On the steady two-dimensional flow of blood with heat transfer in the presence of a stenosis

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Abstract: In the present study, we consider the steady two-dimensional flow of blood in an axisymmetric artery having a constriction, referred to as a stenosis, of cosine shape with heat transfer. Blood is assumed to behave like an incompressible Newtonian fluid. The governing Navier-Stokes equations are transformed and solved analytically by the regular perturbation technique. The results thus obtained are presented graphically in the form of stream lines, wall shear stress, separation point, pressure distribution, velocity components and temperature distribution. It is observed that an increase in the height of the constriction, increases the velocity of blood, wall shear stress, pressure and temperature. A parametric study of the blood flow behavior is presented and some of the results are compared with published literature.

Key–Words: Heat transfer, wall shear stress, pressure distribution

1 Introduction

The flow of blood in the circulatory system in humans is driven by the heart, which pumps blood producing oscillatory flow in blood vessels. It is well known that the deposit of cholesterol and proliferation of connective tissue may be responsible for the abnormal growths in the lumen of the artery. The progression of such a constriction, referred to as a stenosis, in an artery is caused by irregularities on the boundary. The flow of blood has been discussed by many authors theoretically, experimentally as well as numerically for better understanding of the flow properties in the presence of a constriction. For instance Forrester and Young [1] presented the analytical solution of a Newtonian fluid for axisymmetric, steady, incompressible flow and considered a mild constriction for the flow of blood both theoretically and experimentally in a converging and diverging vessel. Morgan and Young [2] presented the approximate analytical solution of axisymmetric, steady flow which is applicable to both mild and severe constrictions by using an integral method. K. Haldar [3] presented the analysis of blood flow treating it as a viscous fluid flowing through an axisymmetric artery having constriction of cosine shape. Chow and Soda [4] presented the analytical solution for Newtonian fluid flow in an axisymmetric tube valid for the case where the spread of roughness is large compared with the mean radius of the tube. Chow *et al.* [5] analyzed the steady laminar flow of a Newtonian fluid for different physical quantities by considering a sinusoidal wall surface variation. A more recent analysis has been done by Misra and Shit [6] where heat transfer is considered.
Previous work found in the literature and cited above is limited to the flow pattern, pressure gradient, separation and reattachment points. In the present investigation blood is assumed to be Newtonian fluid of constant density flowing through an axisymmetric artery with a constriction of cosine shape and constant volume flow rate. In the present paper the stream lines, wall shear stress, pressure, separation and reattach-

ment points, velocity profile and temperature distribution of blood flowing through an artery is analyzed. It is worth noting that the temperature distribution is important in most living creatures, as it is well known that the flow of blood controls the temperature of the body. The results, where possible, are compared with published data and found acceptable.

Nomenclature

$\mathbf{V}$	Velocity vector (m/s)
∇	del operator
p	scalar pressure(Pa)
d/dt	material time derivative
c_p	specific heat(J/kgK)
T	temperature(C^o)
u, w	velocity components(m/s)
r, z	coordinate axis(m)
$\mathbf{A}_1$	first Rivlin-Ericksen tensor
$l_o/2$	length of stenosis(m)
$R(z)$	variable width between the stenosis(m)
R_o	radius of unobstructed artery(m)
u_o	average velocity(m/s)
T_1, T_o	temperatures on boundary of stenosis and fluid(C^o)
Re	Reynolds number
Br	Brinkman number
Pe	Peclet number
$f(z)$	dimensionless boundary profile
τ	extra stress tensor
ρ	density
κ	thermal conductivity
ϕ	viscous dissipation function
μ	dynamic viscosity(Pa/s)
$\top$	transpose
λ, ϵ	maximum height of stenosis
θ	dimensionless temperature
ν	kinematic viscosity(m^2/s)
ψ	stream function
δ	ratio of R_o and l_o
y	ratio of r and f
τ_ω	wall shear stress

2 Governing Equations

The governing equations of motion for a non-isothermal, incompressible linearly viscous fluid consist of the conservation of mass, momentum and energy, given as

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{V}} = 0, \tag{1}$$

$$\rho \frac{d\widetilde{\mathbf{V}}}{dt} = -\widetilde{\nabla}\widetilde{p} + div\widetilde{\tau} + \rho\widetilde{b}, \tag{2}$$

$$\rho c_p \frac{d\widetilde{T}}{dt} = \kappa \widetilde{\nabla}^2 \widetilde{T} + \phi. \tag{3}$$

Where $\widetilde{\mathbf{V}}$ is the velocity vector, ρ the constant density, $\widetilde{p}$ the hydrodynamic pressure, $\widetilde{b}$ the body force per unit mass, $\widetilde{\tau}$ the extra stress tensor, c_p the specific heat, κ the thermal conductivity, $\widetilde{\nabla}^2$ denotes the Laplacian operator, $\widetilde{T}$ the temperature, ϕ symbolizes the viscous dissipation function which is defined as $\phi = \widetilde{\tau} \cdot \widetilde{\nabla}\widetilde{\mathbf{V}}$ and d/dt the material time derivative given as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \widetilde{\mathbf{V}} \cdot \widetilde{\nabla}. \tag{4}$$

The constitutive equation for the extra stress tensor $\widetilde{\tau}$ for a linearly viscous fluid is given by

$$\widetilde{\tau} = \mu \widetilde{\mathbf{A}}_1, \tag{5}$$

where μ is the dynamic viscosity and $\widetilde{\mathbf{A}}_1$ the first Rivlin-Ericksen tensor defined as

$$\widetilde{\mathbf{A}}_1 = \widetilde{\nabla}\widetilde{\mathbf{V}} + \left(\widetilde{\nabla}\widetilde{\mathbf{V}}\right)^\top, \tag{6}$$

where the superscript $\top$ defines the transpose of the tensor.

3 Problem Formulation

It is assumed that blood behaves like a homogeneous, incompressible, Newtonian fluid and the flow field is independent of time. At the inlet and outlet sections of the artery the flow is assumed to be Poiseuille or fully developed. It is further assumed that the flow in the artery has a symmetric constriction of cosine shape [3]. The width of the unobstructed region is $2R_o$ and $R(\widetilde{z})$ is the variable radius of the artery in the constricted region. The $\widetilde{z}$-axis is considered along the axial direction of the artery and $\widetilde{r}$-axis normal to it. The profile for the constriction in dimensional form is

$$R(\widetilde{z}) = \begin{cases} R_o - \frac{\lambda}{2}\left(1 + \cos\left(\frac{4\pi\widetilde{z}}{l_o}\right)\right), & -\frac{l_o}{4} < \widetilde{z} < \frac{l_o}{4}, \\ R_o, & \text{otherwise}, \end{cases} \tag{7}$$

where λ is the maximum height of the constriction and $l_o/2$ is the length of the constricted region. According to the geometry of the problem, the boundary conditions on the velocity components take the form

$$\begin{aligned} &\widetilde{u} = \widetilde{w} = 0, \text{ at } \widetilde{r} = R(\widetilde{z}); \frac{\partial \widetilde{w}}{\partial \widetilde{r}} = 0, \text{ at } \widetilde{r} = 0, \\ &\text{and } \int_0^{R(\widetilde{z})} \widetilde{r}\widetilde{w} d\widetilde{r} = \frac{1}{2} u_o R_o^2, \end{aligned} \tag{8}$$

where the first two conditions are no slip, third is symmetry, fourth is the constant volume flow rate and u_o

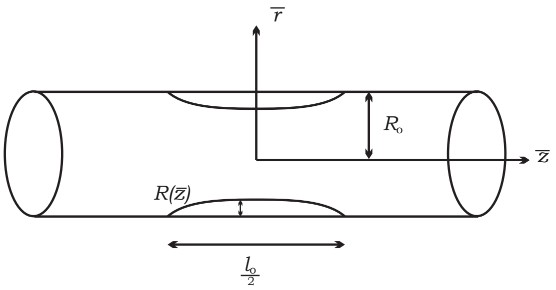


Figure 1: Cylindrical tube representing the artery with stenosis.

is the characteristic velocity. The boundary conditions on the temperature distribution are

$$\widetilde{T} = T_1 \quad at \quad \widetilde{r} = R(\widetilde{z}), \qquad \frac{\partial \widetilde{T}}{\partial \widetilde{r}} = 0 \quad at \quad \widetilde{r} = 0, \tag{9}$$

where the first condition denotes the temperature at the boundary and second at the center of the artery. For the steady axisymmetric flow of blood in cylindrical coordinates, the velocity vector $\widetilde{\mathbf{V}}$ is assumed to be of the form

$$\widetilde{\mathbf{V}} = (\widetilde{u}(\widetilde{r}, \widetilde{z}), 0, \widetilde{w}(\widetilde{r}, \widetilde{z})). \tag{10}$$

Introducing the dimensionless variables

$$u = \frac{\widetilde{u}}{u_o}, w = \frac{\widetilde{w}}{u_o}, r = \frac{\widetilde{r}}{R_o}, z = \frac{\widetilde{z}}{l_o}, p = \frac{R_o^2}{\mu u_o l_o}\widetilde{p}, \quad \theta = \frac{\widetilde{T} - T_o}{T_1 - T_o}, \tag{11}$$

where T_o and T_1 are the temperature of the blood at the center and boundary respectively. The component form of the equations of motion are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \delta \frac{\partial w}{\partial z} = 0, \tag{12}$$

$$Re\delta \left(u\frac{\partial u}{\partial r} + \delta w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial r} + \delta^2 \frac{\partial \Omega}{\partial z}, \tag{13}$$

$$Re \left(u\frac{\partial w}{\partial r} + \delta w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \Omega, \tag{14}$$

$$Pe \left(u\frac{\partial \theta}{\partial r} + \delta w \frac{\partial \theta}{\partial z} \right) = \nabla^2 \theta + Br \left[2 \left\{ \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{u}{r} \right)^2 + \delta^2 \left(\frac{\partial w}{\partial z} \right)^2 \right\} + \left(\delta \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 \right], \tag{15}$$

where

$$\Omega = \delta \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \delta^2 \frac{\partial^2}{\partial z^2}, \tag{16}$$

and

$$Re = \frac{u_o R_o}{\nu}, \delta = \frac{R_o}{l_o}, Br = \frac{\mu u_o^2}{\kappa(T_1 - T_o)}, \quad Pe = \frac{\rho c_p u_o R_o}{\kappa}, \tag{17}$$

where Re is the Reynolds number, Br the Brinkman number and Pe the Peclet number. The dimensionless form of the boundary profile for the constricted region is

$$f(z) = \begin{cases} 1 - \frac{\epsilon}{2}(1 + \cos 4\pi z) & -\frac{1}{4} < z < \frac{1}{4}, \\ 1 & \text{otherwise}, \end{cases} \tag{18}$$

where $f = R(\widetilde{z})/R_o$ and $\epsilon = \lambda/R_o$ is the dimensionless measure of the constriction in reference to the artery.

Now introducing the stream functions as

$$u = \frac{\delta}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{19}$$

satisfying the continuity equation (12) identically. After eliminating the pressure gradient term from the momentum equations (13) and (14), the compatibility equation is obtained as

$$Re\delta \frac{\partial \left(\psi, \frac{E^2\psi}{r^2} \right)}{\partial (z, r)} = \frac{1}{r} E^4 \psi, \tag{20}$$

where $E^2 = \dfrac{\partial^2}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial}{\partial r} + \delta^2 \dfrac{\partial^2}{\partial z^2}$, and the energy equation in terms of the stream function takes the form

$$Pe \frac{\delta}{r} \frac{\partial(\theta, \psi)}{\partial(r, z)} = \nabla^2 \theta + Br \left[2\delta^2 \left\{ \left(\frac{\partial}{\partial r} \left(\frac{1}{r}\frac{\partial \psi}{\partial z} \right) \right)^2 + \left(\frac{1}{r^2}\frac{\partial \psi}{\partial z} \right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial r} \right) \right)^2 \right\} + \left(\frac{\delta^2}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r} \left(\frac{1}{r}\frac{\partial \psi}{\partial r} \right) \right)^2 \right]. \tag{21}$$

The boundary conditions in terms of the stream functions become

$$-\frac{\partial}{\partial r} \left(\frac{1}{r}\frac{\partial \psi}{\partial r} \right) = 0, \psi = 0, \frac{\partial \theta}{\partial r} = 0 \quad \text{at} \quad r = 0, \qquad -\frac{1}{r}\frac{\partial \psi}{\partial r} = 0, \quad \psi = -\frac{1}{2}, \quad \theta = 1 \quad \text{at} \quad r = f, \tag{22}$$

Now our aim is to find the solution of the above equation along with the specified boundary conditions.

4 Solution to the Problem

To solve the compatibility equation (20) and energy equation (21) along with boundary conditions (22), we apply the regular perturbation technique on ψ and θ in terms of δ, being the small parameter of the form

$$\psi = \psi_o + \delta\psi_1 + \delta^2\psi_2 + \cdots, \text{ and } \theta = \theta_o + \delta\theta_1 + \delta^2\theta_2 + \cdots. \tag{23}$$

By making use of equation (23) in (20)-(22), one can obtain the system of equations corresponding to the different orders of δ as follows.

4.1 Zeroth order problem and solution

The Zeroth order system of equations is obtained by equating the coefficients of δ^0 on both sides of the compatibility and energy equations of the form

$$\frac{1}{r}D^2\psi_o = 0, \quad \text{where} \quad D^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}, \tag{24}$$

$$\begin{aligned} \nabla_1\theta_o &= -Br\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right)\right)^2, \\ \text{where} \qquad \nabla_1 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \end{aligned} \tag{25}$$

with the zeroth order boundary conditions

$$\begin{aligned} &-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right) = 0, \psi_o = 0, \frac{\partial\theta_o}{\partial r} = 0 \quad at \quad r = 0, \\ &-\frac{1}{r}\frac{\partial\psi_o}{\partial r} = 0, \quad \psi_o = -\frac{1}{2}, \quad \theta_o = 1 \quad at \quad r = f. \end{aligned} \tag{26}$$

The solution of (24) is obtained by integrating and making use of the respective boundary conditions and is given as

$$\psi_o = \frac{y^2}{2}\left(y^2 - 2\right) \quad \text{where} \quad y = \frac{r}{f}, \tag{27}$$

which is similar to [3, 4]. For the zeroth order temperature, we proceed by substituting (27) in (25) and integrating subject to the boundary conditions on temperature, thus giving

$$\theta_o = 1 - \frac{Br}{f^4}\left(y^4 - 1\right), \tag{28}$$

which depends upon the ratio of the heat production by viscous dissipation to heat transport by conduction.

4.2 First order problem and solution

The first order system is obtained by comparing the coefficient of δ on both sides of (20)-(21) and is given as

$$\frac{1}{r}D^2\psi_1 = Re\frac{\partial(\psi_o, \frac{D\psi_o}{r^2})}{\partial(z, r)}, \tag{29}$$

$$\begin{aligned} \nabla_1\theta_1 &= \frac{Pe}{r}\frac{\partial(\psi_o, \theta_o)}{\partial(z, r)} \\ &- 2Br\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)\right\}, \end{aligned} \tag{30}$$

with corresponding boundary conditions

$$\begin{aligned} &-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right) = 0, \psi_1 = 0, \frac{\partial\theta_1}{\partial r} = 0 \text{ at} \quad r = 0, \\ &-\frac{1}{r}\frac{\partial\psi_1}{\partial r} = 0, \quad \psi_1 = 0, \quad \theta_1 = 0 \quad \text{at} \quad r = f. \end{aligned} \tag{31}$$

The solution of equation (29) is obtained by integrating and using (27) along with the corresponding boundary conditions in (31), and is given as

$$\psi_1 = \frac{Ref'y^2}{36f}\left(y^6 - 6y^4 + 9y^2 - 4\right), \tag{32}$$

which is similar to [3, 4]. The solution of the first order temperature is obtained by substituting (27)-(28) and (32) in (30) and integrating along with the corresponding boundary conditions, thus

$$\begin{aligned} \theta_1 = \tfrac{Brf'}{72f^5}&\left(1 - y^2\right)\left[3\left(3Pe + 4Re\right)y^6 - \left(7Pe + 52Re\right)y^4\right. \\ &\left. - \left(43Pe - 20Re\right)y^2 + 101Pe + 20Re\right], \end{aligned} \tag{33}$$

which also depends upon the ratio of heat transport by convection to conduction.

4.3 Second order problem and solution

To obtain the solution for the second order system, we equate the coefficients of δ^2 on both sides of equations (20)-(22) thus

$$\begin{aligned} \frac{1}{r}D^2\psi_2 &= -\frac{2}{r}D\left(\frac{\partial^2\psi_o}{\partial z^2}\right) + \\ Re&\left[\frac{\partial\left(\psi_o, \frac{D\psi_1}{r^2}\right)}{\partial(z, r)} + \frac{\partial\left(\psi_1, \frac{D\psi_o}{r^2}\right)}{\partial(z, r)}\right], \end{aligned} \tag{34}$$

$$\nabla_1\theta_2 = -\frac{\partial^2\theta_o}{\partial z^2} + \frac{Pe}{r}\left[\frac{\partial(\psi_o,\theta_1)}{\partial(z,r)} + \frac{\partial(\psi_1,\theta_o)}{\partial(z,r)}\right]$$
$$- 2Br\left[\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial z}\right)\right)^2 + \left(\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right)\right)^2\right.$$
$$+ \frac{1}{2}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)\right)^2 + \left(\frac{1}{r^2}\frac{\partial\psi_o}{\partial z}\right)^2 +$$
$$\left.\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right) - \frac{1}{r}\frac{\partial^2\psi_o}{\partial z^2}\right\}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right)\right], \tag{35}$$

subject to corresponding boundary conditions

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right) = 0, \psi_2 = 0, \frac{\partial\theta_2}{\partial r} = 0 \quad \text{at} \quad r = 0,$$
$$-\frac{1}{r}\frac{\partial\psi_2}{\partial r} = 0, \quad \psi_2 = 0, \quad \theta_2 = 0 \quad at \quad r = f. \tag{36}$$

Solution of (34) is obtained by integrating and making use of (27) and (32) along with the boundary conditions and is given as

$$\psi_2 = \frac{y^2\left(y^2-1\right)^2}{6}\left[f'^2\left\{-5 + \frac{Re^2}{3600f^2}\left(38y^6 - \right.\right.\right.$$
$$\left.\left. 314y^4 + 759y^2 - 818\right)\right\}$$
$$\left. + f''\left\{f - \frac{Re^2}{1200f}\left(2y^6 - 16y^4 + 41y^2 - 52\right)\right\}\right], \tag{37}$$

which is slightly different from [3], but the graphical representation is similar. The second order temperature is obtained by substituting (27)-(28), (32)-(33) and (37) in (35) along with the boundary conditions is given as

$$\theta_2 = -\frac{Br\left(y^2-1\right)}{f^6}\left[\frac{Pe^2}{576}\left(5f'^2 - ff''\right)\left(y^{10} - 3y^8\right.\right.$$
$$\left. -8y^6 + 72y^4 - 173y^2 - 231\right) - \frac{f^2}{9}\left\{f'^2\left(26y^4 + \right.\right.$$
$$\left.\left. 47y^2 - 106\right) - ff''\left(14y^4 - 7y^2 - 16\right)\right\} +$$
$$f'^2 PeRe\left\{\frac{1}{648}\left(11y^{10} - 79y^8\right) + \frac{1069}{5184}y^6 + \right.$$
$$\left.\frac{1}{1728}\left(47y^4 - 685y^2 + 899\right)\right\} + Re^2 f'^2\left\{\frac{1}{540}\right.$$
$$\left(13y^{10} - 107y^8\right) + \frac{1147}{2160}y^6 - \frac{3679}{6480}y^4 + \frac{163}{1296}$$
$$\left.\left(y^2+1\right)\right\} - ff''PeRe\left(\frac{1}{648}y^{10} - \frac{209}{16200}y^8 + \right.$$
$$\left.\frac{3953}{129600}y^6 - \frac{1}{1728}\left(61y^4 - 439\right) - \frac{683}{14400}y^2\right)$$
$$- Re^2 ff''\left(\frac{1}{540}y^{10} - \frac{43}{2700}y^8 + \frac{1}{10800}\left(503y^6\right.\right.$$
$$\left.\left.\left. -697y^4 + 173\left(y^2+1\right)\right)\right)\right], \tag{38}$$

which depends on the ratio of the heat production by viscous dissipation to heat transport along with heat transport by convection to conduction. Now one can easily find the stream function ψ with the help of (27), (32) and (37). The velocity components u, w can be found easily by substituting ψ in (19). The expression for the temperature distribution θ can be obtained by substituting (28), (33) and (38) in (23).

4.4 Pressure Distribution

To find the pressure distribution within the artery, we apply the perturbation technique on the pressure and substitute $p = p_o + \delta p_1 + \delta^2 p_2 + \cdots$, in equations (13)-(14) along with (19) and comparing the coefficients of the zeroth, first and second powers of δ, we obtain the corresponding systems and solve these systems by using the relation

$$p = \int_0^r \frac{\partial p}{\partial r}dr + \int_0^z \frac{\partial p}{\partial z}dz, \tag{39}$$

to get the different order of pressure in the subsections that follow.

4.4.1 Zeroth order pressure

Comparing the coefficients of δ^0, we get

$$\frac{\partial p_o}{\partial r} = 0, \tag{40}$$

$$\frac{\partial p_o}{\partial z} = -\nabla_1 \left(\frac{1}{r} \frac{\partial \psi_o}{\partial r} \right). \tag{41}$$

Using relation (39), we obtain the explicit expression for the zeroth order pressure

$$p_o = \frac{1}{12\pi} \left[\frac{3}{(1-\epsilon)^{\frac{7}{2}}} \left(5\epsilon^3 - 18\epsilon^2 + 24\epsilon \right.\right.$$
$$\left. -16 \right) \tanh^{-1} \left(\frac{\tan 2\pi z}{\sqrt{1-\epsilon}} \right) + \frac{\epsilon \sin 4\pi z}{8f^3(\epsilon-1)^3} \left\{ (\epsilon-1)^2 \right.$$
$$\left.\left. +40f(\epsilon-1)(\epsilon-2) + 4f^2 \left(15\epsilon^2 - 44(\epsilon-1) \right) \right\} \right]. \tag{42}$$

The zeroth order pressure depends upon the constriction height.

4.4.2 First order pressure

Equating the coefficients of δ on both sides of (13)-(14) after substitution of (19), we get

$$\frac{\partial p_1}{\partial r} = 0, \tag{43}$$

$$\frac{\partial p_1}{\partial z} = \frac{Re}{r} \frac{\partial \left(\psi_o, \frac{1}{r}\frac{\partial \psi_o}{\partial r} \right)}{\partial (z,r)} - \nabla_1 \left(\frac{1}{r} \frac{\partial \psi_1}{\partial r} \right). \tag{44}$$

The first order pressure is obtained by substituting the above calculated expressions and applying the relation (39) as

$$p_1 = -Re \left(\frac{1}{f^4} - \frac{1}{(1-\epsilon)^4} \right). \tag{45}$$

It is observed that the first order pressure depends on Re and the height of the constriction.

4.4.3 Second order pressure

Comparing coefficients of δ^2, we obtain

$$\frac{\partial p_2}{\partial r} = \nabla_1 \left(\frac{1}{r} \frac{\partial \psi_o}{\partial r} \right) - \frac{1}{r^3} \frac{\partial \psi_o}{\partial z}, \tag{46}$$

$$\frac{\partial p_2}{\partial z} = \frac{Re}{r} \left[\frac{\partial \left(\psi_o, \frac{1}{r}\frac{\partial \psi_1}{\partial r} \right)}{\partial (z,r)} + \frac{\partial \left(\psi_1, \frac{1}{r}\frac{\partial \psi_o}{\partial r} \right)}{\partial (z,r)} \right]$$
$$- \nabla_1 \left(\frac{1}{r} \frac{\partial \psi_2}{\partial r} \right) - \frac{\partial^2}{\partial z^2} \left(\frac{1}{r} \frac{\partial \psi_o}{\partial r} \right). \tag{47}$$

Solution of the above equations is obtained by substituting ψ_o, ψ_1 and ψ_2 in expressions (46)- (47) and making use of (39), thus

$$p_2 = -\frac{\pi\epsilon}{518400} \left[\frac{1}{(\epsilon-1)^{\frac{9}{2}}} \left\{ 240\epsilon(\epsilon-2) \right.\right.$$
$$\tanh^-1 \left(\frac{\tan 2\pi z}{\sqrt{\epsilon-1}} \right) \left(11520(\epsilon-1)^2 + \right.$$
$$\left.\left. 11Re^2 \left(7\epsilon^2 - 16(\epsilon-1) \right) \right) \right\} + \frac{16\sqrt{(f-1)(f+\epsilon-1)}}{f^5\epsilon(\epsilon-1)^4}$$
$$\left\{ 96(\epsilon-1)^4 \left(10800r^2 + 121Re^2 \right) + \right.$$
$$\left. 528Re^2 f(\epsilon-2)(\epsilon-1)^3 \right\} + 88f^2(\epsilon-1)^2$$
$$\left\{ -7200(\epsilon-1)^2 + Re^2 \left(7\epsilon^2 - 12(\epsilon-1) \right) \right\}$$
$$+ 2f^3(\epsilon^2 - 3\epsilon + 2) \left\{ 57600(\epsilon-1)^2 + 11Re^2 \left(35\epsilon^2 \right.\right.$$
$$\left.\left. -24(\epsilon-1) \right) \right\} + f^4 \left\{ 57600(\epsilon-1)^2 \left(3\epsilon^2 - 4(\epsilon-1) \right) \right.$$
$$\left.\left. +11Re^2 \left(105\epsilon^4 - 308\epsilon^3 + 476\epsilon^2 - 192\epsilon + 96 \right) \right\} \right]. \tag{48}$$

It is found that the second order pressure depends on the height of the constriction.

5 Wall shear stress

The dimensionless form of the wall shear stress is given as

$$\tau_\omega = \delta \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \tag{49}$$

and making use of the velocity components computed up to the second order in δ, the wall shear stress is obtained as

$$\tau_\omega = -\frac{1}{f^3} + \frac{Re\delta f'^2}{6f^4} + \delta^2 \left[\frac{Re^2}{2160f^5} \left(67f'^2 - 15ff'' \right) \right.$$
$$\left. + \frac{1}{3f^3} \left(8f'^2 - ff'' \right) \right]. \tag{50}$$

The points of separation and reattachment at the wall are those points where reverse flow occurs and are obtained by setting the wall shear stress equal to zero. The resulting equation is quadratic in Re, the solution in terms of the Reynolds number is

$$Re = \frac{6}{\delta \left(15ff'' - 67f'^2 \right)} \left[15ff' \pm \sqrt{A} \right], \tag{51}$$

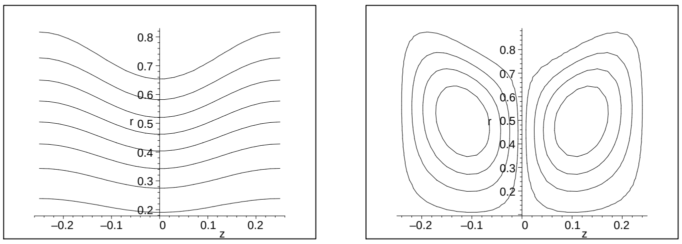
where

$$A = 5 \left(246f^2f'^2 - 45f^3f'' - \delta^2 f^2 \left(536f'^4 \right.\right.$$
$$\left.\left. -187ff'^2f'' + 15f^2f''^2 \right) \right). \tag{52}$$

Now from the above relation, our aim is to find the critical Reynolds number graphically at which the separation and reattachment points occur.

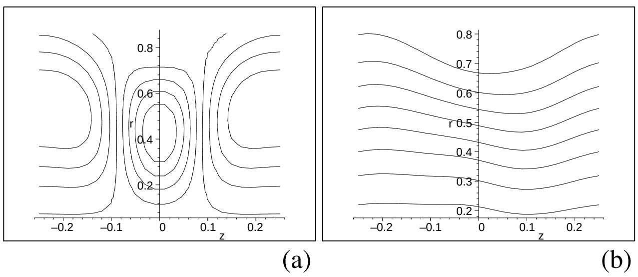
6 Graphical Discussion

In this section, the effect of various parameters that control the fluid flow are discussed. The geometry of the proposed model for the study of constricted artery is depicted in Figure 1. The radii of the obstructed and unobstructed sections are $R(z)$ and R_o respectively. In Figures 2 and 3, the stream lines for the flow pattern are shown by taking the z-axis parallel to the axis of the artery and the r-axis perpendicular to the axis of the artery. The zeroth order solution corresponds to the flow with vanishing wall slopes and reduces to Poiseuille for $\epsilon = 0$. The stream lines are relatively straight lines in the center for $\epsilon = 0.20, Re = 12, \delta = 0.1$. The first order solution is depicted in Figure 2(b), which induces the clockwise and counterclockwise rotational motion in the converging and diverging region of the artery. This shows that the separation point lies in the converging region of the artery and reattachment point lies in the diverging region. Figure 3(a) presents graphically the



(a) (b)

Figure 2: The zeroth order stream lines for $\epsilon = 0.20, Re = 12, \delta = 0.1$ are shown in (a) and the first order stream lines are shown in (b).



(a) (b)

Figure 3: The second order stream lines are shown in (a) and the streamlines correct up to the second order in δ are shown in (b).

flow pattern for the second order stream lines which reinforce the first order solution and shows the rotational motion indicating the separation and reattachment points. Figure 3(b) shows the flow pattern of streamlines up to second order in δ in the converging and diverging region of the artery. It is observed that the stream lines are relatively straight at the center. The distribution of wall shear stress across the

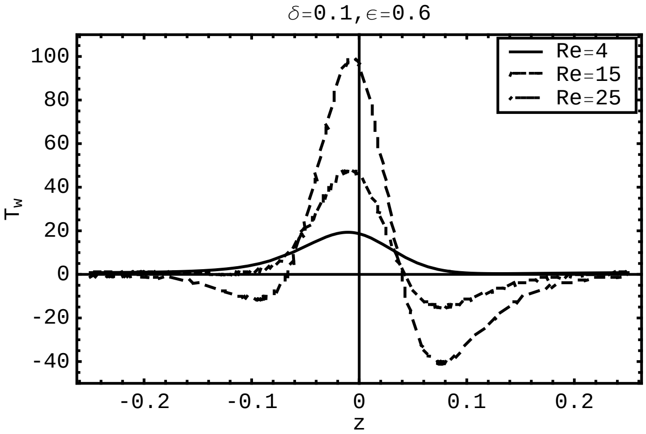


Figure 4: The effect of Re on wall shear stress.

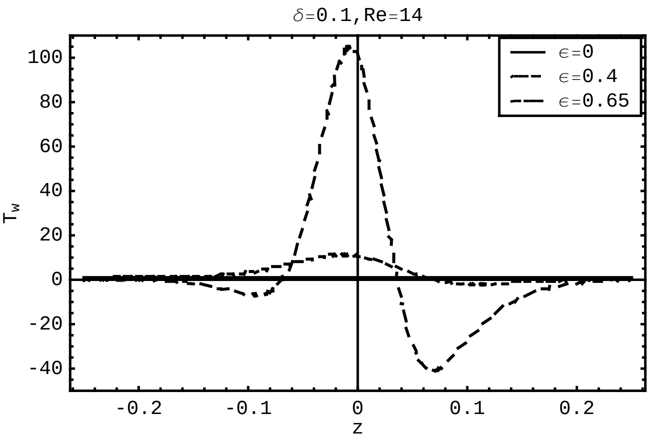


Figure 5: The effect of ϵ on wall shear stress .

constricted region has been described for the variation of Re in Figure 4 for $\epsilon = 0.6$ and $\delta = 0.1$. As Re increases, wall shear stress increases over the constriction and becomes negative in the converging and diverging sections of the artery. The negative shearing indicates the occurrence of separation point in the converging section and reattachment point in the diverging section. Figure 5 presents the effect of ϵ on the wall shear stress for $\delta = 0.1$ and $Re = 14$. As there is no constriction, the flow is Poiseuille flow or fully developed flow. Increases in ϵ lead to an increase in the wall shear stress and it becomes negative in the converging and diverging sections of the artery, which is due to back flow and indicates the points of separation and reattachment. Figure 6 presents the phenomena of the separation point. The separation point

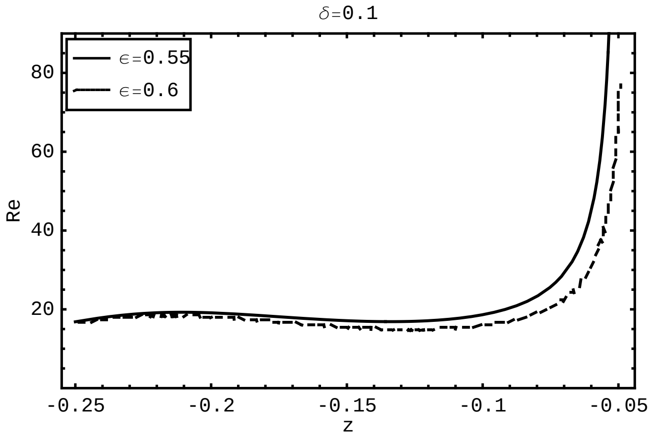


Figure 6: The separation points in converging region .

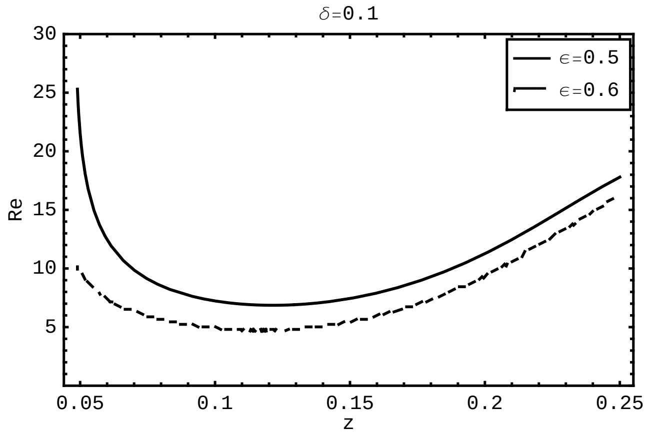


Figure 7: The reattachment points in diverging region .

occurs as there is no wall shear stress i.e., zero wall shear stress. From the graphical presentation for the separation point, our aim is to find the critical Re at which the separation occurs. It is observed that with the increase in ϵ, the critical Re decreases in the converging section of the artery for $\delta = 0.1$. Figure 7 depicts the reattachment point in the diverging region, an increase in height of constriction, the critical Re decreases with a fixed value of $\delta = 0.1$. It is noted that the theory that the critical Reynold number decreases with the increase in ϵ is verified.

Figures 8 and 9 shows the effect of the pressure on the flow at the center line of the artery for different Re and ϵ. An increase in the value of Re for fixed values of $\epsilon = 0.25$ and $\delta = 0.1$, the pressure increases in the diverging region of the artery. Similarly, figure 9 presents the behavior of ϵ for fixed $Re = 15$ and $\delta = 0.1$, as there is no constriction, pressure appears in a straight line and with an increase in ϵ, it is found that the pressure increases.

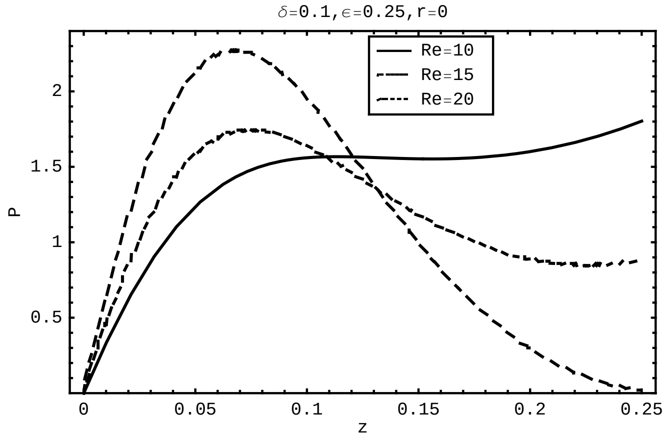


Figure 8: The pressure distribution for Re .

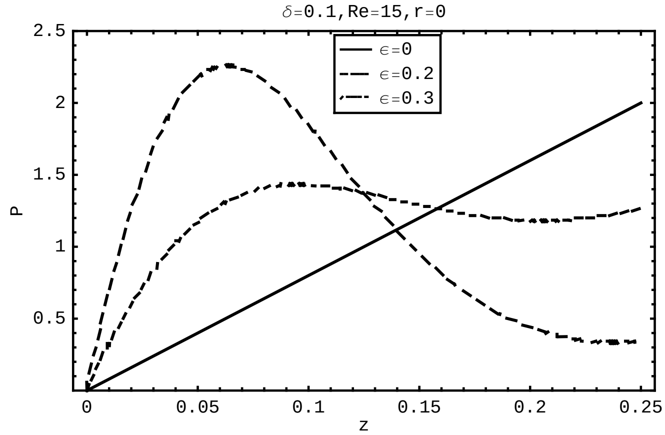


Figure 9: The pressure distribution for ϵ .

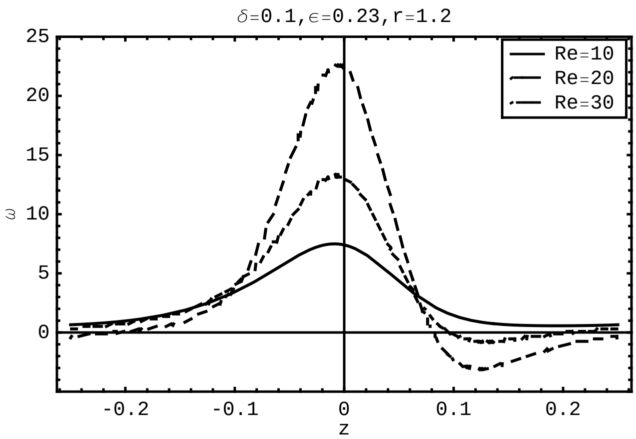


Figure 10: The axial velocity distribution for Re .

Figures 10 and 11 depicts the effect of axial velocity along the axis of artery for the variation of Re and ϵ with fixed values of $r = 1.2$ and $\delta = 0.1$. An

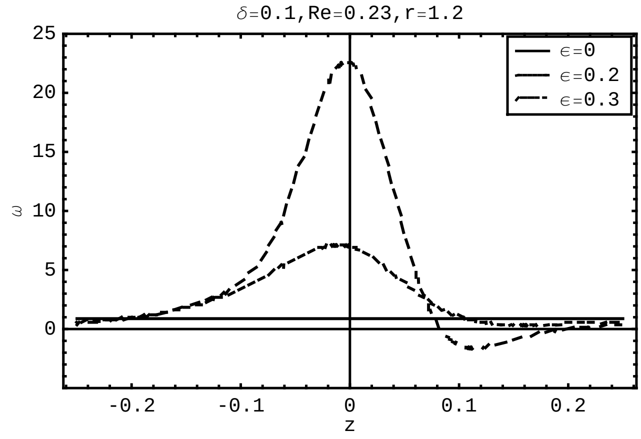


Figure 11: The axial velocity distribution for ϵ.

increase in Re, leads to increase in the axial velocity and it becomes maximum over the constricted region for $\epsilon = 0.23$. The velocity becomes negative in the converging as well as in the diverging sections of the artery. Negative velocity indicates reverse flow, expressing the separation and reattachment points in the converging and diverging sections. It is observed from Figure 11 that as there is no constriction, the velocity profile is a straight line for $Re = 15$. An increase in the height of constriction causes velocity increases on the constricted region and becomes negative in the diverging section of the artery predicting reverse flow.

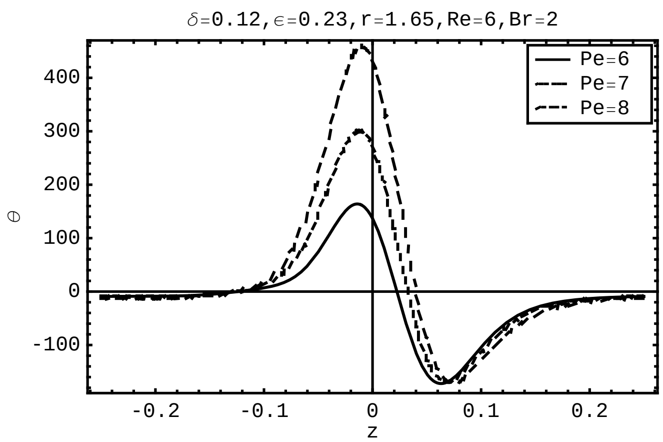


Figure 12: The effect of Pe on temperature distribution.

Figure 12 depicts the effect of Pe on temperature distribution. It is observed that an increase in Pe with other parameters fixed, leads to increase in the temperature over the constriction. The temperature becomes negative where reverse flow occurs in the converging and diverging region of the artery. Similarly, Figure

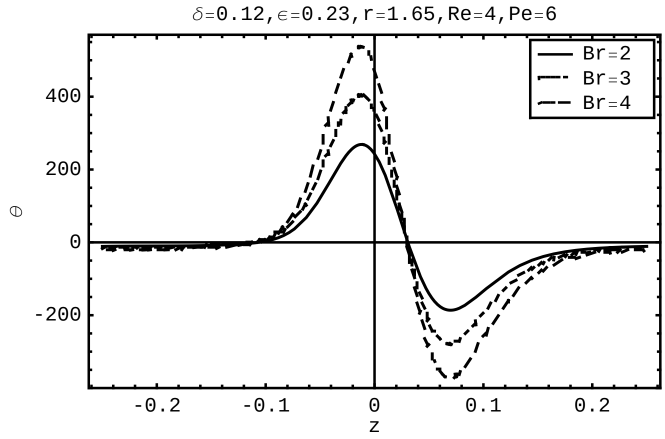


Figure 13: The effect of Br on temperature distribution .

13 presents the behavior of Br on the temperature distribution for the fixed values of other parameters. Increase in Br increases the temperature over the constricted region and becomes negative in the converging and diverging region of the artery due to reverse flow.

Figure 14 shows the effect of ϵ on the temperature distribution. An increase in ϵ leads to increase in temperature over the constricted region and the temperature becomes negative in the converging and diverging regions of the artery, which indicates that there are separation and reattachment points. It is important to note that negative values of temperature are to be taken as indicators and not literally.

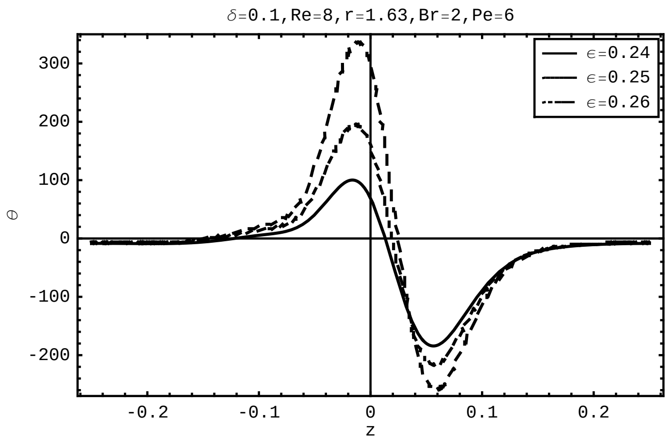


Figure 14: Effect of ϵ on the temperature distribution.

7 Conclusion

In the present paper the steady state flow of blood is considered with heat transfer in an axisymmetric artery having a constriction of cosine shape. It is assumed that blood behaves like an homogeneous, incompressible Newtonian fluid in a tube representing an artery. The governing equations are transformed and solved analytically with the help of the regular perturbation technique. The solutions are obtained for velocity components, pressure, wall shear stress, separation and reattachment points and temperature distribution. It is observed that with the increase in height of constriction wall shear stress, pressure, velocity and temperature increases while the separation and reattachment points decrease. It is found that with the increase in Re wall shear stress, pressure and velocity increase. It is also observed that the solutions obtained in this analysis are compatible with the results existing in the literature. The general pattern of streamlines is similar to [4, 5], the wall shear stress is similar to [1, 3] and separation and reattachment points are similar to [3].

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