About solutions of some nonlinear singular model problems taking into account functional initial conditions and extreme regime

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Abstract—The work is to devoted investigation of nonlinear singular models objects with functional initial conditions and determination of solutions of problems, which are connected with these models. The proposed method is similar to the method of separating variables for nonlinear parabolic problems. It is also considered its some applications in field of technology, ecology, hydromechanics, economics and soon. In these problems, the initial condition of Coushy type is replaced by the initial condition of functional initial type. On base of producing functionals are solved some problems in extreme regimes

Index Terms—Nonlinear singular models, models equations, functional initial conditions, producing functionals with initial conditions.

As you know, many problems that arise in our daily life are related to nonlinear and singular problems. Such problems are the problems of thermal conductivity, diffusion, heat and population waves, the number of particles and populations, and many other related so-called parabolic problems with functional initial conditions in which chaotic processes occur[1-2]. For example, the problem of finding the solution of ordinary differential equations

$$\dot{x} = f(x, \tau), \ 0 < \tau \leq L, \ x(0) = \int_0^L \varphi(x(\xi), \xi)d\xi$$

is an example of such problems. Here $f(\cdot), \varphi(\cdot)$ are given functions, L is given number. We shall investigate nonlinear models that are described by the help of partial differential equation of first and second orders. Solutions of linear and some singular nonlinear problems with functional initial conditions are obtained form by means of models parameters and in particular are considered in some works[3-12].

1. About some nonlinear singular model problems. Under mathematical modeling of ecological and others processes we have a different variant of next problem. Define the function $N = N(x, a, t), N = (N_1, N_2)$, which is a solution of the equation

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial a} + \sum_{i=1}^{2} V_i \frac{\partial N}{\partial x_i} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(D_i \frac{\partial N}{\partial x_i}) + F(N, a, t),$$
(1)

$x \in G, 0 < a < \infty, 0 < t \leq t_k$ and satisfies initial, boundary conditions:

$$N\Big|_{t=0} = N_0(x, a), \ \ 0 \leq a < \infty,$$
(2)

$$N(x, 0, t) = \int_0^\infty B(N(x, \xi, t), \xi, t)d\xi, \ 0 \leq t \leq t_k, \ x \in \overline{G},$$
(3)

$$\alpha \frac{\partial N}{\partial n} + \beta N|_s = \varphi,$$
(4)

where $F(\cdot), B(\cdot), N_0(\cdot), \varphi(\cdot)$ are given vector functions, V_i, D_i are diagonal matrices, $F(\cdot) = F(N, a, t)N, B(\cdot) = B_0(N, a, t), F_0(\cdot), B(\cdot)$ are diagonal matrix of 2-th odor

$$B_0 = b(a, t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b(\cdot) = 0, \ \overline{G} = G + S, \ G \subseteq E^n, S$$

is the boundary of G. It is noticed that the problem (1)-(4), when $N = N(a)$ i.e.

$$\frac{dN}{da} = F(N, a), \ 0 < a < \infty, \ N(0) = \int_0^\infty B(N(\xi), \xi)d\xi$$

is called stationary problem and it was investigated in the work[1]. In the same place it was introduced the concert of problems with functional initial conditions and it was justified mathematically. The case N=N(a , t) were investigated in numerous scientific work ([2]). Moreover in this work is constructed the theory of nonlinear age -dependent population dynamics which is connected with last case. The general models of type (1)-(4) were item of investigations in works [1]. These are devoted to questions of existence and uniqueness of solutions of (1)-(4), presentations of solution for linear problems of (1)-(4): $(D_i = const, F = F_0(a, t)N, B = B_0(a, t)N)$ Moreover in these works are obtained a priori estimates and theorems of comparisons for solutions are proved and were founded new conditions of stability of stationary solutions of the problem (1)-(4). These conditions of stability are connected with concept of so- called potential of system (1)-(4): $h = ||B||$, where

$$B(a, t) = B_0(a, t)X(a, t), \ \frac{\partial X}{\partial a} = -F_0(a, t)X, \ X|_{a=0} = I,$$

I is unit matrix of the m-th order.

The aim of this work consist of investigation nonlinear singular and degenerate models of (1)-(4) in bounded and unbounded domains.

We now consider the problem of (1)-(4) in the form

$$
\begin{cases}
\frac{\partial N}{\partial t} + \frac{\partial N}{\partial a} = D_0 \frac{\partial}{\partial x}(N^q \frac{\partial N}{\partial x}) + qN^\beta, \\
-\infty < x < \infty, \ a > 0, \ t > 0 \\
\\
N(0,0,t) = \int_0^\infty B_0(\xi)N((0,\xi,t),\xi,t)d\xi, \ t \ge 0,
\end{cases}
\tag{5}
$$

where D_0, q, σ, β are given constants, $B_0(a)$ is nonnegative function, $N = N(x,a,t)$ is unknown scalar function.

Statement 1. Let $B_0(\alpha) \ge 0$, $\int B_0(a)da < 1$, and the function $f = f(y)$ is solution of the equation

$$(f^\delta f_y)_y - \frac{\delta+1-\beta}{2}yf_y - f(1-f^{\beta-1}) = 0, \ -\infty < y < \infty.$$

then the solution of problem (5) is represented in the form:

$$N(x,a,t) = \mu(t-a)[1-q(\beta-1)\mu^{\beta-1}(t-a)a]^{-\frac{1}{\beta-1}}f(y), \tag{6}$$

where $y = x/\psi(a,t-a)$, $\psi(a,t)$ is defined next way:

$$\varphi(a,t-a) = \sqrt{\frac{D_0}{q}}\mu^{\frac{\delta+1-\beta}{2}}(t-a)[1-q(\beta-1)\mu^{\beta-1}(t-a)a]^{\frac{\delta+1-\beta}{2(\beta-1)}}$$

and the function $\mu = \mu(t)$ is solution of the integral equation

$$\mu(t) = \int_0^\infty B_0(a)[1-q(\beta-1)\mu^{\beta-1}(t-a)a]^{-\frac{1}{\beta-1}}\mu(t-a)da \tag{7}$$

The wave is defined with the help of (6) we shall call mowing wave, when $\beta = \delta+1$ we have so-called steady wave of type

$$N(x,a,t) = \mu(t-a)[1-q\mu^\delta(t-a\mu^{\beta-1}(t-a)a)]^{-\frac{1}{\delta}}f(\sqrt{\frac{q}{D_0}}x),$$

where $\mu(t)$ is a solution of an equation (7) under the condition $\beta = \delta+1$ it is noticed the If $\beta \to 1$ then we have next moving wave

$$N(x,a,t) = \mu(t-a)e^{-qa}f(\frac{x}{\varphi(a,t-a)}),$$

where $\mu(t) = \sum_j c_j e^{\delta_j t}$ and δ_j are roots of the equation $\int_0^\infty B(a)^{-\delta a}da = 1$, $B(a) = B_0(a)e^{-qa}$

Moreover last equation has one maximal real root δ_{max} which satisfy the condition.

$$
\delta_{max} = \begin{cases}
< 0 \ if \ h = \int_0^\infty B_0(a)e^{-qa}da < 1 \\
= 0 \ if \ h = 1 \\
> 0 \ if \ h > 1
\end{cases}
$$

and others roots δ_j are complex conjugates : $\delta_j = \alpha_j \pm i\beta_j$ Hence

$$\mu(t) = C_0 e^{\delta_{max}t} + \sum_{j=1}^\infty C_j e^{d_j t}\cos\beta_j t.$$

Example 1. Now we consider the problem

$$
\begin{cases}
\frac{\partial N}{\partial a} = D_0\frac{\partial}{\partial x}\left(N^\sigma \frac{\partial N}{\partial x}\right), a > 0, \\
\\
N(0,0) = \int_0^\infty B_0(\xi)N(\xi,0)d\xi
\end{cases}
$$

It has a solution of type

$$N(x,a) = \left(\frac{\psi_0\delta}{D_0(\delta+2)}\right)^{\frac{1}{\delta}}(1-\delta a)^{-\frac{1}{\delta+2}}f(\frac{x}{\psi(a)}),$$

where δ is maximal and real root of the equation $\int_0^\infty B_0(a)(1-\delta a)^{-\frac{1}{\delta+2}}da = 1$ and $f(Y)$ is a solution of the equation $(f^\sigma f_y)_y - yf_y = f$.

Thus under the condition

$$f_y\big|_{y=0} = 0, \ f\big|_{y=0} = 0, \ \left(\frac{\partial N}{\partial x}\big|_{x=0} = 0, N\big|_{x=0} = 0\right)$$

we have next solution

$$
N(x,a) = \begin{cases}
\left[\frac{\sigma\delta}{2D_0(\sigma+2)}\right]^{\frac{1}{\delta}}(1-\delta a)^{-\frac{1}{\sigma}}x^{\frac{2}{\sigma}}, \ if \ a < \frac{1}{\delta} \\
0, \ in \ other \ cases.
\end{cases}
$$

It is noticed that a corner frequency $\beta(t)$ and components wave vector $\alpha_i(t)$ are determined from some additional conditions. For example, its may be an initial and boundary condition.

2. One class model nonlinear equations. We consider next equations ([13-24]):

$$\sum_{i=1}^m X_{im}^n = Z_m^n \tag{8}$$

or the equation

$$Z_m = \max_{\alpha \in A}\mu_s(\alpha), \ \mu_s(\alpha) = (\sum_{i=1}^m \alpha_i X_{im}^s)^{1/s}, \tag{8_0}$$

where $X_{im}, Z_m, i = \overline{1,m}, m = 2,...,M, M < \infty$ are unknown values and $n > s$ is natural number, s is natural, $A = [\alpha_i : 0 < \alpha_i < 1, \sum_{i=1}^m \alpha_i^{n/(n-s)} = 1]$. It is justly next.
Statement 2. For any natural n between to set the solutions of adjacent by m equations (8) (i.e. under $m = k-1$ and $m = k$) it takes place next presentations:

$$Y_{ik} = xX_{ik-1}, Y_{kk} = yZ_{k-1}, Y_k = zZ_{k-1}, i = 1,2,...,k-1 \tag{9}$$

where $k = 2,3...$; (x,y,z) are some of solution (8) under $m = 2$, i.e. $x^n + y^n = z^n$. Really let $(X_{1k-1}...,X_{k-1k-1},Z_{k-1})$ are solutions (8) under $m = k-1$. We shall that $(Y_{ik},...,Y_{kk},Y_k)$ obtained by of (9) and are the solutions (8) under $m = k$. So that $\sum_{i=1}^{k-1} X_{ik-1}^n = Z_{k-1}^n$, then multiplide both part of the last identify on x^n we have: $x^n \sum X_{ik-1}^n = x^n Z_{k-1}^n$. From here $\sum_{i=1}^{k-1}(xX_{ik-1})^n = (z^n - y^n)Z_{k-1}^n$, and hence, using (9) we have: $\sum_{i=1}^k Y_{ik}^n = Y_k^n$, i.e equation (8) under $m = k$. Analogous, if $(Y_{ik}...,Y_{kk},Y_k)$ also satisfied the equations (8), then it is easy to see $(X_{ik-1},...,X_{k-1k-1},Z_{k-1})$ also satisfied (8) under $m = k-1$.

It is notice that transformation of (9) transfers any point of $(X_{1m-1},...,X_{m-1m-1}) \in E^{m-1}$ with metric Z_{m-1} to some point of $(Y_{1m},...,Y_{mm}) \in E^m$ with metric Y_m and conversely and what is more $(x,y) \in E^2 \to (X_{13},X_{23},X_{33}) \in E^3 \to ...$ $(Y_{1m},...,Y_{mm}) \in E^m$ and

$$\lim_{m \to \infty} [Y_m, \max_{1<i<m} Y_{im}] = \begin{cases} 0 & at \quad z < 1 \\ \infty & at \quad z > 1 \end{cases}$$

i.e some moving object on this path may be pass to initial coordinate of space E^m at $m \to \infty$ or it is receded from it. Besides if the light velocity is constant for all spaces E^m i.e.$Y_{1m} = ct_m, X_{1m-1} = ct_{m-1}$ or ($Y_m = ct_m, Z_{m-1} = ct_{m-1}$) then $x = \sqrt{\frac{c-v_m}{c-v_m}}, u_m = xu_{m-1}, t_m = xt_{m-1}$ (or $z = \sqrt{\frac{c-v_m}{c+v_m}}$) and what is more all properties of the theory of relativity are correct with respect to transformation of (8), where c, v, u are velocity, t is the time.
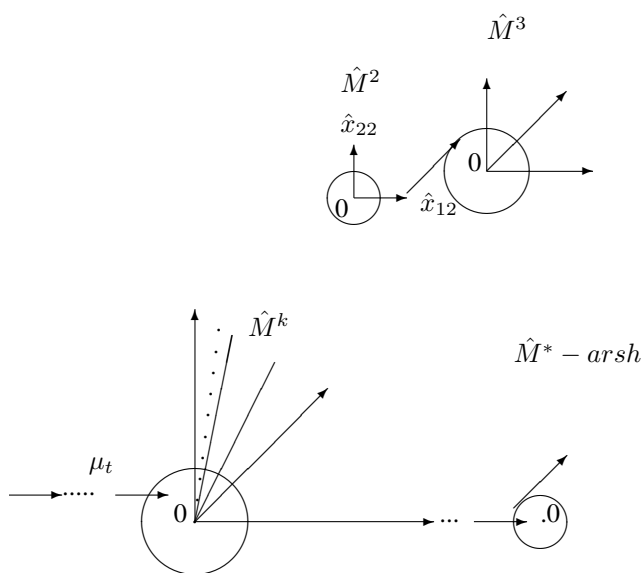
Besides the transformation of (9) may be represented in the form of[13-18]:

$$Y = MX,$$

where $X = (X_{1m-1}, ..., X_{m-1m-1}, X_{m-1}X_{m-1})$, $X_{m-1} = (\sum_{i=1}^{m-1} X_{im-1}^n)^{1/n}$ and $Y = (Y_{1m}, ..., Y_{mm}, Y_m)$, $Y_m = (\sum_{i=1}^{m} Y_{im}^n)^{1/n}, m = 1, 2...,$ and M is defined in the following way

$$M = \begin{pmatrix} x & 0 & \dots & 0 & 0 & 0 \\ 0 & x & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & x & 0 & 0 \\ 0 & 0 & \vdots & 0 & y & 0 \\ 0 & 0 & \vdots & 0 & 0 & z \end{pmatrix}$$

and x, y is some point of E^2 with metric $z = (x^n + y^n)^{1/n}$ at $n \geq 2$ which transfer into $(Y_{1m}, ..., Y_{mm})$ with metric Y_m i.e

Statement 2. The transformation B_p (matrices of $k+1$ order),

where

$$B_p = \begin{pmatrix} x^p & 0 & \dots & 0 & 0 & 0 \\ 0 & x^p & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & x^p & 0 & 0 \\ 0 & 0 & \vdots & 0 & y^p & 0 \\ 0 & 0 & \vdots & 0 & 0 & z^p \end{pmatrix}$$

also transfers any point $(x_{1k-1}, ..., x_{k-1k-1}, z_{k-1}, z_{k-1})$ into point $(x'_{1k}, ..., x'_{kk}, z'_k)$, i.e $E^{k-1} \to E^k$, where $x + y = z$, $z_{k-1} = (\sum_{j=1}^{k-1} x_{jk-1}^n)^{1/n}$, $z'_k = (\sum_{j=1}^{k} x'^n_{jk})^{1/n}$, and what is more: 1). $B_{p+s} = B_p B_s, s \geq 0, B_0 = I, \lim_{p \to p_0} B_p X = B_0 X$ for any vector $X \in E^{k+1}$. 2). B_p is linear transformation and $B_p = M^p$, i.e. $M = B_p^{1/p}, p = 1/n$. 3). Eigenvalues of transformation B_p are represented in the form of: $\lambda_j = x^p, j = 1, 2, ..., k - 1, \lambda_k = y^p, \lambda_{k+1} = z^p$ and $\max_j \lambda_j = z^p, \|B_p\| = z^p, \|B_p^{-1}\| = x^{-1/p}$. 4). The infinitesimal generating operator $B = \lim_{t \to 0} t^{-1}(B_t - I)$ is diognal matrix and represented in the following way: $a_{ii} = \ln x, i = 1, ..., k - 1, a_{kk} = \ln y, a_{k+1k+1} = \ln z$, and what is more $R(m, B) = (qI - B)^{-1}, B_t = e^{tB}$, $B = \ln B_t^{1/t}, B = \ln M$.

It is noticed that the inverse transformation is not unique. For example some point of Minkovsky space (x_1, x_2, x_3, x_4), where $x_1 = dx, x_2 = dy, x_3 = dz, x_4 = c_0 ds$ with metric $z = c_0 dt$, c_0 is the light velosity is transfered into points of three circles with radius $xds/y, c_0^2 xdz/yz, c_0^2 x^2 dt/z$ accordingly, $(x, y, z) \in M_0$.

The equation $x^n + y^n = z^n$ is called basic equation for equation (8), $n = 1, 2, 3...$.

The case of $x^n + y^n = z^n$. Now we consider the function(μ- function):

$$\mu_s(\alpha) = [\alpha X^s + \left(1 - \alpha^{n/(n-s)}\right)^{(n-s)/n} Y^s]^{1/s}, \quad (10)$$

and the equation

$$Z = \max_{0<\alpha<1} [\alpha X^s + (1 - \alpha^{n/(n-s)})^{(n-s)/n} Y^s]^{1/s}, \quad (9_0)$$

where X, Y, Z are positive numbers, n, s are natural, s is some fixed number, $0 < \alpha < 1$.

Lemma. The function $\mu_s(\alpha), 0 < \alpha < 1, n > s, s \geq 1$, under $\alpha^0 = (X^n/(X^n + Y^n)^{(n-s)/n}$ has maximal value $Z = \mu_s(\alpha^0) = (X^n + Y^n)^{1/n}$ i.e. points X, Y, Z corresponding to value α^0 are solutions of the equation

$$X^n + Y^n = Z^n \quad (11)$$

Really, it is easy to show that

$$\frac{d\mu_s(\alpha)}{d\alpha} = 0$$

From here $\alpha^0 = (X^n/(X^n + Y^n))^{(n-s)/n}$ and $d^2(\mu_s)/d^2\alpha < 0$ only under $s > 1$. For value $\alpha = \alpha^0$

from (1) after not difficult culculations we have the equation of (4).

Statement 3. All solutions of the equations (11) (i.e. all α^0 are a maximal points of the function $\mu_s(\alpha)$, $\alpha \in [0,1], s \geq 1$).

The proof. We assume that equation (4) has positive solution (X, Y, Z), i.e $X^n + Y^n = Z^n$. It is noticed that $X < Z, Y < Z, Z < X + Y$. As $X^{n-s}X^s + Y^{n-s}Y^s = Z^{n-s}Z^s$, then introducing notations $\alpha = (X/Z)^{n-s}, \beta = (Y/Z)^{n-s}$ we have

$$\alpha X^s + \beta Y^s - Z^s = 0, \quad \alpha + \beta > 1, \quad \alpha + \beta < 2$$
$$X^s - \alpha^{s/(n-s)}Z^s = 0, \quad\quad\quad (12)$$
$$Y^s - \beta^{s/(n-s)}Z^s = 0,$$

This system has solution $(X^s, Y^s, Z^s) > 0$ only in case when it's determinant is equals to 0 (det = 0!), i.e $\alpha^{n/(n-s)} + \beta^{n/(n-s)} - 1 = 0$. Hence $\beta = \left(1 - \alpha^{n/(n-s)}\right)^{(n-s)/n}$ and from the first equation (5) we have the function $\mu_s(\alpha)$ and the corresponding equation of (3_0): $Z = \max_{0<\alpha<1} [\alpha X^s + \left(1 - \alpha^{n/(n-s)}\right)^{(n-s)/n} Y^s]^{1/s}$, and $\alpha^0 = (X/Z)^{(n-s)} = (X^n/(X^n + Y^n))^{(n-s)/n}$. The proof of backwards statement is corollary of Lemma.

Now we shall find all solutions f the equation (4). By virtue of $\alpha^{n/(n-s)} = X^n/(X^n + Y^n)$, $\beta^{n/(n-s)} = Y^n/(X^n + Y^n)$ according X^n, Y^n we have next homogeneous linear algebraic equations:

$$(1 - \alpha^{n/(n-s)})X^n - \alpha^{n/(n-s)}Y^n = 0$$

and

$$-\beta^{n/(n-s)}X^n + (1 - \beta^{n/(n-s)})Y^n = 0$$

or

$$(1 - \alpha^{n/(n-s)})X^n - \alpha^{n/(n-s)}Y^n = 0$$

and

$$-(1 - \alpha^{n/(n-s)})X^n + \alpha^{n/(n-s)}Y^n = 0.$$

As determinant of this system equals to zero then its has nontrival solution of type:

$$X^n = \alpha^{n/(n-s)}, Y^n = (1 - \alpha^{n/(n-s)}), 0 < \alpha < 1.$$

Thus all solutions are represented in the form of:

$$X^n = k\alpha^{n/(n-s)}, Y^n = k(1 - \alpha^{n/(n-s)})$$

or

$$X = k^{1/n}\alpha^{1/(n-s)}, Y = k^{1/n}(1 - \alpha^{n/(n-s)})^{1/n}, Z = k^{1/n}$$

Such all maximal points of the function $\mu_s^s(\alpha)$ satisfy the equation of curvilinear ellipse: $[\alpha X^s + \left(1 - \alpha^{n/(n-s)}\right)^{(n-s)/n} Y^s] = Z^s$ for all $0 < \alpha < 1$. On the other hand its are the solutions of next curvilinear

circle: $X^s + Y^s = [\alpha^{s/(n-s)} + \left(1 - \alpha^{n/(n-s)}\right)^{s/n}]Z^s$. From here we have equation:

$$X^s + Y^s = R^s, \quad\quad\quad (*)$$

where

$$R^s = [\alpha^{s/(n-s)} + \left(1 - \alpha^{n/(n-s)}\right)^{s/n}]Z^s.$$

We denote $r = \sqrt{\alpha^{2/(n-2)} + \left(1 - \alpha^{n/(n-2)}\right)^{2/n}}$. The equation of (*) under $s > 2$ turn into the initial equation (2) and therefore we consider only case $s = 2$. It easy to see that $Z^2 < \max_\alpha R^2 = 2^{(n-2)/n}Z^2$ and from here $r \in J$ where $J = (1, \sqrt{2}\ 2^{-1/n}]$. Such the question of the integrity solutions $X^n + Y^n = Z^n$ is reduced to the integity of $X^2 + Y^2 = R^2$. On the other hand all solutions of the equation (*) are presented in the form of $X = \sqrt{2kY + k^2}$, Y, $R = Y + k$, $k = 1, 2...$ and what is more we have $X_k = jk$, $Y_k = \frac{(j^2-1)k}{2}$, $R_k = \frac{(j^2+1)k}{2}$, $j = 1, 2, ...$, and these solutions may be represented in the following way:

X_k Xk	Y_k Yk	R_k Rk	!	X_m Xm	Y_m Ym
$3k$	$4k$	$5k$	!	$4m$	$15m/2$
$5k$	$12k$	$13k$	!	$6m$	$35m/2$
$7k$	$24k$	$25k$	!	$8m$	$63m/2$
$9k$	$40k$	$41k$	!	$10m$	$99m/2$
...	....	.....	..	....	.....
jk	$(j^2-1)k/2$	$(j^2+1)k/2$	!	jm	$(j^2-1)m/2$
...	.....	.....	....	.....	......

where X, Y, R are basic solutions and k, m are natural and $m = 2k$, when j is a even number.
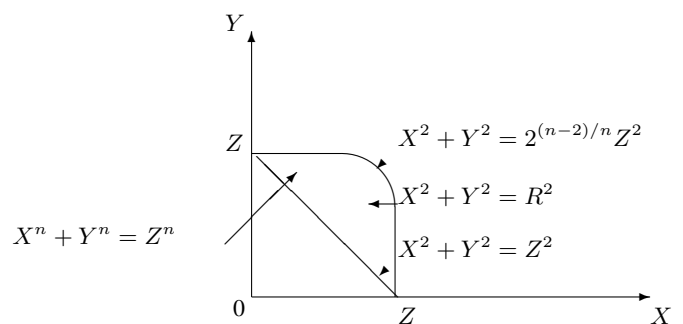
$$Z^2 < X^2 + Y^2 \leq 2^{(n-2)/n}Z^2, \quad X^2 + Y^2 = R^2, R = rZ$$
$$Z = R/r, \quad r \in (1, \sqrt{2}\ 2^{-1/n}]$$

The Domain of Solutions

Corollary 1. The solutions of (*) are not positive integer numbers $X, Y, Z = R/r$ for any $n > 2, s = 2$. Really let X, Y, Z some positive integer solution of (*). Let X, Y are integer and R is some corresponding number then we have

$Y < Z = \frac{(j^2+1)k}{2r} < R$ and Z may be integer only when among Y and R is exist natural. In the case of when j is a odd number all $X = j, Y = (j^2 - 1)/2, R = (j^2 + 1)/2$ are natural but among Y, R is not exist natural and then we have Z is not integer. When j is a even number then $X = 2j, Y = (j^2 - 1), R = (j^2 + 1)$ and among Y, R exist only one integer number j^2. But in this case $X = 2j, Y = j^2 - 1, Z = j^2$ is not the solution of the equation (4). If R is not natural then Y is also not natural. This result is correct for any solution of X_k, Y_k, Z_k.

3. Application to solutions of nonlinear equations

We consider some set G from E^m. Let the function $u = u(x_1, ..., x_m, z), (x_1, ...x_m, z) \in G$ is characterizing the density of some "process" (or information law, some "substance" and so on), $L_j, L, j = 1...m$ are some operators, which are realizing changes of given "process". We have the equation

$$Lu = \max_{\alpha \in A} \sum_{i=1}^{m} \alpha_i L_i u, \qquad (*_0)$$

where $A = \left[\alpha_i : \sum_{i=1}^{m} \alpha_i^{\frac{n}{n-s}} = 1, \quad 0 < \alpha_i < 1, i = 1, ..., m\right]$, s, is natural, s is a some number. Correctly next statement. Let $\alpha_i = \alpha_i^0$ is corresponding value under which the right of part of the equation $(*_0)$ has the maximal value. Then the equation of $(*_0)$ under $\alpha = \alpha^0$ and the equations of

$$(Lu)^n = \sum_{i=1}^{m} (L_i u)^n, \qquad (**)$$

are equivalent.

The equations (**) has solution if and only if when the predetermined system

$$L_j u = \phi_j, Lu = \phi, \qquad (***)$$

where $\phi_j = \phi_j(x_1, x_2...x_m, z), j = 1, 2...m, \phi(x_1, x_2, ..., x_m, z)$ are solutions of the next functional equations

$$\sum_{j=1}^{m} \phi_j^n = \phi^n, \qquad (****)$$

has some solution. Having taken $\phi = Z_m = c_m, \phi_j = X_{im} = c_{im}$ and using transformation (2) we have all "simple" solutions of the equations of (**) (i.e. $(*_0)$ under $\alpha = \alpha^0$). For example let c_j, c are some numbers solutions of equation $\sum_{j=1}^{m} c_j^n = c^n$ then $\phi_j = c_j \phi(x_1, ..., x_m, z), \phi = c\phi(x_1, ..., x_m, z)$ for all $\phi(.) > 0$ and $\phi \in C$ are solutions of the equation (****). We may be also take $\phi_j(.) = x_j^{2/n}, \phi = c_0 t^{2/n}$ and considering process will be define in the set of $G_0 \in G$, where

$G_0 = (x_1, ..., x_m, z) : \sum_{j=1}^{m} x_j^2 = c_0^n t^2$.

The set G_0 is series orbit with radius $R = c_0^{n/2} t, 0 < t < \infty$. We consider the equation[7]:

$$\sum_{i=1}^{m} \frac{\partial u}{\partial x_{im}}^n = \frac{\partial u}{\partial z_m}^n, m = 2, 3... \qquad (**_0)$$

The corresponding over determined system is defined next way: $\frac{\partial u}{\partial x_i} = c_i, \frac{\partial u}{\partial z} = c, i = 1, 2...m$ where c_i, c are solutions of equations (1). It is easy to see that we have next solutions of this equation:

1). $n = 2, m = 2$: $u(x_1, x_2, z) = u_0 + 3x_1 + 4x_2 + 5z$,

2). $n = 2, m = 3$: $u(x_1, x_2, x_3, z) = u_0 + 9x_1 + 12x_2 + 20x_3 + 25z$,

3). $n = 2, m = 5$: $u(x_1..x_5, z) = u_0 + 81x_1 + 108x_2 + 180x_3 + 300x_4 + 500x_5 + 625z$,

4). $n = 3, m = 2$: $u(x_1, x_2, z) = u_0 + 31.62278x_1 + 98.93459x_2 + 100z$,

5). $n = 5, m = 2$: $u(x_1, x_2, z) = u_0 + 3x_1 + 8.9928x_2 + 9z$,

6). $n = 10, m = 2$: $u(x_1, x_2, z) = u_0 + 5x_1 + 7.992694x_2 + 8z$, ... In general case we have[14]:

$$u(x_1, ..., x_m, z) = u_0 + \sum_{i=1}^{m} c_i x_i + cz, \qquad (7)$$

where u_0 is a value of the function u at the point $(x_1, ..., x_m, z) = 0$. Usually value of the u_0 is determined with help of Coushy condition. But for great many important practical problems it is not defined really. For example, populations initial numbers in ecosystems; the initial numbers of elementary particles in physics and soon. In connection with it we shall consider the condition for determination initial state in the following way[8-13,16]:

$$u(0, ..., 0, 0) = \int_{G} \phi(x_1, ..., x_m, z)u(x_1, ..., x_m, z)dxdz, \quad (8)$$

$$u(0, ..., 0, z) = \int_{G_0} \phi(x_1, ..., x_m, z)u(x_1, ..., x_m, z)dxdz,$$

where $\phi(.)$ is a usual law of behavioral distributions of considered processes. For example, $\phi(.)$ is the beath-function in eclogical and others processes[7-16]. G is a given set from E^{m+1}.

Statement 4. Let G is defined next way: $G = \{(x_1, ..., x_m, z) : \sum_{i=1}^{m} x_i^2 = z^2\}$ then the function

$$u(x_1, ..., x_m, z) = u_0 + \sum_{i=1}^{m} x_i^{1+2/n}/(1+2/n) + z^{1+2/n}/(1+2/n),$$
$$(9)$$

is the general solution of the equation $(**_0)$. Theorem 4 is proved directly by usual operations. It is noticed that under $n = 1$ we have: $u(x, z) = u_0 + \sum_{i=1}^{m} x_i^3/3 + z^3/3, (x, z) \in G$ and at $n \to \infty, u(x_1, ..., x_m, z) = u_0 + \sum_{i=1}^{m} x_i + z$. It should be pointed out that the solutions (7) and (9) are equivalent for all points $(x, z$ from G. Really, for all points $(x, z) \in G$ it is easy to see that $c_i = x_i 2/n, c = z^{2/n}$ are solutions of the equations (3). Now we shall consider equations with variable coefficients:

$$\sum_{i=1}^{m} \left(\frac{\partial u}{a_i(x_i)\partial x_i}\right)^n = \left(\frac{\partial u}{a(z)\partial z}\right)^n, \qquad (10)$$

where $a_i(x_i), a(z)$ are given functions and its may be has singular points for example, $a_i(x_i) = (x_i - x_i^0)^\alpha, \alpha > 0$ is

constant, $a(z) = (z - z_0)^\beta, \beta > 0$ is constant. It is easy to see that function

$$u(x, z) = u_0 + \sum_{i=1}^{m} c_i \int_0^{x_i} a_i(t)dt + c \int_0^{z} a(t)$$

is a solution of the equation (10). Here c_i, c are the solutions of equations (****) under some m, n; u_0 is defined from (8). This solution equals to the solution of next type:

$$u(x, z) = u_0 + \sum_{i=1}^{m} \int_0^{x_i^{(n+1)/n}} a_i(t)dt + \int_0^{z^{(n+1)/n}} a(t)dt,$$

which satisfies the equation (10) with coefficients $a_i(x_i^{(n+1)/n}$ and $a(z^{(n+1)/n})$, $(x, z) \in G$, $m \geq 2$, $n = 1, 2....$

The equation of th k-th order. We consider next equation[15]:

$$\sum_{i=1}^{m} \left(\frac{\partial^k u}{\partial x_i^k}\right)^n = \left(\frac{\partial^k u}{\partial z^k}\right)^n,$$

General solution of this equation is represented in the form of:

$$u(x_1, ..., x_m, z) = \varphi(x_1, ..., x_m, z) + \sum_{i=1}^{m} c_i \frac{x_i^k}{k!} + \frac{cz^k}{k!},$$

or $u(x_1, ..., x_m, z) = \varphi(x_1, ..., x_m, z) + \sum_{i=1}^{m} \frac{x_i^{k+2/n}}{(1+2/n)...(k+2/n)} + \frac{z^{k+2/n}}{(1+2/n)...(k+2/n)}$, where $\sum_{i=1}^{m} x_i^2 = z^2$, $-\infty < x_i < \infty$, $-\infty < z < \infty$, c_i, c are the solutions of the equation (1), the function $\varphi(.)$ is polinom of the $k - 1$-th order. Coefficients of this polinom we shall define with help of boundary and initial conditions. For example at $m = 2, k = 2$ if we are given

$$\frac{\partial u}{\partial z}\Big|_{z=0} = u_1(x_1, x_2), u\Big|_{z=0} = u_0(x_1, x_2)$$

for definition of the function $\varphi(.)$ we have: $\varphi(x_1, x_2, z) = u_0(0,0) + u_1(0,0)z + \frac{\partial u_0(0,0)}{\partial x_1}x_1 + \frac{\partial u_0(0,0)}{\partial x_2}x_2 + \frac{\partial u_1(0,0)}{\partial x_1}x_1z + \frac{\partial u_1(0,0)}{\partial x_2}x_2z + \frac{\partial^2 u_0(0,0)}{\partial x_1\partial x_2}x_1x_2 + \frac{\partial^2 u_1(0,0)}{\partial x_1\partial x_2}x_1x_2z.$

Corollary 2.. *The solutions of considering equations with constant coefficients and connected with predetermined system (***) are polinoms of k-th order, where k equals to the order of these differential equations. The leading coefficients satisfy the equations of (1) and the others are defined from corresponding boundary and initial conditions.*

Remark 1. All solutions of the equations (1): $Z^n = \sum_{i=1}^{m} X_i^n$ are represented in the following way: $X_i = k^{1/n}\alpha_i^{1/(n-s)}, i = 1, ..., m - 1$ and $X_m = k^{1/n}(1 - \sum_{i=1}^{m-1} \alpha_i^{\frac{n}{n-s}})^{1/n}, Z = k^{1/n}$. Really, we consider the function $\mu_s(\alpha) = [\sum_{i=1}^{m} \alpha_i X_i^s]^{1/s}, \alpha \in A, A = [\alpha_i : 0 < \alpha_i < 1, \sum_{i=1}^{m} \alpha_i^{\frac{n}{n-s}} = 1]$, and solving the problem of $\max_{\alpha \in A} \mu_s(\alpha)$ we have mentioned above solutions of the equations (1). It is clear that these solutions are not positive integer numbers under $n > 2$.

In work the new model of growth of the population, so-called energetic model of growth for population
$$\frac{dL}{dt} = \delta L, \quad \delta : \int_G B(a)e^{-\lambda a}da = 1, \quad \lambda = \delta + \beta + \gamma r, \quad L(t) = \left(\sum_{j=0}^{\infty} c_j^p e^{\alpha_j t} \cos(\beta_j t)\right)^{1/p}, (p=2),$$

is constructed and energetic theory of population growth is also investigated. This model is received for one and n countries, where $L(t) = \left(\int_G \varphi(\eta) N^p(\eta) d\eta\right)^{1/p}, 0 < p < \infty$.

Here
$$B(a) = B_0(a)e^{-p\int_0^a F_0(\xi)d\xi}$$ is function of survival rate of the population, $\varphi(\eta)$ is some non-negative function with a condition, $\varphi(x, a, t) = \int_a^{\infty} e^{-\int_a^{\xi} F_o^*(\eta)d\eta + \delta(a-\xi)} B_0^*(\varsigma) \varphi(x + r(\varsigma - a), 0, t - a + \xi)d\xi, \int_G \varphi(\eta) d\eta = 1; N(\eta) = N(x, a, t), N(\eta) = \sqrt[p]{\frac{\varphi(\eta)}{\int_0^{\infty} \varphi^2(\varsigma) d\varsigma}} L(t)$, and values of α_j β_j - are solutions of system $\int_G B(a)e^{-\alpha_j a} \cos \beta_j a \, da = 1$, $\int_G B(a)e^{-\alpha_j a} \sin \beta_j a \, da = 0$.

It is should out that energetic model is constructed on the base of some initial groups of the models, describing growth of a population in view of age structure and spatial distributions:

1.
$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) N = -F_0(a)N, \ 0 < a < \infty, \ 0 < t < t_r \\ N(a, 0) = N_0(a), \quad 0 \leq a < \infty, \\ N(0, t) = \sqrt[p]{\int_0^{\infty} B_0(a)N^p(a, t) \, da}, \\ \quad (a, t) \in G \end{cases}$$

2.
$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} + r\frac{\partial}{\partial x}\right) N = -F_0(a)N, \ 0 < a < \infty, 0 < t < t_k \\ N(x, a, 0) = N_0(x, a), \quad 0 \leq a < \infty, \ 0 < x < L, \\ N(x, 0, t) = \sqrt[p]{\int_0^{\infty} B_0(a)N^p(x, a, t) \, da}, \ 0 < x < L, \\ N\big|_{x=0} = 0 = N\big|_{x=0}, \end{cases}$$

3. *Models in view of a diffusion distributions*, $(x, a, t) \in G$.

The series of computer experiments were carried out for initial functions described with help of uniformly and normal distributed laws.

Remark 2. The method is also used for differential equations of type:

$$\sum_{i=1}^{m} \left(\frac{\partial^k u}{a_i(x_i)\partial x_i^k}\right)^n = \left(\frac{\partial^p u}{a(z)\partial z^p}\right)^n,$$

$$\sum_{i=1}^{m} \left(\frac{\partial^k u}{a_i(x_i)\partial x_i^{k_1}\partial x_i^{k_2}}\right)^n = \left(\frac{\partial^p u}{\partial z^p}\right)^n,$$

$$\sum_{i=1}^{m} \left(\frac{\partial^i u}{a_i(x_i)\partial x_i^i}\right)^n = \left(\frac{\partial^p u}{a(z)\partial z^p}\right)^n, ...;$$

where $k = 1, 2, 3..., p = 1, 2, 3..., k_1 + k_2 = k$ and others may be also solved with help of this method.

Example 2. Let $Lu = i\frac{\partial u}{\partial t}$, $L_j u = \sum_{j=1}^{m} \frac{\partial^k u}{\partial x_j^k}$ then we have $u = v + iw$, where

$v = u_0(x_1, x_2, t)\cos(\int_0^t F(a)da) +$
$+c_0 \int_0^t a^{2/n} \cos(\int_a^t F(a')da')da$,

$w = -u_0(x_1, x_2, t)\sin(\int_0^t F(a)da) -$
$c_0 \int_0^t a^{2/n} \sin(\int_a^t F(a')da')da + \frac{\sum_{j=1}^{m} x_j^{k+2/n}}{(1+2/n)...(k+2/n)}$ and $u_0(.)$ is polinom of $(k-1)$-th order, $F(t)$ is given function, $(x_1, ..., x_m, t) \in G$,

$G = \{\sum_{j=1}^{m} x_j^2 = c_0^n t^2\}$

4. Computer experiments of the equation $\sum_{j=1}^{m} X_j^n = Z^n$

Consider the case $n = 2$ and define positive integer solutions of the equations (1) under different values $m = 3, 4, 5, 6....$ It is clear that basic solution is the solution $x^2 + y^2 = z^2$. For example $x = 3, y = 4, z = 5$.

Computer experiments are given in the form of:

$m = 3$	$m = 4$	$m = 5$
$X_1 = 9$	$X_1 = 27$	$X_1 = 81$
$X_2 = 12$	$X_2 = 36$	$X_2 = 108$
$X_3 = 20$	$X_3 = 60$	$X_3 = 180$
$Z = 25$	$X_4 = 100$	$X_4 = 300$
	$Z = 125$	$X_5 = 500$
		$Z = 625$

$m = 9$	$m = 10$	$m = 11$
$X_1 = 6561$	$X_1 = 19683$	$X_1 = 59049$
$X_2 = 8748$	$X_2 = 26244$	$X_2 = 78732$
$X_3 = 14580$	$X_3 = 43740$	$X_3 = 131220$
$X_4 = 24300$	$X_4 = 72900$	$X_4 = 218700$
$X_5 = 40500$	$X_5 = 121500$	$X_5 = 364500$
$X_6 = 67500$	$X_6 = 202500$	$X_6 = 607500$
$X_7 = 112500$	$X_7 = 337500$	$X_7 = 1012500$
$X_8 = 187500$	$X_8 = 562500$	$X_8 = 1687500$
$X_9 = 312500$	$X_9 = 937500$	$X_9 = 2812500$
$Z = 390625$	$X_{10} = 1562500$	$X_{10} = 4687500$
	$Z = 1953125$	$X_{11} = 7812500$
		$Z = 9765625$
...	....	....
...	....	....

Computer experiments under $n = 3, 4, ..., s = 1$. **1.** $n = 3$. Under all experiments in this case we shall consider that the basic solution is equals to $x = 3, z = 9$. Then we have:

Under $m = 2$; $x_i : 3$; 8.887488; $z = 9$; $\sum x_i^3 = 729.0001$; $z^3 = 729$
Under $m = 5$; $x_i : 81$; 239.9622; 719.8866; 2159.66; 6478.979; $z = 6561$
$\sum x_i^3 = 2.824298.10^{11}$, $z^3 = 2.824298.10^{11}$
Under $m = 9$; $x_i : 6561$; 19436.94; 58310.81; 174932. 4; 524797.3; 1574392 4723176; $1,416953.10^7$; $4,250858.10^7$; $z = 4,304672.10^7$
$\sum x_i^3 = 7,976644.10^{22}$; $z^3 = 7,976644.10^{22}$.
Under $m = 2; z = 100; \alpha = 0.1$; $x_1 = 31.62278; x_2 = 98.93459; z = 100$
$\sum x_i^3 = 999999.9; z^3 = 1000000$
2. $n = 5$ The basic solution: $x = 3, z = 9$
Under $m = 2$; x_i: 3, 8,9928; $z = 9$;
$\sum x_i^5 = 59049; z^5 = 59049$

Under $m = 3$; x_i: 9; 26.97774; 80.93323; $z = 81$
$\sum x_i^5 = 3,486785.10^9; z^5 = 3,486785.10^9$
Under $m = 5$; x_i: 81; 242,7997; 728,399; 2185,197; 6555,591; $z = 6561$
$\sum x_i^5 = 1,215767.10^{19}; z^5 = 1,215767.10^{19}$
Under $m = 2; z = 100; \alpha = 0.1$; $x_1 = 56.23413; x_2 = 98.84913; z = 100$
$\sum x_i^5 = 9,999999.10^9; z^5 = 10^{10}$
3. $n = 10$ The basic solution: $x = 5, z = 8$
Under $m = 2$; $x_i : 5$, 7,992694; $z = 8$;
$\sum x_i^{10} = 1,073742.10^9; \quad z^{10} = 1,073742.10^9$
Under $m = 3$; $x_i : 25$; 39,96347; 63,94155; $z = 64$
$\sum x_i^{10} = 1,152921.10^{18}; z^{10} = 1,152922.10^{18}$
Under $m = 5$; $x_i : 625$; 999,0868; 1598,539; 2557,662; 4092,259; $z = 4096$
$\sum x_i^{10} = 1,329228.10^{36}; z^{10} = 1,329228.10^{36}$
Under $m = 2; z = 100; \alpha = 0.1$; $x_1 = 77.42637; x_2 = 99.139736$
$\sum x_i^{10} = 9,999999.10^{19}; z^{10} = 10^{20}$
Under $n = 15; m = 2; z = 100; \alpha = 0.1; x_1 = 84.83429; x_2 = 99.41074$
$\sum x_i^{15} = 10^{30}; z^{15} = 10^{30}$

The dependence of the solutions from n. Now we shall consider the case when parameters m, α, $z = k^{1/n}$ are constants and we define the dependence of the solutions (1) from n.

1. Let $m = 2$, $\alpha = .5$, $k^{1/n} = 1$, then we have:
$n = 3$: $x_1 = .7071067, x_2 = .884657$;
$n = 100$: $x_1 = .993027, x_2 = .9931616$;
$n = 1000$: $x_1 = .993064, x_2 = .993078$;
$n = 2000$: $x_1 = .9996533, x_2 = .9996536$.
2. Let $m = 2$, $\alpha = .1$, $k^{1/n} = 1$, then we have:
$n = 3$: $x_1 = .3162278, x_2 = .989346$;
$n = 1000$: $x_1 = .9976978, x_2 = .9998949$;
$n = 10000$: $x_1 = .9997697, x_2 = .9999894$;
$n = 100000$: $x_1 = .999977, x_2 = .9999989$;
$n = 1000000$: $x_1 = .9999977, x_2 = .9999999$;
$n = 10000000$: $x_1 = .9999998, x_2 = 1$.
3. $m = 2$, $\alpha = .5$, $k^{1/n} = 3$, then:
$n = 3$: $x_1 = 2.12132, 2.593973$;
$n = 20$: $x_1 = 2.892528, x_2 = 2.902913$;
$n = 40$: $x_1 = 2.947152, x_2 = 2.949749$;
$n = 60$: $x_1 = 2.964962, x_2 = 2.966116$;
$n = 80$: $x_1 = 2.973793, x_2 = 2.974443$.

From these experiments it follows that under increase value of n the solutions of the equation (1) tend to value of z. When $\alpha = .5$ all x_i tend to z simultaneously and when $\alpha < .5$ then x_2 rapidly tend to z than x_1.

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