Investigation of Queuing Systems in System Structure Management

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Abstract: Queuing theory is used to develop recommendations for constructing queuing systems efficiently, organizing the associated operations and functionalities, and regulating request flows for optimal performance. This paper presents a study of the income functional for two specific cases of controlled queuing systems: the M/G*/1/N* system for a controlled service duration and number of waiting spaces, and the G*/M/n/m queuing system with a controlled arrival flow. The construction of a controlled semi-Markov process and the construction of an income functional on its trajectories were used as the basis for this study. The task is to find the optimal control strategy in the given queuing systems. An algorithm for finding optimal strategies applicable to similar queuing systems to increase their functioning efficiency when controlling the system's main characteristics was developed for both systems.

Key-Words: - semi-Markov process; queuing system; income functional; controlled arrival flow; optimal strategy

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1 Introduction

Queuing theory mainly deals with the construction of mathematical models linking a system's operating conditions with its performance indicators that show how effectively the system copes with the request flow.

A queuing system is understood as a system consisting of an incoming stream, queue rules, and a service mechanism.

Developing a mathematical model of the process allows for carrying out mathematical calculations, creating a simulation model, and finding the necessary numerical properties to further optimize and improve the operation and functioning of real processes.

Various QS elements are used to describe a system:

Incoming request flow: probability distribution of request arrival.

Service time: a random process determines the request service time.

Number of service devices: can be a person (cashier, hairdresser, etc.) or some space (airport runway, parking lot, hospital ward, etc.).

Queue: request accumulator; there may be no waiting space in the system, or the queue (accumulator) may be infinitely large.

A queuing model may be represented by the letters A/B/S/R/D, where A is the probability distribution of the request arrival time, B is the probability distribution of the service time, S is the number of channels (or service stations), R is the system capacity or the number of spaces to wait in the queue (for $R \to \infty$, R is not specified), and D is the queue discipline. For example, M/M/1/∞/FIFO represents a queuing system in which the time between arrivals and the service time are exponentially distributed when the system has one server, with a first-in-first-out basis, and the number of clients allowed in the system may be infinite. The first three characteristics in the above notation were introduced by D. Kendall in 1953. Later, in 1966, A. Lee added the fourth and fifth characteristics to the notation. In the classical case of the notation, Kendall's symbolism providing for four digits is used.

Queuing theory is applied in various spheres, from economic to financial and informational spheres [1,2], work in production and technical workshops [3,4], and ending with such fields as human services, work with personnel, transport, hospitals, and the sale of products and various articles [5-7].

In each of these areas, when working with queuing systems, various tasks arise requiring not only the calculation of the stationary characteristics of systems, but also control of these characteristics, which may depend on various factors [8,9].

Various methods are used to study queuing systems, including both obtaining analytical expressions for specific systems and using software tools that allow for obtaining numerical values of indicators without deriving analytical expressions

For a long time, the method using the theory of Markov chains with continuous time was considered the most widespread modeling method. This method allows for calculating and analyzing how efficiently a queuing system operates and the probability of losing an incoming request. Similarly, methods allowing not only the calculation of a system's performance indicators, but also control of a studied queuing system are becoming increasingly popular [10,11].

The scientific article [12] by Ryzhikov and Ulanov presents an analysis of the structure of information flows in data transmission systems and a method modeling the service process with the Pareto distribution. In studying this method, by constructing a nested Markov chain, the authors obtained the expressions necessary for calculating the distribution of queuing system requests, and the Laplace–Stieltjes transform was used to obtain the expression necessary for the waiting time distribution.

The method described in the above article allows for obtaining analytical expressions to study the probabilistic–temporal characteristics of a multichannel QS and, consequently, to develop a computational program that allows the queuing system's operation to be more accurately studied. The method proposed by the above authors will help in the development and study of mathematical models of information and telecommunication networks, as well as allowing the fractal nature of traffic to be taken into account.

The article [13] authored by Sasanuma, Hampshire, and Scheller-Wolf discusses a queuing model for an M/M/s system with exits. In the event that this QS is overloaded, many customers and requests exit the queue, thus lowering the operating efficiency and effectiveness. To solve this problem, the authors proposed using an overload management scheme to manage customer arrival and service speeds during overload.

The authors used the Markov chain decomposition method in their article. This method allows for decomposing a complex Markov chain into simpler sub-chains that are then analyzed individually. In the above article, the authors implemented an overload-based management scheme that allows the system to change the arrival and service speeds under overload conditions, thus making the system less sensitive to parameter changes.

The work by Yousry H. [14] considers a Markov queuing model with one server and an infinite system queue. The article presents some additional performance indicators that depend on order statistics methods. The expected value and variance of the minimum (maximum) number of clients in the system (queue), as well as the moments of the minimum (maximum) waiting time in the queue, are displayed. Although this work is currently limited to the M/M/1 model, it can be applied to other queuing models.

The following question often arises: how can a system's quality be assessed, and how can its performance be optimized? For example, a study by Park Y-J and Yi C-Y [15] presented a simulation model of construction. In this work, a method for assessing quantitative quality indicators was used. The method comprises a prediction of the quality of indicators of the duration and cost of a specific construction operation when determining the production plan. The advantages of using the developed system include the following: firstly, this study presented a method for the simultaneous evaluation and qualitative performance of a construction operation taking into account the duration and cost; secondly, the system helps establish the optimal alternative plan for creating a construction plan that guarantees maximum quality for a limited period of time and cost of construction work. This study offers a useful method for managing performance indicators by evaluating design performance and quality for practical use.

In [2], Usmanov and Jarský explored the topic of applying queuing theory in construction. The task is for the investor (customer) to meet the planned completion dates and estimates. In practice, detailed and laborious optimization calculations are not allowed. The above study created technical and mathematical models leading to the optimization of construction processes (minimizing labor and costs, fuel consumption, construction time, etc.).

The work by Salawu et al. [1] describes a controlled M/M/S queuing model. The model describes a virtual production scenario in which robots are used as servers in the packaging stage. The system model is a multi-stage production system. The problem with this system is the delay at some stages of the process, leading to the formation of queues. This research work led to the development of a model and mathematical expression that are useful for manufacturers when making decisions at an early stage of production.

The procedure for modeling queuing systems is relevant and necessary when solving problems both related to eliminating the loss of requests due to downtime or channel overload [16,17], and the efficient management of incoming requests [18–20]. It can therefore be concluded that, thanks to various methods for modeling queuing systems, they are an

integral part of the analysis of ongoing work in many areas of activity.
One of the main problems in solving which queuing theory methods should be used is the search for management that takes into account both the improvement in the quality of service and the allowable costs associated with improving the system operation.
The purpose of this work is to demonstrate the application of the theory of controlled semi-Markov processes to system characteristics management. The method is based on the construction of the accumulation functional on the trajectories of a controlled semi-Markov process [21], representing an efficient method of increasing system efficiency by managing its structure.

2 Materials and methods

In this section we repeat some definitions presented [22-24] and then derive some relations for the semi-Markov kernel and probability measures used in construction of the accumulation functional. The classic case [22] has a semi-Markov process that is $\xi(t)$ defined by a homogeneous two-dimensional Markov chain or a homogeneous Markov renewal process:

$$(\xi_n, \theta_n), \quad n \geq 0, \quad \xi_n \in E, \quad \theta_n \in R^+ \tag{1}$$

Without limiting generality, the set of states E can be identified with the set $E = \{1,2,\ldots,N\}, \quad N < \infty$.

The homogeneous Markov chain (ξ_n, θ_n) is defined by transition probabilities called semi-Markov kernels:

$$P\{\xi_{n+1} = j, \theta_{n+1} < t/\xi_n = i, \theta_n = \tau\} = P\{\xi_{n+1} = j, \theta_{n+1} < t/\xi_n = i\} = Q_{ij}(t), \tag{2}$$

where $n \geq 1, \quad i,j \in E, \quad t \in$, and by some initial distribution:

$$p_i = P\{\xi_0 = i\} \geq 0, \quad i \in E, \quad \sum_{i \in E} p_i = 1. \tag{3}$$

In what follows, we assume $P\{\theta_0 = 0\} = 1$.
From Definition (2), we obtain a conditional distribution of random variables θ_n,

$$F_i(t) = P\{\theta_n < t/\xi_{n-1} = i\} = \sum_{j \in E} P\{\xi_n = j, \theta_n < t/\xi_{n-1} = i\} = \sum_{j \in E} Q_{ij}(t) \tag{4}$$

due to the incompatibility of events $\{\xi_n = j\}$ for different j.

For any pair $i,j \in E$, the inequality $Q_{ij}(t) \leq F_i(t)$ is true. Therefore, by virtue of the Radon–Nikodym theorem [23], there is a function $p_j(x,i)$ for which the relation $Q_{ij}(t) = \int_0^t p_j(x,i) dF_i(x)$ holds. The function $p_j(x,i)$ can be interpreted as a conditional probability:

$$p_j(x,i) = P\{\xi_{n+1} = j/\xi_n = i, \theta_{n+1} = x\}. \tag{5}$$

For $t \to \infty$, from definition (2), we obtain the matrix of transition probabilities of the nested Markov chain:

$$p_{ij} = lim_{t \to \infty} Q_{ij}(t) = P\{\xi_{n+1} = j/\xi_n = i\}, \quad i,j \in E \tag{6}$$

At $i,j \in E$, for which $p_{ij} > 0$, the conditional probability can be determined:

$$F_{ij}(t) = P\{\theta_n < t/\xi_n = j, \xi_{n-1} = i\} = \frac{Q_{ij}(t)}{p_{ij}} \tag{7}$$

and in the new notation, the following equality can be further used:

$$Q_{ij}(t) = p_{ij} F_{ij}(t) \tag{8}$$

If for $i,j \in E$, the equality $p_{ij} = 0$ is true, then Relation (7) is not defined and the conditional distribution $F_{ij}(t)$ needs to be further defined; for example, it can be considered a degenerate distribution.

Thus, we come to the conclusion that a Markov renewal process can be set in three ways:
1. By specifying the semi-Markov kernel $Q_{ij}(t)$;
2. By setting the transition probability matrix of the nested chain
$P = \{p_{ij}\}$ and functions $F_{ij}(t) = P\{\theta_n < t/\xi_n = j, \xi_{n-1} = i, \quad i,j \in E, \quad t \geq 0$;
3. By setting the conditional distributions $F_i(t) = P\{\theta_n < t/\xi_{n-1} = i\}$ and

$$p_j(x,i) = P\{\xi_{n+1} = j/\xi_n = i, \theta_{n+1} = x\}, \quad i,j \in E, x \geq 0, t \geq 0.$$

Note that all of the above cases require specifying the initial distribution of the first component of the Markov renewal process (3).
The semi-Markov kernel $Q_{ij}(t)$ has, under any conditions $i,j \in E, \quad t \geq 0$, properties arising from probability properties.

Next, we pay attention to the fact that elements of the semi-Markov kernel $Q_{ij}(t)$ determine the behavior of the Markov renewal process for one period (step) between neighboring moments of the state change, and the transition probabilities are assumed to be homogeneous (no dependence on the step number *n*).

The controlled semi-Markov process $X(t) = \{\xi(t), u(t)\}$ is defined by a homogeneous three-dimensional Markov chain or a *homogeneous controlled Markov renewal* process:

$$(\xi_n, \theta_n, u_n), \quad n \geq 0, \quad \xi_n \in E, \quad \theta_n \in R^+, u_n \in U \quad (9)$$

In the above notation (9), we consider:
$E = \{e_1, e_2, \ldots, e_N\}, \quad N < \infty$ - a finite set of states (in what follows, we will often identify set E with the set $E = \{1,2,\ldots,N\}, \quad N < \infty$), where the first component ξ_n of the homogeneous controlled Markov renewal process takes discrete values from this set.
R^+— a set of positive real numbers; therefore, we identify the second component θ_n of the homogeneous controlled Markov renewal process with time, and in the space R^+, we define the Borel σ-algebra.
U is the control space with σ-algebra $\boldsymbol{A}$ of subsets of this space.

The homogeneous Markov chain (ξ_n, θ_n, u_n) is defined by transition probabilities:

$$P\{\xi_{n+1} = j, \theta_{n+1} < t, u_{n+1} \in B / \xi_n = i, \theta_n = \tau, u_n = u\},$$
$$i, j \in E, \quad t, \tau \in R^+, u \in U, \quad B \in A \quad (10)$$

and the initial distribution:

$$p_i = P\{\xi_0 = i, \theta_0 < \infty, u_0 \in U\} \quad (11)$$

In the studied case, we assume that the Markov chain is given by transition probabilities of a specific type:

$$P\{\xi_{n+1} = j, \theta_{n+1} < t, u_{n+1} \in B / \xi_n = i, \theta_n = \tau, u_n = u\} = P\{\xi_{n+1} = j, \theta_{n+1} < t, u_{n+1} \in B / \xi_n = i\}, \quad (12)$$

in which there is no dependence on the parameters τ and *u*—values of the second and third components in the previous step—and the step number *n*. Consequently, the resulting behavior of a homogeneous controlled Markov renewal process depends only on the value of the first component, and this controlled Markov renewal process is homogeneous.
In what follows, we will use the following notation:

$$P\{\xi_{n+1} = j, \theta_{n+1} < t, u_{n+1} \in B / \xi_n = i\} = \tilde{Q}_{ij}(t, B) \quad (13)$$

Since the resulting course of the process depends only on the first component, it is possible to make the model more complex and assume that the domain of definition of the function $\tilde{Q}_{ij}(t, B)$ by the variable B depends on state i. This means that, for each $i \in E$, there is a set of controls U_i and σ-algebra $\boldsymbol{A}_i$ of subsets of this space U_i.
It is easy to see that, for $t \to \infty$ and $B = U_i$, we obtain a transition probability:

$$p_{ij} = \tilde{Q}_{ij}(\infty, U_i) = P\{\xi_{n+1} = j / \xi_n = i\} \quad (14)$$

for a *nested Markov chain* characterizing the evolution of the first component of the introduced three-dimensional Markov chain.
Further, it is easy to see that, for $B = U_i$, we have the equality

$$Q_{ij}(t) = \tilde{Q}_{ij}(t, U_i), \quad (15)$$

where the function $Q_{ij}(t)$ is defined by Equality (2).
From Definition (13), it follows that

$$\tilde{Q}_{ij}(\infty, B) = lim_{t \to \infty} \tilde{Q}_{ij}(t, B) = = P\{\xi_{n+1} = j, u_{n+1} \in B / \xi_n = i\} \quad (16)$$

Consequently, the family of functions $\tilde{Q}_{ij}(t, B)$ generates a family of probability measures for $i \in E, \quad B \in A_i$ in the measurable space (U_i, B_i):

$$G_i(B) = = \sum_{j \in E} \tilde{Q}_{ij}(\infty, B) = P\{u_{n+1} \in B / \xi_n = i\} \quad (17)$$

Since for any $j \in E$, $t \in R^+$, $B \in A_i$ the inequality $G_i(B) \geq \tilde{Q}_{ij}(t, B)$ is valid, based on the Radon–Nikodym theorem, there are measurable functions $Q_{ij}(t, u)$ such that the equality

$$Q_{ij}(t, B) = \int_B Q_{ij}(t, u) G_i(du) \quad (18)$$

holds.
The functions $Q_{ij}(t, u)$ are the conditional probabilities:

$$Q_{ij}(t, u) = = P\{\xi_{n+1} = j, \theta_{n+1} < t / \xi_n = i, u_{n+1} = u\} \quad (19)$$

Thus, a homogeneous controlled Markov renewal process can be defined by the family of matrices

$\{Q_{ij}(t,u)\}$, the set of probability measures $G_i(B)$, and an initial probability distribution of states p_i, $i,j\in E$, $t\in R^+$, $u\in U_i$, $B\in \boldsymbol{A}_i$.
We will call the family of matrices $\{Q_{ij}(t,u)\}$ the semi-Markov kernel of the controlled semi-Markov process, and we will call the family of probabilistic measures $\vec{G} = \{G_1(B), G_2(B), \dots, G_N(B)\}$ the family of control measures.
It is easy to establish a connection between the semi-Markov kernel of a controlled semi-Markov process and the family of probabilistic control measures:
$Q_{ij}(t) = \int_{U_i} Q_{ij}(t,u)G_i(du)$ (20)
Next, the controlled semi-Markov process $X(t)$ is defined as the pair

$$X(t) = \{\xi(t), \quad u(t)\},$$

where $\xi(t) = \xi_{\nu(t)}$, $u(t) = u_{\nu(t)+1}$, and the counting process $\nu(t)$ is defined by the equality

$$\nu(t) = n: \{\sum_{k\leq n} \theta_k \leq t\}, \quad \theta_0 = 0.$$

It is clear that the process $\xi(t)$ coincides with the standard semi-Markov process. The second component of the controlled semi-Markov process $u(t)$ determines the trajectory of the decisions made.
To build a controlled semi-Markov process in a queuing system, the following parameters must be determined: Markov moments, state space, control space, semi-Markov kernel. To formulate and solve the control problem, the functional on the trajectories of this controlled semi-Markov process must be determined [24].

Control objectivity is determined by the process trajectory. This requires building a functional. This means that the trajectory or its parts must be matched with a number.
The trajectories of a controlled semi-Markov process with a finite set of states comprise step functions that are set at an arbitrary interval $(0,T)$ by the sequence $t_0 = 0 < t_1 < t_2 < \ldots < t_n \leq T < t_{n+1}$ of jump moments, the sequence $i_0, i_1, i_2, \ldots, i_{n+1}, i_k \in E, 0 \leq k \leq n+1$ determining the states of the process at the moments t_k, and the sequence $u_0, u_1, u_2, \dots, u_{n+1}, \xi(t_k) = i_k, n(T)$ that shows the number of state shifts (jumps) in the interval $(0,T)$: $\sum_{s=0}^{n(T)} t_s < T < \sum_{s=0}^{n(T)+1} t_s$.
Given the numerical functions $R_{ij}(t,u), i,j \in E$, the value of the functional corresponding to the trajectory defined above is determined by

$\sum_{k=0}^{n(T)-1} R_{i_k i_{k+1}}(t_{k+1}, u_{k+1}) + R_{i_{n(T)} i_{n(T)+1}}$ (22)

The function $R_{ij}(t,\tau,u)$ is the mathematical expectation of the accumulated effect (income) over time τ, provided that the process is in state i and the next state will be state j, and the time of this transition is t, provided that the decision u is made at the time of the transition from state i.
The function $R_{ij}(t,u)$ is the mathematical expectation of the accumulated effect (income) for the entire period t, provided that the process is in state i and the next state will be state j, provided that the decision u is made at the time of the transition.
The accumulation functional $MS(T)$ is defined as the mathematical expectation of the accumulated effect over the time interval $(0,T)$.
Assuming that the process starts from state i, $P\{\xi(0) = i\} = 1$, the mathematical expectation defined above will depend on i. We denote the conditional mathematical expectation of the accumulated income for the period $(0,t)$ as $MS_i(t)$, provided that the process $\xi(t)$ starts from state i for $t = 0$. Therefore, according to the formula of the complete mathematical expectation, we obtain a system of integral equations with respect to the sought functions $S_i(t) = MS_i(t)$:

$$S_i(t) = s_i(t) + \sum_{j\in E} \int_0^t S_i(t-x)dQ_{ij}(x), \qquad (23)$$

$$s_i(t) = \sum_{j\in E} \int_{u\in U_i} \{\int_0^t R_{ij}(x,u)dQ_{ij}(x,u) + \int_t^\infty R_{ij}(x,t,u)dQ_{i,j}(x,u)\}G_i(du) \qquad (24)$$

The following notation will be used in subsequent sections of this paper:

$$s_i = \lim_{t\to\infty} s_i(t) = \sum_{j\in E} \int_{u\in U_i} \int_0^\infty R_{ij}(x,u)d_xQ_{ij}(x,u)G_i(du) \quad (25)$$

The values s_i are the mathematical expectations of the accumulated income for the full period of the process's stay in state i, and we will assume that they are finite.
If the process $\xi(t)$ is observed for a long time, the accumulated effect grows indefinitely. Consequently, the specific accumulated effect is taken as a characteristic of the functioning quality, that is, $\frac{\lim_{t\to\infty} S_i(t)}{t}$ (where this limit exists).
According to the accumulation functional theorem [22], if there is a semi-Markov process with a finite set of states, the nested Markov chain is ergodic, at

least one of the distributions $Q_{ij}(t)$ is indecisive, and the average revenues s_i are finite for any $i \in E$, then it can be defined as follows:

$$S = \frac{\lim_{t\to\infty} s_i(t)}{t} = \frac{\sum_{k\in E} s_k \pi_k}{\sum_{k\in E} m_k \pi_k}, \qquad (26)$$

where π_i represents the stationary probabilities of the distribution of states of the nested Markov chain, that is, the solution to the system of equations:

$$\pi_l = \sum_{k\in E} \pi_k p_{kl}$$

$$\sum_{k\in E} \pi_k = 1$$

where

$p_{ij} = \lim_{t\to\infty} Q_{ij}(t)$ — transition probabilities of the nested Markov chain;

$s_i = \lim_{t\to\infty} s_i(t)$—conditional mathematical expectations of the accumulated income for the full period of the process's stay in state *i*, $s_i < \infty$;

$m_i = \lim_{t\to\infty} m_i(t) = \int_0^\infty [1 - \sum_{j\in E} Q_{ij}(x)]dx$ — conditional mathematical expectation of the time of the process's continuous stay in state i.

Thus, to solve the problem, the following algorithm is used to calculate the quality indicators of the system functioning and choose the optimal control.

1. The probabilistic characteristics of the semi-Markov process $Q_{ij}(t) = Q_{ij}(t, U_i) = \int_{U_i} Q_{ij}(t,u) G_i(du)$ are determined.

The semi-Markov kernel $Q_{ij}(t,u)$ is the probability that the semi-Markov process will transition to state *j* and the time before this transition will not exceed *t*, provided that the process is in state *i*, and in this state, the decision *u* is made from a set of controls.

2. The distributions $G_i(u)$ are set to define the solution selection rule in state $i, j \in E$.

3. The transition probabilities of the nested Markov chain are determined $p_{ij} = Q_{ij}(\infty) = Q_{ij}(\infty, U_i)$.

4. The normalized solution to the algebraic system of equations is determined $\pi_k = \sum_{j\in E} \pi_j p_{jk}, \sum_{k\in E} \pi_k = 1$.

5. Conditional mathematical expectations of the time of the process's continuous stay in state *i* are determined:

$$m_i = \int_{x\in(0,\infty)} \left[1 - \sum_{j\in E} Q_{ij}(x)\right] dx.$$

Conditional mathematical expectations of the accumulated income for the full period of the process's stay in state *i* are determined:

$$s_i = \sum_{j\in E} \int_{x\in(0,\infty)} R_{ij}(x) d_x Q_{ij}(x).$$

The function $R_{ij}(t)$ is the conditional mathematical expectation of the accumulated income, provided that the process is in state *i* and after time *t* will transition to state *j*.

6. The value of the functioning quality indicator is calculated according to Formula (26).

Considering that the functional defined by (26) is linear-fractional with respect to the distributions defining the accumulation structure, according to the linear-fractional functional theorem [25], the maximum of the linear-fractional functional must be sought in the set of degenerate distributions. The search for strategies in which the value of the quality of the performance indicator is maximum is carried out.

Note that, depending on the type of system and its constituent structures and controlled parameters, the parameters required for calculation, starting from Markov moments and ending with analytical formulas for the accumulation functional, will differ. We will now demonstrate the sequence of the algorithm for constructing the accumulation functional on the trajectories of a controlled semi-Markov process for various queuing systems when controlling various characteristics of the system.

3 Practical Part of the Study: Queuing System Management

A queuing system in Kendall's symbolism notation can be written as M/G*/1/N*. The symbol M means that the moments of numbers' arrival form the simplest flow of homogeneous events with the parameter λ. The symbol G* means that the service duration distribution function varies depending on the state of the system, that is, it is a control action. The system is single-channel. The symbol N* indicates the maximum number of waiting spaces. This parameter is also a control action, that is, the number of waiting spaces varies depending on the state of the process.

Markov moments are the moments of the end and the beginning of service of the request. At these times, the system is either freed or a new request begins to be served. In the first case, the resulting behavior of the sequence of requests depends only on the flow of requests arriving after the end of service and on the duration of their service. Since the incoming flow is the most elementary, these factors do not depend on the past. In the second case, the sequence behavior does not depend on the past due to the absence of an after-effect for the incoming elementary stream, the independence of the service duration, and the fact that a new request is just starting to be served at the moment in question.

We define the set of states of system E: state $\{i\}$ - one request is being served in the system, and *i-1* requests are in the queue. Therefore, for example, state {0} means that there are no requests in the system, and the extreme state $\{N\}$ means that one request is being served in the system, and *N-1* requests are in the queue.

We define the control space. In state *i*, we select the duration of the request service time and decide to create additional places in the queue (except for state {0}). Therefore, we use two control parameters from the given set of controls: $\{u_i, v_i\} \in U_i$, where there is a change in the service duration $u_i \in [0, \infty)$. The selection range for the number of additional places in the queue is $0 \le v_i \le N - i + 1$.

One of the following decisions can be made:

$v_0 = 0$—we do not add empty spaces, and the decision is made with probability $p_0{}^{[i]}$;

$v_1 = 1$—we add one space to wait in the queue, and the decision is made with probability $p_1{}^{[i]}$;

…….

$v_i = N - i + 1$—we create a queue of the maximum possible size, and the decision is made with probability $p_{N-i+1}{}^{[i]}$.

Considering that in state *i*, we make one of the decisions on adding free places, we have: $\sum_{l=0}^{N-i+1} p_l{}^{[i]} = 1$. Hence, the state space can be defined as follows: $U_i = [0,\infty] \times [0, N-i+1]$.

We can write $\sum_{l=0}^{N-i+1} \int_{u \in U_i} dG_i(u)\, p_l{}^{[i]} = 1$, where the distributions $G_i(u)$ determine the decision-making rule (service duration).

Taking into account that $U_0 = \emptyset$, when the system is free, we do not make management decisions, since putting up additional seats will result in unjustified expenses.

Given that the transition is carried out from state i to j, and in state i, we make decisions from the control set on the service duration and adding places to the queue, we can write the semi-Markov kernel $Q_{ij}(t, \{u_i, v_i\})$:

$$Q_{ij}(t, \{u_i, v_i\}) = \begin{cases} 0, j < i - 1 \\ 0, t < u_i \\ \frac{(\lambda u_i)^{j-i+1}}{(j-i+1)!} e^{-\lambda u_i}, i - 1 \le j < i - 1 + v_i, t > u_i \\ \sum_{k=v_i}^{\infty} \frac{(\lambda u_i)^k}{k!} e^{-\lambda u_i}, j = i - 1 + v_i \end{cases} \quad (29)$$

Since $0 \le v \le N - i + 1$, and $i - 1 \le j \le N$, for the last state of the system, we write $Q_{i,i+v-1}(t, \{u_i, v_i\}) = \sum_{k=v}^{\infty} \frac{(\lambda u_i)^k}{k!} e^{-\lambda u} \; i$, taking into account the arrival of requests that found the system full.

To exit state {0, at least one request must arrive; therefore,

$$Q_{01}(t, \{u_i, v_i\}) = 1 - e^{-\lambda t}$$

When checking the calculations for the semi-Markov kernel, we find that the calculations are correct, since the normalization condition is satisfi ed: $\sum_{j \in E} p_{ij} = 1$, where $p_{ij} = \lim_{t \to \infty} Q_{ij}(t)$.

To find the type of income functionality, we carry out additional calculations using Formulas (23)–(28).

We will now demonstrate the algorithm on a specific example for *N* = 2*.

Therefore, the problem in Kendall's notation has the following form: M/G*/1/2*.

One of three states is possible in the system.

State {0}—no requests in the system;

State {1}—one request is being served in the system, and there are no requests in the queue;

State {2}—one request is being served in the system, and one request is in the queue.

In state *i*, we select the duration of the request service time and decide to create additional spaces in the queue (except for state {0}).

In state 1, we can add 0, 1, or 2 additional spaces to the queue. In state 2, we can add 0 additional seats or 1 additional seat. We find that the number of possible combinations of strategies to choose v is 6.

We write down the pairs of strategies for state 1 and state 2: (0,0); (0,1); (1,0); (1,1); (2,0); (2,1). The last pair of strategies refers to the classic case of the task when all the spaces are exposed and only the service duration is controlled.

Given the form of the semi-Markov kernel (29), the transition probabilities take the form

$$p_{ij} = \begin{cases} 0, j < i - 1 \\ \frac{(\lambda u_i)^{j-i+1}}{(j-i+1)!} e^{-\lambda u_i}, i - 1 \le j < i - 1 + v_i, \\ \sum_{k=v_i}^{\infty} \frac{(\lambda u_i)^k}{k!} e^{-\lambda u_i}, j = i - 1 + v_i \end{cases} \quad (30)$$

Analytical formulas can be obtained for all transition probabilities in the explicit form.

Therefore, for example, for state $i = 1$:

$$p_{1j} = \begin{cases} \frac{(\lambda \tau_1)^j}{j!} e^{-\lambda \tau_1}, o \le j < v_1 \\ \sum_{k=v_1}^{\infty} \frac{(\lambda \tau_1)^k}{k!} e^{-\lambda \tau_1}, j = v_1 \end{cases} \quad (31)$$

Possible values for selecting additional spaces in state $\{1\}$ have the form $0 \le v_1 \le 2$.

We denote the transition probability when choosing the specific value v as $p_{ij}^{[v]}$.
If we choose $v = 0$, we get: $p_{10}^{[0]} = 1$
If we choose $v = 1$, we get:

$$p_{1j}^{[1]} = \begin{cases} p_{10}^{[1]} = e^{-\lambda\tau_1} \\ p_{11}^{[1]} = \sum_{k=1}^{\infty} \frac{(\lambda\tau_1)^k}{k!} e^{-\lambda\tau_1} \end{cases}$$

If we choose $v = 2$, we get:

$$p_{1j}^{[2]} = \begin{cases} p_{10}^{[2]} = e^{-\lambda\tau_1} \\ p_{11}^{[2]} = \lambda\tau_1 e^{-\lambda\tau_1} \\ p_{12}^{[2]} = \sum_{k=2}^{\infty} \frac{(\lambda\tau_1)^k}{k!} e^{-\lambda\tau_1} \end{cases}$$

As noted in Section 2, the calculation of the functionality (26) is carried out when substituting degenerate distributions.
It is possible to find all possible variants of the values s_i depending on the state and the number of exposed spaces in a given state, taking into account the substitution of degenerate distributions.
We select the number of additional places v_i with probability 1 in the state, and the service duration distribution will take the form

$$G_i(u) = \begin{cases} 1, u_i < u \\ 0, u_i > u \end{cases}$$

To construct a conditional mathematical expectation of the accumulated income $R_{ij}(t,u)$, provided that the process is in state *i* and after time *t* will transition to state *j*, we introduce constants characterizing income and expenses:
$-c_1$—payment per unit time of device operation during the request service period;
$-c_2$—payment per unit downtime of the device;
$-c_3$—payment per unit time spent by the request in the queue;
$-c_4$—income received for serving one request;
$-c_5$—payment for one lost request.
$-c_6$ - payment per one added spot in the queue.
Development of a cost model with consideration of economic applications is a common practice in many similar researches [27,28].

Therefore, for example, for $i = 2, v = 0$, the conditional mathematical expectation of the accumulated income for the full period of the process's stay in state *i* is
$s_2 = c_4 - c_1\tau_2 - c_3\tau_2 - c_5\lambda\tau_2$ *(32)*
For $i = 2, v = 1$, the conditional mathematical expectation of the accumulated income for the full period of the process in state *i* is

$$s_2 = c_4 - c_1\tau_2 - c_3\left(2\tau_2 - \frac{1}{\lambda} + \frac{1}{\lambda}e^{-\lambda\tau_2}\right) - c_6 - c_5\left(\lambda\tau_2 - 1 + e^{-\lambda\tau_2}\right) \quad (33)$$

Additionally, to calculate the functional, there is a need to find the stationary distributions π_i. Consider the solutions of the system depending on the strategies. When applying different pairs of strategies, we get different transition probability matrices.
Therefore, for example, with the strategy (0,0), the matrix P_{ij} has the form

$$\begin{pmatrix} 0 & p_{01} & 0 \\ p_{10}^{[0]} & 0 & 0 \\ 0 & p_{21}^{[0]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

In this case, when solving the system, we have $\pi_0 = \pi_1 = \frac{1}{2}, \pi_2 = 0$.
When applying strategy (1,0), the matrix P_{ij} has the form

$$\begin{pmatrix} 0 & p_{01} & 0 \\ p_{10}^{[0]} & p_{11}^{[1]} & 0 \\ 0 & p_{21}^{[0]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ p_{10}^{[0]} & p_{11}^{[1]} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, the solution to the system will be
$\pi_0 = \frac{e^{-\lambda\tau_1}}{1+e^{-\lambda\tau_1}}, \pi_1 = \frac{1}{1+e^{-\lambda\tau_1}}, \pi_2 = 0$
When applying strategy (2,0), the matrix P_{ij} has the form

$$\begin{pmatrix} 0 & p_{01} & 0 \\ p_{10}^{[2]} & p_{11}^{[2]} & p_{12}^{[2]} \\ 0 & p_{21}^{[0]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ p_{10}^{[2]} & p_{11}^{[2]} & p_{12}^{[2]} \\ 0 & 1 & 0 \end{pmatrix}$$

The solution of the system will be

$$\pi_0 = \frac{e^{-\lambda\tau_1}}{2-\lambda\tau_1 e^{-\lambda\tau_1}},$$
$$\pi_1 = \frac{1}{2-\lambda\tau_1 e^{-\lambda\tau_1}},$$

$$\pi_2 = \frac{1-\lambda\tau_1 e^{-\lambda\tau_1} - e^{-\lambda\tau_1}}{2-\lambda\tau_1 e^{-\lambda\tau_1}}.$$

For all other strategies, it is similarly easy to obtain the stationary probability values.

Thus, for different strategies, we obtain different analytical expressions for the accumulation functional.

For example, for strategy (0,1), the functional has the following form:

$$S = \frac{\frac{-c_2}{\lambda}+c_4-c_1\tau_1-c_5\lambda\tau_1}{\frac{1}{\lambda}+\tau_1} \quad (34)$$

For strategy (1,0), the following expression is obtained:

$$S = \frac{-c_2 e^{-\lambda\tau_1}+\lambda S_\lambda}{e^{-\lambda\tau_1}+\lambda\tau_1}, \quad (35)$$

where

$$S_{\lambda 1} = c_4 - c_1\tau_1 - c_3\left(\tau_1 - \frac{1}{\lambda} + \frac{1}{\lambda}e^{-\lambda\tau_1}\right) - c_6 - c_5\left(\lambda\tau_1 - 1 + e^{-\lambda\tau_1}\right)$$

For strategy (1,1), the following values are obtained:

$$S = \frac{-c_2 e^{-\lambda\tau_1}+\lambda S_{\lambda 2}}{e^{-\lambda\tau_1}+\lambda\tau_1}, \quad (36)$$

$$S_{\lambda 2} = c_4 - c_1\tau_1 - c_3\left(\tau_1 - \frac{1}{\lambda} + \frac{1}{\lambda}e^{-\lambda\tau_1}\right) - c_6 - c_5\left(\lambda\tau_1 - 1 + e^{-\lambda\tau_1}\right).$$

Corresponding functionals can be written for all other pairs of strategies. Each of the obtained functions should be maximized by τ_1, τ_2. The maximum value is then selected from the ones obtained. Finding the maximum value gives the answer to the problem, namely, strategy $((v_1^*, v_2^*), \tau_1^*\ \ , \tau_2^*)$. Then, in i state, we make the decision $\{\tau_i^*, v_i^*\} \in U$. The choice of this solution determines the most efficient operation of the system.

Consider one more variant of the system with restrictions on the number of spaces in the queue and the number of service devices specified in Kendall's symbolism: G*/M/n/m. The symbol G* means that the incoming flow is distributed arbitrarily, and the intervals between neighboring moments of requests are distributed arbitrarily. This is a control action, that is, it is possible to control the system's incoming request flow. The symbol M means that the service duration distribution function is given by an exponential distribution law with a given parameter. The symbol n represents the number of service devices. The symbol m represents the number of spaces in the queue.

Consider a brief version of the problem: the maximum possible number of service channels for the given system is 2, each of which can simultaneously serve only one request. The maximum possible number of waiting spaces in the studied system is 2.

The studied queuing system model can serve no more than one request at any given time. The system has a limited number of waiting spaces. A request that finds the system free immediately begins to be served. A request that finds the system busy enters the queue if there are free spaces and waits for the service to start. If there are no free spaces in the queue, the request leaves the system (the request is lost).

Markov moments are the moments of the conditions' arrival in the system.

We will now describe the set of states of this queuing system.

State {1}—one request is being served in the system;

State {2}—two requests are being served in the system;

State {3}—two requests are being served in the system, and one request is in the queue;

State {4}—two requests are being served in the system, and two requests are in the queue.

Since the control is carried out by the incoming flow, the distribution of incoming requests can be arbitrary. We denote them as follows: $F_1(x)$ is the distribution of the moment of arrival of the first request; $F_2(x)$ is the distribution of the moment of arrival of the second request; $F_3(x)$ is the distribution of the moment of arrival of the third request; $F_4(x)$ is the distribution of the moment of arrival of the fourth request.

We build a semi-Markov kernel. We write out the matrix form $Q_{ij}(t,u)$:

$$\begin{matrix} Q_{11}(t,u) & Q_{12}(t,u) & 0 & 0 \\ Q_{21}(t,u) & Q_{22}(t,u) & Q_{23}(t,u) & 0 \\ Q_{31}(t,u) & Q_{32}(t,u) & Q_{33}(t,u) & Q_{34}(t,u) \\ Q_{41}(t,u) & Q_{42}(t,u) & Q_{43}(t,u) & Q_{44}(t,u) \end{matrix}$$

We describe each of the given values:

$Q_{11}(t,u)$—one request has been served, and a new request has arrived;

$Q_{12}(t,u)$—the request was not served, and a new request has arrived;

$Q_{21}(t,u)$—two requests have been served, and a new request has arrived;

...

$Q_{43}(t,u)$—two requests have been served, two requests have not been served, and a new request has arrived;

$Q_{44}(t,u)$—one request has been served, three requests have not been served, and a new request has arrived.

$$Q_{11}(t,u) = \begin{cases} 0, t < u \\ P\{\xi < u\}, t > u \end{cases} \quad (38)$$

Given that $P\{\xi < u\} = 1 - e^{-\lambda u}$, we can write

$$Q_{11}(t) = \int_0^t (1 - e^{-\lambda u}) dF_1(u) \quad (39)$$

Therefore, for example, for $Q_{12}(t,u)$:

$$Q_{12}(t,u) = \begin{cases} 0, t < u \\ P\{\xi > u\}, t > u \end{cases} \quad (40)$$

Given that $P\{\xi > u\}$=$e^{-\lambda u}$, we can write

$$Q_{12}(t) = \int_0^t (e^{-\lambda u}) dF_1(u) \quad (41)$$

For $Q_{43}(t,u)$:

$$Q_{43}(t,u) = \begin{cases} 0, t < u \\ P\{\xi_1 + \xi_2 < u, \xi_1 + \xi_2 + \xi_3 > u\}, t > u \end{cases} \quad (42)$$

Given that $P\{\xi_1 + \xi_2 < u, \xi_1 + \xi_2 + \xi_3 > u\}$=$2\lambda^2 u^2 e^{-2\lambda u}$, we get

$$Q_{43}(t) = \int_0^\infty Q_{43}(t,u) dF_4(u) = \int_0^t (2\lambda^2 u^2 e^{-2\lambda u}) \, dF_4(u) \quad (43)$$

For $Q_{44}(t,u)$:

$$Q_{44}(t) = \int_0^\infty Q_{44}(t,u) dF_4(u) = \int_0^t (2\lambda u e^{-2\lambda u} + e^{-2\lambda u}) \, dF_4(u) \quad (44)$$

Similar calculations can be performed for all other cases.
To ensure that the calculations are carried out correctly, one can resort to checking the normalization condition, which is satisfied in this case.
Analytical expressions for the conditional mathematical expectation of the accumulated income can be obtained for all possible cases.
Example:

$$R_{11}(t) = c_1 - 2c_3 t + (c_3 - c_2) \frac{(\lambda t + 1)e^{-\lambda t} - 1}{\lambda(e^{-\lambda t} - 1)} \quad (45)$$

$$R_{21}(t) = -2c_3 + 2c_1 + 2(c_3 - c_2) \frac{(\lambda t + 1)e^{-\lambda t} - 1}{\lambda(e^{-\lambda t} - 1)} \quad (46)$$

$$R_{43}(t) = -2c_2 t + 2c_1 - c_4 \frac{(\lambda t + 1)e^{-\lambda t} - 1}{\lambda(e^{-\lambda t} - 1)} \quad (47)$$

$$R_{44}(t) = c_1 - 4c_2 t - \frac{7}{2} c_4 t - c_5 \quad (48)$$

To calculate the value of s_i, the following formula is used:

$$s_i(t) = \int_{u \in U_i} \{ \sum_{j \in X} \int_0^t R_{ij}(\mathrm{x,u}) \mathrm{d}Q_{ij}(\mathrm{x,u})) + + \sum_{j \in X} \int_t^\infty R_{ij}(\mathrm{x,u}) \mathrm{d}Q_{ij}(\mathrm{x,u})) \} F_i(du) \quad (49)$$

The values $Q_{ij}(t,u)$ and $R_{ij}(t,u)$ calculated earlier should be substituted into Formula (49).
To calculate the value of m_i, the following formula is used:

$$m_i = \int_{u \in U_i} \{ \int_0^\infty [1 - \sum_{j \in X} Q_{ij}(t,u)] dt \} F_i(du) = \int_0^\infty \overline{F}_i(x) dx = u_i \quad (50)$$

To calculate πij, one needs to calculate the determinant of a given matrix of a homogeneous system:

p_{11}-1	p_{12}	0	0
p_{21}	p_{22}-1	0	0
p_{31}	p_{32}	p_{33}-1	p_{34}
p_{41}	p_{42}	p_{43}	p_{44}-1

where $p_{ij} = \int_{u \in U_i} Q_{ij}(\infty, u) F_i(du)$, normalizing condition $\sum_{i \in E} \pi_i = 1$. After solving the system of algebraic equations for calculating stationary probabilities, we can proceed to searching for the maximum of the functional S: $S = \frac{\sum_{i \in E} s_i \pi_i}{\sum_{i \in E} m_i \pi_i} \to max.$
Thus, decisions should be made in accordance with the established maximum of the functional S. That is, after entering state i, it is necessary to choose the decision u_i. When making the decision u_i, that is, deciding that the next request will arrive after time u_i, the income will be guaranteed to be no less than the value of S, corresponding to this value.

4 Discussion

In this paper, an approach based on the theory of controlled semi-Markov processes was implemented that allows for selecting the optimal parameters of a queuing system, thus increasing the system's efficiency and profitability. Using this approach for various networks, analytical expressions can be obtained for the conditional mathematical expectation, the conditional mathematical expectation of the accumulated income, and the stationary probabilities of the state distribution of the nested Markov chain for a controlled system. Special cases were implemented for two controlled systems.

Queuing theory methods, particularly the theory of controlled semi-Markov processes, can be used to solve many tasks of planning, evaluating, and optimizing customer service quality. In particular, recommendations can be developed for optimal service system management to generate maximum income.

This paper presents specific cases of two G*/M/n/m and M/G*/1/N* managed queuing systems. Here are a few examples of using queuing systems in various fields: call centers, production lines, transportation systems, and internet services.

The implementation of the algorithm and the achievement of a result depending on various control parameters are demonstrated. Despite the fact that the analytical conclusions of the formulas represent a time-consuming process for each new system, it is possible to implement the algorithm for other types of queuing systems with the possibility of obtaining analytical formulas for each specific system and implementing the algorithm using software applications.

The use of management, and especially the expansion of the state space in solving optimization problems for queuing systems, can bring tangible results. Raising the level of income from the functioning of the system is advisable, since the increase in the funds received can be used in the future.

In further studies, it is possible to expand the state space and control three parameters simultaneously in some systems.

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