Notes on Various Transforms Identified by Some Special Functions with Complex (or Real) Parameters and Some of Related Implications

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Abstract – The fundamental aim of this special research is first to introduce certain essential information in regards to some special functions, which are the Gamma function and the Beta function and play a big role in both (applied) mathematics and most engineering sciences, and then to present both a number of their familiar properties and several relationships between them. Afterward, various possible-undeniable effects of those special functions in the transformation theory, their special implications, and suggestions for the relevant researchers will be also considered as special information.

Key-Words: - The Gamma function, the Beta function, power functions, fractional calculus, series expansions, improper integrals, integral transformations, operators.

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1 Introduction and Certain Special Information

As it is well known, the Gamma function and the Beta function play a large part in both the family of special functions and the theory of integral transformation(s). Apart from this, these two functions are frequently encountered in their forms with both real parameters and complex parameters as various theoretical research and nearly all applied sciences in the literature. As special purpose applications, they are very important tools for modeling situations comprising continuous changes, with various applications to calculus, differential equations, complex analysis, and statistics. As some examples, for each one of them, it can be checked over the scientific studies presented in, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

In addition, in cases where our classical analysis information is not sufficient, as various useful tools, the importance of those special functions cannot be ever denied for science and technology as their applications. Specially, as natural results of unclassical examples, without using any one of those related functions which is the possible solutions of the following improper integrals cannot be concluded:

$$\int_0^\infty e^{-\kappa^{m/n}} d\kappa \tag{1}$$

and

$$\int_0^1 \left(1-\kappa^{m/n}\right)^{1/n} d\kappa \tag{2}$$

where the concerned parameters m and n are in the set $\mathbb{N}-\{1\} := \{2,3,4,\cdots\}$ with $m \neq n$. Quite simply, as is frequently encountered in the fields of mathematics and even many other sciences, it can easily be seen that when the mentioned parameters used in (1) and (2) are chosen appropriately, denumerably infinite types of either the definite type integrals and the convergent types of the improper integrals will be also encountered. For more (simple) examples, it even suffices for them to simply focus on the (basic) classical analysis books (cf., e.g., [12], [13], [14], [15]).

As the most basic definitions of the Gamma function and the Beta function, for the complex parameter ρ, as is known, the gamma function is denoted by $\mathbf{\Gamma}(\rho)$ and also defined by the improper integral given by

$$\mathbf{\Gamma}(\rho)=\int_0^\infty \kappa^{\rho-1}e^{-\kappa}d\kappa, \qquad (3)$$

where $\Re e(\rho)>0$.

At the same time, for the other, the beta function (consisting of the complex parameters σ and ρ) is also denoted by $\mathbf{B}(\sigma,\rho)$, which is called the Euler integral of the first kind, and then defined by the (improper) integral given by

$$\mathbf{B}(\sigma,\rho)=\int_0^1 \kappa^{\sigma-1}(1-\kappa)^{\rho-1}\,d\kappa \qquad (4)$$

where $\Re e(\sigma)>0$ and $\Re e(\rho)>0$.

In terms of literature, the special functions, defined by (3) and (4), possess several essential properties. In particular, an extensive property of the beta function is its close relation to the Gamma function, which is given by

$$\mathrm{B}(\sigma,\rho)=\frac{\Gamma(\sigma)\Gamma(\rho)}{\Gamma(\sigma+\rho)} \qquad (5)$$

where $\Re e(\sigma)>0$ and $\Re e(\rho)>0$.

Privately, *when*

$$\sigma,\rho\in\mathbb{N}:=\{1,2,3,\ldots\},$$

the familiar relation given by (5) immediately arrives at the assertion given by

$$\mathrm{B}(\sigma,\rho)=\frac{(\sigma-1)!(\rho-1)!}{(\sigma+\rho-1)!}. \qquad (6)$$

In particular, for the details of those special functions and other relevant identities, formulas, properties, relations, and also some of the related transforms to some of the other special functions, one may concentrate on the fundamental works accentuated in, [2], [3], [4], [10], [14], [16], [17], [18], [19], [20], [21], [22], [23]. In addition, for the relevant researchers, since this particular investigation will be related to various forms of special functions with complex parameters, for the researchers who are interested in the theory of complex functions and related areas of special scientific interest, we think that the main works given in the references in [7], [9], [12], [13], [15], [22], [24], [25], [26], will be also important as extra special information.

In the meantime, under the conditions that the integrals given by (1) and (2) are either in any forms of the definite integrals or in any convergent forms of the improper integrals and by means of the special definitions given by (3) and (4), the main object of this extensive research is to construct some of (general type) transformations in certain domains of the complex plane and then to make some possible inferences by taking into account both the gamma function being of the form given in (1) together with (3) and the beta function having the form given in (2) along with (4). Now, in light of the information between (1)-(6), let us first create them in order and then represent some of their possible implications as our elementary results.

2 The Gamma Function and Some of the Related Implications

The first implication is directly related to various comprehensive applications of the gamma function being of the form given in (3). For this special function and its details, one may refer to the works in, [1], [5], [7], [16], [18], [20]. The main references for its characterization, when two expedient functions like $v(\tau)$ and $u(\tau)$ are given, it will be enough to consider any instrumental-extensive forms of any function $\omega(\tau)$ defined by

$$\omega(\tau):=v(\tau)e^{-u(\tau)}, \qquad (7)$$

which concurrently identifies processes that change exponentially in time (or related space).

Let us now begin to designate its details.

For those functions $v:=v(\tau)$ and $u:=u(\tau)$ that satisfy the necessary-convergence criteria (for improper integral just below), by the following notation:

$$\mathbb{G}[v(\tau);u(\tau)],$$

we firstly denote a general transformation of the function $\omega(t)$ created by (7) and then define it as the improper integral given by

$$[v(\tau);u(\tau)]:=\int_0^\infty \omega(\tau)d\tau. \qquad (8)$$

As a result of a simple focus, for all the scalars C_1 and C_2, and for all the functions $v_1(\tau)$ and $v_2(\tau)$, the following propositions can be easily presented.

Remark 2.1. For the mentioned functions v_1 and v_2, the following are also asserted:

i) $[C_1v_1(\tau);u(\tau)]\equiv C_1\mathbb{G}[v_1(\tau);u(\tau)]$ (9)

and

ii) $$\begin{aligned}\mathbb{G}[C_1 v_1(\tau) + C_1 v_2(\tau) u(\tau)] &\equiv \mathbb{G}[C_1 v_1(\tau); u(\tau)] \\ &\quad + \mathbb{G}[C_2 v_2(\tau); u(\tau)] \\ &\equiv C_1 \mathbb{G}[v_1(\tau); u(\tau)] \\ &\quad + C_2 \mathbb{G}[v_2(\tau); u(\tau)].\end{aligned} \tag{10}$$

By considering the convergence of the generalized integral constituted in (8) and in the light of our classical analysis knowledge, a large number of the familiar transformations can be easily constructed by choosing various suitable forms of those functions expressed by u and v. For example, by taking the function u as the form given by

$$u \coloneqq u(\tau) := -\lambda\tau \quad (\lambda > 0),$$

the related transform, which is well recognized as the Laplace transformation of the mentioned function v, is easily arrived at the familiar definition being of

$$\begin{aligned}\boldsymbol{\mathcal{L}}[v(\tau)] &\equiv \boldsymbol{\mathcal{L}}[v](\tau) \\ &\equiv \mathbb{G}\left[v(\tau); e^{-\lambda\tau}\right] \\ &= \int_0^\infty v(\tau) e^{-\lambda\tau} d\tau ,\end{aligned} \tag{11}$$

where $\lambda > 0$ and $\left|v(\tau)e^{-\lambda\tau}\right| < \infty$ for all of the values of τ belonging to the interval $[0, \infty)$.

We point out here that when considering the special information between Laplace transform and the relevant improper integral indicated as in (11), it can be easily re-emphasized the familiar relations related to

$$\boldsymbol{\mathcal{L}}[v(\tau)] = \mathcal{V}(\lambda)$$

and

$$v(\tau) = \boldsymbol{\mathcal{L}}^{-1}[\mathcal{V}(\lambda)].$$

As various possible applications in relation to the mentioned transformation are presented by the definition given in (11) (or (8) together with (7)), it can be also concentrated on a great number of its elementary implications. We want to present only some of those consisting of (real) parameters as remarks just below.

Remark 2.2. Let

$$v(\tau) \coloneqq \tau^\beta \quad \text{and} \quad u(\tau) \coloneqq \lambda\tau.$$

As it is well known, *when* $\beta \in \mathbb{N}$, the transform of the function $v(\tau)$ can be easily calculated by making use of our classical analysis information. But, *when* β is a real number with $\beta > -1,$ of course, it is impossible to calculate (or determine) the transform of that function. At this time, with the help of the gamma function, it can be also determined. Then, the related-special assertions can be easily represented as the equivalent forms given by

$$\begin{aligned}\mathcal{L}\left[\tau^\beta\right] &\equiv \mathbb{G}\left[\tau^\beta; e^{-\lambda\tau}\right] \\ &= \int_0^\infty \tau^\beta e^{-\lambda\tau} d\tau \\ &= \frac{\Gamma(\beta+1)}{\lambda^{1+\beta}},\end{aligned}$$

where $\lambda > 0$ and $\beta > -1$.

Naturally, from the special result just above, the following assertions can easily be revealed:

$$\boldsymbol{\mathcal{L}}\left[\tau^\beta\right] = \frac{\Gamma(\beta+1)}{\lambda^{1+\beta}}$$

and

$$\tau^\beta = \boldsymbol{\mathcal{L}}^{-1}\left[\frac{\Gamma(\beta+1)}{\lambda^{1+\beta}}\right].$$

Most particularly, as it is known, when $\beta \in \mathbb{N}$, we immediately get that

$$\boldsymbol{\mathcal{L}}\left[\tau^\beta\right] = \frac{\beta!}{\lambda^{1+\beta}}$$

and

$$\tau^\beta = \boldsymbol{\mathcal{L}}^{-1}\left[\frac{\beta!}{\lambda^{1+\beta}}\right].$$

Remark 2.3. Let

$$v(\tau) \coloneqq C \quad \text{and} \quad u(\tau) \coloneqq \lambda\tau,$$

where $C \neq 0$ and $\lambda > 0$. Then, the Laplace transformation of the constant function C $(C \neq 0)$ can be easily determined as

$$\begin{aligned}\boldsymbol{\mathcal{L}}[C] &\equiv \mathbb{G}\left[C; e^{-\lambda\tau}\right] \\ &= C\int_0^\infty e^{-\lambda\tau} d\tau \\ &= C\lambda^{-1}.\end{aligned}$$

We also get that

$$C = \boldsymbol{\mathcal{L}}^{-1}\left[\frac{C}{\lambda}\right] \quad (C \neq 0; \lambda > 0).$$

Remark 2.4. Let

$$v(\tau) \coloneqq 1 \quad \text{and} \quad u(\tau) \coloneqq \lambda\tau^\beta.$$

Then, by taking into account our classical analysis information, it can be easily presented by the special assertions given by

$$\begin{aligned}\mathbf{G}_\lambda(\beta) &\coloneqq \mathbb{G}\left[1; e^{-\lambda\tau^\beta}\right] \\ &= \int_0^\infty e^{-\lambda\tau^\beta} d\tau\end{aligned}$$

where $\lambda > 0$ and $\beta > 0$.

For the improper integral just above, by having regard to the following-parametric-differential changes:

$$\kappa = \lambda\tau^{\beta} \iff \tau = \lambda^{-1/\beta}\,\kappa^{1/\beta}$$
$$\Longrightarrow d\tau = \frac{1}{\beta}\,\lambda^{-1/\beta}\kappa^{1/\beta-1}d\kappa,$$

the pending result:

$$\mathbf{G}_{\lambda}(\beta) = \frac{1}{\beta}\,\lambda^{-1/\beta}\int_0^{\infty}\kappa^{1/\beta-1}\,e^{-\kappa}d\kappa$$
$$= \lambda^{-1/\beta}\,\mathbf{\Gamma}\left(1+\frac{1}{\beta}\right)$$

is then obtained, *where* $\lambda > 0$ and $\beta > 0$.

More specially, by choosing the values of β as $\beta \coloneqq m$ $(m \in \mathbb{N})$ in Remark 2.3, we then receive the following assertions:

$$\mathbf{G}_{\lambda}(1/m) = \int_0^{\infty} e^{-\lambda\sqrt[m]{\tau}}d\tau$$
$$= \lambda^{-m}\,\Gamma(1+m)$$
$$= \frac{m!}{\lambda^{m}},$$

where $\lambda > 0$.

Of course, from Remark 2.3, it follows that the convergence of each one of the improper integrals being of the elementary forms:

$$\mathrm{G}_1(1/m) = \int_0^{\infty} e^{-\sqrt[m]{\tau}}d\tau = m!$$

and

$$\mathrm{G}_{\lambda}(1) = \int_0^{\infty} e^{-\lambda\tau}d\tau = \frac{1}{\lambda}$$

also is obvious.

As an elementary-extensive application of the mentioned forms specified by (7) and (8), in a similar manner to the relevant transformation and its special implications like Remarks 2.1-2.4, a great deal of special transforms can be also composed by choosing different types of those functions u and v in for the related integral forms constituted as in (8). In particular, for both Laplace transform and some of the familiar others transform, one may refer to the source materials in the references of this work given in, [4], [9], [17], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], as various differ-integral transformations or numerous related different type operators.

3 The Beta Function and Some of the Related Implications

The second implication is directly associated with numerous extensive applications of the beta function possessing the form constituted in (4). Specifically, for this special function and related information, one looks over the works given in, [1], [2], [4], [7], [16], [18], [19], [20], [21]. For its construction, when two expedient functions like $v \coloneqq v(\tau)$ and $u \coloneqq u(\tau)$ are given, it will suffice to take into account any function like $\omega_{w,V}(\tau)$ being of the form given by

$$\omega_{w,V}(\tau) \coloneqq v(\tau)\big(w - u(\tau)\big)^{V}, \qquad (12)$$

where $w \in \mathbb{R}$ and $V \in \mathbb{R}$. As it is known, the notation $\mathbb{R}$ denotes the familiar set of the real numbers. At the same time, of course, the special parameter V will also be any members belonging to the set $\mathbb{C}$, which also is the set of complex numbers.

Especially, in terms of convergence and light of those functions $v \coloneqq v(\tau)$ and $u \coloneqq u(\tau)$ that should satisfy the necessary criteria, by $\mathbb{B}[v(\tau); u(\tau)]$, we also denote the second general transformation of the function consisting of the form $\omega(t)$ indicated as in (7) and then define it as the integral form given by

$$\mathbb{B}_{w,V}[v(\tau); u(\tau)] = \int_0^{w}\omega_{w,V}(\tau)d\tau. \qquad (13)$$

We especially note here that, by selecting those parameters w and V differently, it can also receive various integral representations consisting of many rational forms of those improper integrals. In such cases, by considering the convergence of those integrals constituted by (13) along with (12), and also making use of our classical analysis knowledge, a great number of the familiar transforms can be easily created in the light of various different types of those mentioned functions u and v. At the same time, if the value of the parameter w is allowed as real or complex, then that general integral transformation will lead to various fields with extensive implications. Now let us focus on some of its essential properties and/or possible implications, and also constitute those as remarks again.

Remark 3.1. Let C_1 and C_2 be any scalar. For the functions v_1 and v_2, the following statements are also satisfied:

i) $\mathbb{B}_{w,V}[C_1 v_1(\tau); u(\tau)] \equiv C_1\mathbb{B}_{w,V}[v_1(\tau); u(\tau)]$

and

ii) $\mathbb{B}_{w,V}[C_1 v_1(\tau) + C_1 v_2(\tau); u(\tau)]$

$$\equiv C_1 \mathbb{B}_{w,V}[v_1(\tau); u(\tau)] + C_2 \mathbb{B}_{w,V}[v_2(\tau); u(\tau)].$$

Remark 3.2. Let the functions v and u be of the form given by

$$v \coloneqq v(\tau) := \tau^{\lambda} \quad \text{and} \quad u \coloneqq u(\tau) := w\tau.$$

The special form of that mentioned transform can be received as follows:

$$\mathbb{B}_{w,V}\left[\tau^{\lambda}; w\tau\right] \equiv w^{V} \int_0^{w} \tau^{\lambda} (1-\tau)^{V} d\tau, \qquad (14)$$

where each one of the values of all those parameters located just above is also selected as their sensible values for the (improper) integral form given in (12).

Especially, as one of its essential forms, namely, in Remark 3.2, by setting $w \coloneqq 1$ in (14), that integral transform then arrives at the well-known Beta function, which is one of its equivalent forms given by

$$\mathbf{B}(\lambda+1, V+1) = \int_0^{1} \tau^{\lambda} (1-\tau)^{V} d\tau, \qquad (15)$$

where the familiar notation $\mathbf{B}(\cdot,\cdot)$ is generally called as the Beta function in the mathematical literature. Naturally, due to $\lambda \in \mathbb{C}$ and $V \in \mathbb{C}$ there, it must be $\Re e(\lambda) > -1$ and $\Re e(V) > -1$ (and, of course, $\lambda > -1$ and $V > -1$ *when* $\lambda \in \mathbb{R}$ and $V \in \mathbb{R}$).

Shortly,

$$\mathbb{B}_{1,V-1}\left[\tau^{\lambda-1}; \tau\right] \equiv \mathbf{B}(\lambda, V).$$

Remark 3.3. Let the functions v and u be of the form given by

$$v \coloneqq v(\tau) := \tau^{\lambda} \quad \text{and} \quad u \coloneqq u(\tau) := \tau.$$

Then, the special form of that mentioned transform is easily arrived at:

$$\mathbb{B}_{w,V}\left[\tau^{\lambda}; \tau\right] \equiv \int_0^{w} \tau^{\lambda} (w-\tau)^{V} dt. \qquad (16)$$

At this stage, for the integral given by (16), when considering the change of variable:

$$\tau = wT \implies d\tau = w dT,$$

and by means of the information given in (16), the following relationships:

$$\mathbb{B}_{w,V}\left[(wT)^{\lambda}; wT\right] \equiv \int_0^{1} (wT)^{\lambda} (w-wT)^{V} d(wT)$$

$$= w^{1+\lambda+V} \int_0^{1} T^{\lambda} (1-T)^{V} dT$$

$$= w^{1+\lambda+V}\, \mathbf{B}(1+\lambda; 1+V) \qquad (17)$$

can be also received, *where* λ and V are real numbers greater than -1. Of course, *when* those parameters are any complex numbers, their real parts have to be greater than -1.

Additionally, by taking cognizance of the substantial relationship between the Gamma function and the Beta function, which is the form in (5) (or (6)), the result determined in (17) can be equivalently represented as the form:

$$\mathbf{B}(\lambda, V) \equiv \mathbb{B}_{w,V}\left[(wT)^{\lambda}; wT\right] = w^{1+V+\lambda} \frac{\mathbf{\Gamma}(\lambda+1)\mathbf{\Gamma}(1+V)}{\mathbf{\Gamma}(2+\lambda+V)}. \qquad (18)$$

In addition, *when* the parameters λ and V belong to $\mathbb{N}$, the related assertions presented by (18) equivalently yield that

$$\mathbf{B}(\lambda, V) \equiv \mathbb{B}_{w,V}\left[(w\tau)^{\lambda}; wt\right] = w^{\lambda+V+1} \frac{\lambda!\, V!}{(\lambda+V+1)!}.$$

In addition, only two of the most important roles of the Gamma function and the Beta function appear in both the integral of fractional arbitrary order and the derivative of fractional arbitrary order, which both are classical generalizations of the familiar ordinary integration and the generalization of differentiation to arbitrary non-integer order and are designated for the independent variable with the real τ $(\tau > 0)$, respectively.

Let us now present these two extensive implications, which are closely related to the mentioned information introduced in (3)-(5), within (the scope of) the complex functions as remarks, which are just below.

Remark 3.4. (Fractional Integral Operator): The fractional integral of order φ is denoted, for a function $\theta(z)$, by

$$\mathfrak{D}_z^{-\varphi}[\theta(z)] \equiv \mathfrak{D}_z^{-\varphi}[\theta](z),$$

and also defined by

$$\mathfrak{D}_z^{-\varphi}[\theta(z)] = \frac{1}{\mathbf{\Gamma}(\varphi)} \int_0^{z} \frac{\theta(\tau)}{(z-\tau)^{1-\varphi}} d\tau, \qquad (19)$$

where $\varphi > 0$ and the related function $\theta(z)$ is an analytic function in any simply connected region of z-plane involving the origin, and the multiplicity of $(z-\tau)^{\varphi-1}$ is raised by necessitating $log(z-\tau)$ to be a real number *when* $z-\tau>0$. For more information in relation with fractional differ-intgeral in the references.

Of course, the value of φ, which expresses the fractional order, can also be any complex number. Due to the hypotheses there, it is inevitable that the real part of φ is greater than zero.

To illustrate the definition presented by this remark as only one of the indicated implications, let us consider the mentioned function as the form given by

$$\theta(z) \coloneqq z^{\mathfrak{m}} \tag{20}$$

for some $\mathfrak{m} \in \mathbb{R}$ with $\mathfrak{m} > -1$ (and, of course, $\Re e(\mathfrak{m}) > -1$ when $\mathfrak{m} \in \mathbb{C}$).

Then, in the light of the information between (3) and (5), and also by making use of the variable-differential changes given by

$$\tau = zt \implies d\tau = zdt\,, \tag{21}$$

from (13), its fractional integral of real order φ $(\varphi > 0)$ can be easily determined as the relations given by

$$\begin{aligned}\mathfrak{D}_z^{-\varphi}[z^{\mathfrak{m}}] &\equiv \frac{1}{\Gamma(\varphi)}\mathbb{B}_{z,1-\varphi}[t^{\mathfrak{m}};t]\\ &\equiv \frac{1}{\Gamma(\varphi)}\int_0^z t^{\mathfrak{m}}\,(z-t)^{\varphi-1}dt\\ &= \frac{z^{\mathfrak{m}+\varphi}}{\Gamma(\varphi)}\int_0^1 \tau^{\mathfrak{m}}(1-\tau)^{\varphi-1}d\tau\\ &= \frac{z^{\mathfrak{m}+\varphi}}{\Gamma(\varphi)}\mathrm{B}(1+\mathfrak{m},\varphi)\\ &= \frac{\Gamma(\mathfrak{m}+1)}{\Gamma(\mathfrak{m}+\varphi+1)}z^{\mathfrak{m}+\varphi},\end{aligned} \tag{22}$$

where $\varphi > 0$.

Remark 3.5. (Fractional Derivative Operator): The fractional derivative of order φ is denoted, for a function $\theta(z)$, by

$$\mathfrak{D}_z^{\varphi}[\theta(z)] \equiv \mathfrak{D}_z^{\varphi}[\theta](z),$$

and, also described as

$$\mathfrak{D}_z^{\varphi}[\theta(z)] = \begin{cases}\frac{1}{\Gamma(1-\varphi)}\frac{d}{dz}\int_0^z \frac{\theta(z)}{(z-\kappa)^{\varphi}}d\kappa & (0 \le \varphi < 1)\\ \frac{d^j}{dz^j}\left\{\mathfrak{D}_z^{\varphi-j}[\theta](z)\right\} & (0 \le \varphi - j < 1)\end{cases} \tag{23}$$

where $j \in \mathbb{N}$ and the function $\theta(z)$ is constrained, and the multiplicity of $(z-\kappa)^{-\varphi}$ is also extinct as in Remark 3.4. For the particulars of the definition presented in (23), se the earlier studies in, [39], [40], [41], [42], [43].

Naturally, as it is also emphasized in Remark 3.5, *when* the indicated number φ is any complex number, of course, its real part must be greater than zero, and also $j := 1 + [\![\Re e(\varphi)]\!]$. For more details in relation to the familiar definitions that have been described in Remarks 3.4 and 3.5, one may also refer to some of the essential-earlier results in the investigation presented in, [44], [45], [46], [47], [48].

As an extensive example, we want to consider the function consisting of the complex-exponential form given by (20) again. In parallel with the information indicated between (20) and (21), and by using the basic information between (2) and (4), its fractional derivative(s) of (real) order φ $(0 \le \varphi < 1)$ can be easily re-determined as the elementary result consisting of the relationships given by

$$\begin{aligned}\mathfrak{D}_z^{\varphi}[z^{\mathfrak{m}}] &\equiv \frac{1}{\Gamma(1-\varphi)}\frac{d}{dz}\left(\mathbb{B}_{z,\varphi}[\tau^{\mathfrak{m}};\tau]\right)\\ &\equiv \frac{1}{\Gamma(1-\varphi)}\frac{d}{dz}\left(\int_0^z \tau^{\mathfrak{m}}\,(z-t)^{-\varphi}dt\right)\\ &= \frac{1}{\Gamma(1-\varphi)}\frac{d}{dz}\Big[z^{\mathfrak{m}-\varphi+1}\\ &\qquad\times \int_0^1 \tau^{\mathfrak{m}}\,(1-\tau)^{-\varphi}\Big]d\tau\\ &= \frac{1}{\Gamma(1-\varphi)}\frac{d}{dz}\left[z^{\mathfrak{m}-\varphi+1}\mathbf{B}(\mathfrak{m}+1,1-\varphi)\right]\\ &= \frac{\Gamma(\mathfrak{m}+1)}{\Gamma(\mathfrak{m}-\varphi+1)}\,z^{m-\varphi},\end{aligned} \tag{24}$$

where $0 \le \varphi < 1$.

Indeed, in a similar idea to the elementary result just above and by considering the second case given by (23) in Remark 3.5, its fractional derivative(s) of the (real) order:

$$j + \varphi \;\; (0 \le \varphi < 1\,;\, j \in \mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\})$$

can be also re-determined as in the form given by

$$\begin{aligned}\mathfrak{D}_z^{j+\varphi}[z^{\mathfrak{m}}] &\equiv \frac{d^j}{dz^j}\{\mathfrak{D}_z^{\varphi}[z^{\mathfrak{m}}]\}\\ &= \frac{\Gamma(\mathfrak{m}+1)}{\Gamma(\mathfrak{m}-j-\varphi+1)}z^{m-j-\varphi},\end{aligned} \tag{25}$$

where $0 \le \varphi < 1$.

4 Conclusion and Recommendations

In this special section, numerous possible implications and various sensible suggestions in relation to our essential results will be presented (or emphasized) for our readers, who are also interested in some (generalized) integral transformations and a number of their interesting applications.

As a result of a simple focus on this special-research note in the second and third sections, it is seen that various special information has been presented in which the two relevant special functions play a very active role.

Moreover, in consideration of the special information between (1) and (6), it can be also pointed out that all the specific information passed between (7) and (21) will have quite specific consequences both in the context of the transformations and in some of their various special applications. Of course, in revealing (or determining) those special implications, it will be enough to select the relevant parameters and/or functions in the relevant sections appropriately.

Especially, as more special information in relation to applying the general form in (8), since the Gamma function (with real (or complex) parameters) and the related transformations takes part in a vast number of applications in such diverse areas as astrophysics, fluid dynamics, quantum physics, mathematics, and statistics, it can be also checked those related special studies given in the references dealing with related themas in the references of this special work.

At the same time, a great deal of transformations can be also composed by choosing various different types of those functions u and v in for the integral forms given in (8). For both Laplace transform and some of the others transforms (or their applications), one may also refer to some of the main works given in, [49], [50], [51], [52], [53], [54], [55], [56]. Moreover, as more special information, since the Gamma function (with real (or complex) parameters) and the related transformations takes part in a vast number of applications in such diverse areas as astrophysics, fluid dynamics, quantum physics, mathematics, and statistics, it can be also look over the earlier studies given by the reference in relation with various applications in the other papers in the references of this investigation.

Concurrently, in the light of the special references just above and as more special information relating to a variety of applications of the Beta function with real (or complex) parameters, we point specially that this special function is a quite useful tool in both representing and computing the scattering amplitude for Regge trajectories. Moreover, it was the first known scattering amplitude in string theory, which was first conjectured by Gabriele Veneziano. It also happens in the theory of the preferential attachment process, a type of stochastic urn process. At the same time, it is also an important tool for the Beta distribution and the Beta prime distribution in the theory of statistics. As it has been pointed out this special function is closely connected to the Gamma function and also plays important roles in the well-known classical calculus.

After quite detailed information above, for the relevant researchers, we can present some of those extensive implications and/or examples. Let us now create some of those special results and then make certain extra suggestions.

The first implication is related to various applications of the Laplace transform, which is possible only to be determined with the help of the gamma function. It also is just below.

Implication 4.1. Under the specific conditions designated by the admissible values of the parameters γ, β and λ, let

$$v(t) \coloneqq \mathrm{Sin}(\gamma t^{\beta}) \quad \text{and} \quad u(t) \coloneqq \lambda t.$$

Then, by using the function given by (3), the Laplace transform of the function v (with real variable) can be easily determined. For it, in the light of uniform convergence of the relevant function series and the information expressed in Remark 2.2, and also by the help of the Maclaurin series of the function $v(t)$ (just above), which is

$$\mathrm{Sin}(\gamma t^{\beta}) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \left(\gamma t^{\beta}\right)^{2j+1}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} t^{\beta(2j+1)}, \qquad (26)$$

the improper integral of the function in (26), or, equivalently, the Laplace transformation of the mentioned function, can be firstly specified with the help of those special functions as in the following equivalent forms:

$$\boldsymbol{\mathcal{L}}\left[\mathrm{Sin}(\gamma t^{\beta})\right] \coloneqq \mathbb{G}\left[\mathrm{Sin}(\gamma t^{\beta});\, e^{-\lambda t}\right]$$

$$= \int_0^{\infty} \mathrm{Sin}(\gamma t^{\beta})\, e^{-\lambda t} dt \qquad (27)$$

$$= \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \times \int_0^{\infty} t^{\beta(2j+1)} e^{-\lambda t} dt \right\}$$

$$= \sum_{j=0}^{\infty} \frac{\Gamma[\beta(2j+1)]}{(2j)!} (-1)^j \gamma^{2j+1}$$

where $\lambda > 0$ and $\beta > 0$. As only one of its more special results, *when* setting $\gamma \coloneqq 1$ and $\beta \coloneqq 2$ in both (26) and (27), Implication 4.1 immediately gives us the following relationships given by

$$\mathrm{Sin}(t^2) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} t^{2j+1}, \qquad (28)$$

$$\mathcal{L}[\mathrm{Sin}(t^2)] \coloneqq \mathbb{G}\left[\mathrm{Sin}(t^2);\, e^{-\lambda t}\right] = \int_0^{\infty} \mathrm{Sin}(t^2)\, e^{-\lambda t} dt = \sum_{j=0}^{\infty} \frac{(4j+1)!}{(2j)!} (-1)^j \qquad (29)$$

and

$$\mathrm{Sin}(t^2) = \mathcal{L}^{-1}\left[\sum_{j=0}^{\infty} \frac{(4j+1)!}{(2j)!} (-1)^j\right]. \qquad (30)$$

The second implication deals with various applications of those differ-integral operators of fractional order, for an elementary function with the complex variable z, which also is possible only to be identified by combining the Gamma function and the Beta function. Its creation is just below.

Implication 4.2. Under the conditions accentuated in Remark 3.4 and by using the Gamma function and the Beta function, the second implication can be also determined. For it, it is enough to consider the Maclaurin series of the function given by (26) and make use of the similar-variable changes there. Then, the fractional-order differ-integral of that function given in (26):

$$\mathfrak{D}_z^{-\varphi}\left[\mathrm{Sin}\left(\gamma t^{\beta}\right)\right]$$

$$\equiv \frac{1}{\Gamma(\varphi)} \mathbb{B}_{z,1-\varphi}\left[\mathrm{Sin}\left(\gamma t^{\beta}\right); t\right]$$

$$= \frac{1}{\Gamma(\varphi)} \int_0^z \mathrm{Sin}\left(\gamma t^{\beta}\right) (z-t)^{\varphi-1} dt$$

$$= \frac{1}{\Gamma(\varphi)} \int_0^z \left\{ \left(\sum_{j=0}^{\infty} \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \times t^{\beta(2j+1)} \right) (z-t)^{\varphi-1} \right\} dt$$

$$= \frac{1}{\Gamma(\varphi)} \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \times \int_0^z t^{\beta(2j+1)} (z-t)^{\varphi-1} dt \right\} \qquad (31)$$

$$= \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \frac{\Gamma[\beta(2j+1)]}{\Gamma[\varphi+\beta(2j+1)]} z^{\varphi+\beta(2j+1)} \right\}$$

and

$$\mathfrak{D}_z^{\varphi}\left[\sin\left(\gamma t^{\beta}\right)\right]$$

$$\coloneqq \frac{1}{\Gamma(1-\varphi)} \frac{1}{dz} \mathbb{B}_{z,\varphi}\left[\sin\left(\gamma t^{\beta}\right); t\right]$$

$$= \frac{1}{\Gamma(1-\varphi)} \frac{1}{dz} \int_0^z \sin\left(\gamma t^{\beta}\right) (z-t)^{-\varphi} dt$$

$$= \frac{1}{\Gamma(1-\varphi)} \frac{1}{dz} \int_0^z \left\{ \left(\sum_{j=0}^{\infty} \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \times t^{\beta(2j+1)} \right) (z-t)^{-\varphi} \right\} dt$$

$$= \frac{1}{\Gamma(1-\varphi)} \frac{1}{dz} \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \times \int_0^w t^{\beta(2j+1)} (w-t)^{-\varphi} dt \right\} \qquad (32)$$

$$= \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j \gamma^{2j+1}}{(2j+1)!} \frac{\Gamma[\beta(2j+1)]}{\Gamma[\varphi+\beta(2j+1)]} z^{\varphi+\beta(2j+1)} \right\}$$

are also determined, respectively.

As concluding remarks or various suggestions, first of all, the specific conclusions of all the comprehensive results we have achieved so far can be easily obtained. Also, by using the specific information given between (1) and (5), several different-type integral transformations can be reconstructed. For each of those transformations, just as in the second part and after the main conclusions made in the third part, similar conclusions can easily be drawn to the comprehensive conclusions mentioned in the fourth part, that is, (26) to (32) and their more specific conclusions. We bring the determination of each of such determinations to the attention of relevant researchers.

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