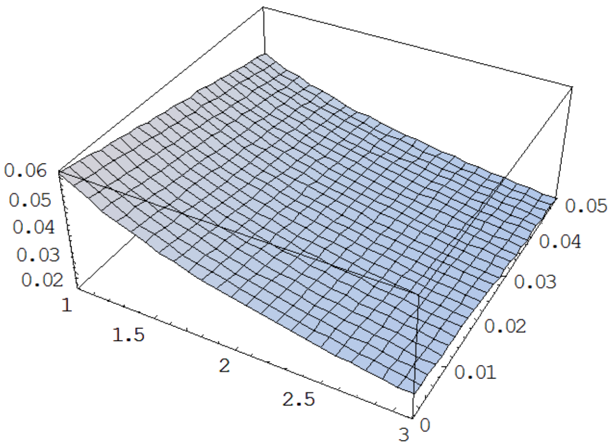

Fig. 8. Marginal stability surface for the case $R = \infty$ and different values of β and B.

Both non-uniform friction and particle loading have stabilizing effect on the flow as in the case of the base flow profile (7). Finally, in Fig. 9 we present the marginal stability surfaces in the range $1 \le \beta \le 3$ and $0 \le B \le 0.05$ for $1/R = 0.03$.

Small curvature also has a stabilizing effect on the flow (it can be seen from the comparison of Fig. 8 and 9). In addition, stabilization for the case of the base flow profile (8) is more pronounced than for the base flow profile (7).

Comparison of the linear stability characteristics for the base flow profiles (7) and (8) shows that the case of vanishing velocity as $y \to -\infty$ is less stable than the case where $U(y)$ approaches non-zero limit as $y \to -\infty$.

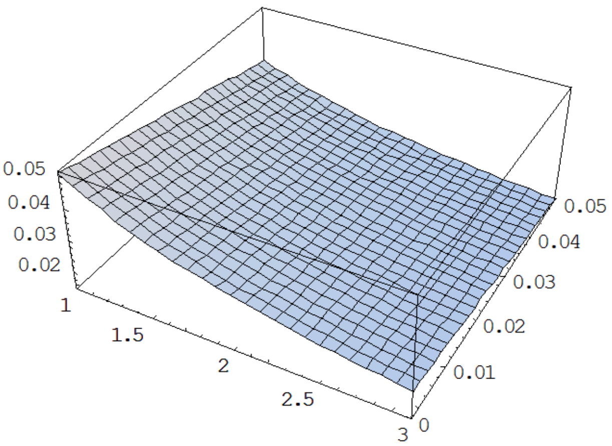

Fig. 9. Marginal stability surface for the case $1/R = 0.03$ and different values of β and B.

6 Numerical Results for the Base Flow Profile (9)

The results for profiles (8) and (9) for the case of plane shallow mixing layers would be exactly the same. Thus, in this section we investigate the effect of small curvature on the linear stability of the base flow (9).

Fig. 10 plots the marginal stability surface for the case $1/R = 0.03$ and different values of β and B.

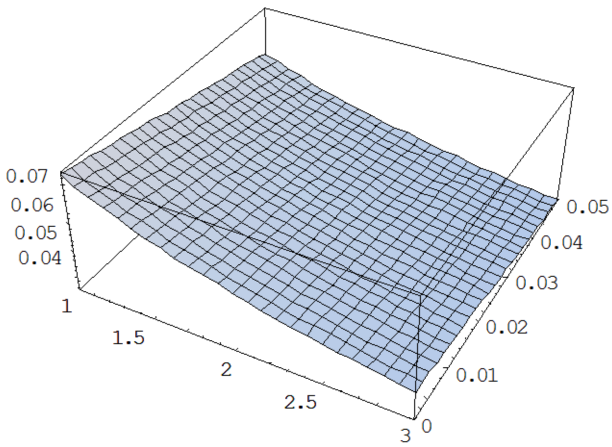

Fig. 10. Marginal stability surface for the case $1/R = 0.03$ and different values of β and B.

Comparing Figs. 9 and 10 we see that unstably curved mixing layer (the base flow profile (9)) is less stable than the base flow (8) which corresponds to stably curved mixing layer. The reason can be associated with centrifugal instability.

7 Conclusion

In this paper we presented an extensive parametric study of linear stability characteristics of shallow mixing layers. The combined effect of several parameters is analyzed: (a) particle loading parameter representing the effect of small heavy particles in the carrier fluid; (b) small curvature; (c) variable friction in the transverse direction. The analysis is performed for two types of base flows: (a) stably curved mixing layer and (b) unstably curved mixing layer. The major results of the study are as follows.

It is shown that for stably curved mixing layers all three parameters: particle loading parameter, small curvature and non-uniform friction have a stabilizing influence on the flow. Unstably curved mixing layer is also stabilized by non-uniform friction while the increase in curvature destabilizes the flow.

Experiments in [11]-[15] show that the base flow profile is not symmetric with respect to the transverse coordinate. The effect of base flow asymmetry on the stability of shallow mixing layers with uniform friction is analyzed in [23]. The authors are currently investigating linear stability of shallow mixing layers with non-uniform friction for the case where the base flow is not symmetric.

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