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INVARIANTS FOR A SYSTEM OF TWO LINEAR HYPERBOLIC EQUATIONS BY COMPLEX METHODS

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Invariants of symmetry groups under transformations of dependent and independent variables lead to simplification of differential equations and their exact solutions if solutions of the transformed equations are known. Though Lie developed his Symmetry Analysis for complex functions of complex variables, he did not explicitly use complex analyticity. We have applied Complex Symmetry Analysis in which we make explicit use of the Cauchy-Riemann equations and find that one can solve systems of differential equations by it for equations not readily amenable to the usual real methods. We show that, via complex methods, one can deduce invariants in a simple manner.

Keywords: invariants, Lie symmetries, hyperbolic equations, complex methods.

I. INTRODUCTION

Symmetry analysis was developed by Lie (Lie, 1883; Lie, 1884; Lie, 1888) in an initial attempt to extend the works of Abel and Galois (see for instance (Schwarz, 2008)) for polynomial equations to differential equations (DEs). Galois managed to classify polynomial equations solvable by means of radicals by utilising the symmetry of the roots of the algebraic equations. Lie did not obtain a corresponding analogue for DEs. Instead of finite groups, Lie found continuous groups which are also differentiable. These are called Lie Groups (Schwarz, 2008; Ibragimov, 1999; Ovsiannikov, 1982) and have been widely used since Lie's initial works for DEs which can be ordinary or partial.

Here we study the invariants for a system of linear hyperbolic PDEs from the complex viewpoint. We find expressions involving the dependent and independent variables (and their derivatives up to some order) that remain invariant under the group considered and its prolongations. Such quantities are called differential invariants. If they are obtained only by a subgroup of transformations, one refers to them as semi-invariants and joint invariants otherwise.

Invertible maps of the dependent and independent variables of the DEs which preserve their form are known as equivalence transformations. They enable the derivation of the invariants and reductions of the corresponding DEs to simpler forms. The semi- and joint

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invariants of a system of two linear hyperbolic equations

$$
\begin{aligned}
u_{tx} &+ a_1(t,x)u_t + a_2(t,x)v_t + b_1(t,x)u_x \\
&+ b_2(t,x)v_x + c_1(t,x)u + c_2(t,x)v = 0, \\
v_{tx} &+ a_3(t,x)u_t + a_4(t,x)v_t + b_3(t,x)u_x \\
&+ b_4(t,x)v_x + c_3(t,x)u + c_4(t,x)v = 0, \quad (1)
\end{aligned}
$$

under an invertible change of the dependent variables were derived (Tsaousi and Sophocleous, 2010) by the usual (real) infinitesimal method, where the subscripts t and x denote partial derivatives with respect to these independent variables. The most general group of equivalence transformations (Ibragimov, 1999; Ovsiannikov, 1982), i.e. invertible change of the dependent and independent variables, that maps a system of hyperbolic equations to itself with, in general, different coefficients is obtained first and then employed to find the associated invariants in (Tsaousi and Sophocleous, 2010).

Complex symmetry analysis (CSA) explicitly uses the complex analyticity of the dependent variables as a function of the real independent variables (Safdar *et al.*, 2011; Ali *et al.*, 2014). It was found that one could linearize systems of ODEs (Safdar *et al.*, 2011), and thereby solve them, even though they may not have enough symmetries to be solvable using the standard (real) methods (Ali *et al.*, 2014). Having this in mind, we wanted to see if the same type of benefits could be obtained for PDEs. Of course, a major difference between ODEs and PDEs is that the former always have a finite number of arbitrary constants, while the latter generally have arbitrary functions, and thus require boundary conditions or initial conditions to limit the number of solutions. As such, one tries to construct invariants (or joint invariants) to obtain classes of solutions. We construct invariants by both the standard (real) methods and the complex methods. We then make a comparison.

Semi-invariants associated with a subclass of a system of hyperbolic equations (1) under a change of only the dependent variables had been obtained by CSA (Mahomed *et al.*, 2011). This subclass of systems is represented by the following two hyperbolic equations

$$
\begin{aligned}
u_{tx} &+ \alpha_1(t,x)u_t - \alpha_2(t,x)v_t + \beta_1(t,x)u_x \\
&- \beta_2(t,x)v_x + \gamma_1(t,x)u - \gamma_2(t,x)v = 0, \\
v_{tx} &+ \alpha_2(t,x)u_t + \alpha_1(t,x)v_t + \beta_2(t,x)u_x \\
&+ \beta_1(t,x)v_x + \gamma_2(t,x)u + \gamma_1(t,x)v = 0. \quad (2)
\end{aligned}
$$

This system of hyperbolic equations corresponds to the scalar complex hyperbolic equation

$$
w_{tx} + \alpha(t,x)w_t + \beta(t,x)w_x + \gamma(t,x)w = 0, \qquad (3)
$$

if $\alpha_1 + \iota\alpha_2 = \alpha$, $\beta_1 + \iota\beta_2 = \beta$, $\gamma_1 + \iota\gamma_2 = \gamma$, and $u + \iota v = w$. Since the transformations used to obtain the semi-invariants of (2) satisfy the Cauchy-Riemann (CR) equations, so do the invariants. These quantities were found to be real parts of the complex invariants associated with the base complex equation by means of a change of the complex dependent variable (Ibragimov, 2004). All the invariants of the scalar linear hyperbolic equation are given in (Ibragimov, 2004; Johnpillai *et al.*, 2002). We present an explicit derivation of invariants of the system (2), using real and complex symmetry approaches to show that these are the same when derived under transformations of only dependent variables, while in case of transformations of only the independent variables and both the dependent and independent variables, they appear different. For the latter cases it is shown that complex symmetry approach provides an alternate invariance criterion for systems of two hyperbolic PDEs.

The plan of the paper is as follows. The second section is on the preliminaries in which the infinitesimal method is demonstrated. The subsequent section contains a derivation of invariants of a system of hyperbolic equations by real symmetry analysis. The fourth section is on obtaining the invariants for the same class of systems by the complex procedure and the comparison of these with the invariants found in the third section. The penultimate section is on the application of the derived invariants. Concluding remarks are presented in the final section.

II. PRELIMINARIES

Semi-invariants of the linear hyperbolic equation (3) (with t, x replaced by z_1, z_2), under the local re-scaling transformation of (only) the dependent variables

$$
w(z_1,z_2) = \sigma(z_1,z_2)u(z_1,z_2), \qquad (4)
$$

are given in (Ibragimov, 2004). The infinitesimal form of the above transformation reads as

$$
w(z_1,z_2) = [1 + \epsilon\eta(z_1,z_2)]u(z_1,z_2), \qquad (5)
$$

which leads to the generator

$$
\mathbf{Z} = \eta_{z_2}\partial_\alpha + \eta_{z_1}\partial_\beta + (\eta_{z_1z_2} + \alpha\eta_{z_1} + \beta\eta_{z_2})\partial_\gamma, \qquad (6)
$$

where η_{z_1}, η_{z_2} denotes partial derivatives of η, i.e. $\frac{\partial\eta}{\partial z_1}$, $\frac{\partial\eta}{\partial z_2}$ and $\partial_\alpha = \frac{\partial}{\partial\alpha}$. The following first-order semi-invariants (called Laplace invariants) are deduced in (Ibragimov, 2004)

$$
h = \alpha_{z_1} + \alpha\beta - \gamma, \quad k = \beta_{z_2} + \alpha\beta - \gamma. \qquad (7)
$$

The first extension of (6) acts on $J(\alpha,\beta,\gamma,\alpha_\rho,\beta_\rho,\gamma_\rho)$, where $\rho \in \{z_1,z_2\}$, to reveal (7) that these are differential invariants as they contain first-order derivatives of the coefficients of scalar PDE (3). Similarly, a change of the independent variables

$$
z_1 = \phi(t), \quad z_2 = \psi(x), \qquad (8)
$$

which can be written in the infinitesimal form as

$$z_1 = t + \epsilon\xi_1(t), \quad z_2 = x + \epsilon\xi_2(x), \tag{9}$$

leaves the original hyperbolic PDE invariant. This change of (only) the independent variables leads to the generator

$$\mathbf{Z} = \xi_1\partial_t + \xi_2\partial_x - \alpha\xi_{2,x}\partial_\alpha - \beta\xi_{1,t}\partial_\beta - \gamma(\xi_{1,t} + \xi_{2,x})\partial_\gamma, \tag{10}$$

i.e., the operator obtained after transforming the linear hyperbolic equation according to (9) and by reading its infinitesimal coordinates from the transformed new coefficients. Applying this generator on $J(\alpha, \beta, \gamma)$ yields a zeroth-order semi-invariant

$$I_1 = \frac{\gamma}{\alpha\beta}. \tag{11}$$

We apply the first extension of (10) on $J(\alpha, \beta, \gamma, \alpha_\varrho, \beta_\varrho, \gamma_\varrho)$, to obtain the first-order semi-invariants, where now $\varrho \in \{t, x\}$. This yields a system of linear PDEs that provides the following invariant quantities when solved, viz.

$$I_2 = \frac{\alpha\beta}{\alpha_t}, \quad I_3 = \frac{\beta_x}{\alpha_t}, \quad I_4 = \frac{\gamma}{\alpha_t}, \quad I_5 = \frac{\alpha(\beta\gamma_t - \gamma\beta_t)}{\beta\alpha_t^2}, \quad I_6 = \frac{\alpha\gamma_x - \gamma\alpha_x}{\alpha^2\alpha_t}. \tag{12}$$

Further, joint invariants of the hyperbolic equation have been derived (Johnpillai *et al.*, 2002) by applying an operator of the form (10) on the space of the Laplace semi-invariants h and k given in (7). Therefore, one needs to transform the operator (10) to the variables h, k, and then apply it on $J(h, k), J(h, k, h_\varrho, k_\varrho)$ and so on, in order to get the zeroth-, first- and higher-order *joint invariants* of the linear hyperbolic equation, respectively. On writing the generator (10) in the space of the Laplace invariants h, k, i.e.

$$\mathbf{Z} = \mathbf{Z}(h)\partial_h + \mathbf{Z}(k)\partial_k, \tag{13}$$

its first extension reads as

$$\begin{aligned}\mathbf{Z}^{[1]} &= \xi_1(t)\partial_t + \xi_2(x)\partial_x - (\xi_{1,t} + \xi_{2,x})h\partial_h - (\xi_{1,t} + \xi_{2,x})k\partial_k - (\xi_{1,tt}h + 2\xi_{1,t}h_t + \xi_{2,x}h_t)\partial_{h_t} \\ &- (\xi_{1,t}h_x + \xi_{2,xx}h + 2\xi_{2,x}h_x)\partial_{h_x} - (\xi_{1,tt}k + 2\xi_{1,t}k_t + \xi_{2,x}k_t)\partial_{k_t} - (\xi_{1,t}k_x + \xi_{2,xx}k + 2\xi_{2,x}k_x)\partial_{k_x}.\end{aligned} \tag{14}$$

It yields the following joint invariants of the original scalar linear hyperbolic equation

$$\begin{aligned} J_1 &= \frac{k}{h}, \\ J_2 &= \frac{(hk_t - kh_t)(hk_x - kh_x)}{h^5}, \\ J_3 &= \frac{kh_{tx} + hk_{tx} - h_tk_x - h_xk_t}{h^3}, \\ J_4 &= \frac{(hk_x - kh_x)^2(hkh_{tt} - h^2k_{tt} - 3kh_t^2 + 3hh_tk_t)}{h^9}, \\ J_5 &= \frac{(hk_t - kh_t)^2(hkh_{xx} - h^2k_{xx} - 3kh_x^2 + 3hh_xk_x)}{h^9}. \end{aligned} \tag{15}$$

of which four form a basis (see (Ibragimov, 2004; Johnpillai *et al.*, 2002)).

III. INVARIANTS OF A SYSTEM OF TWO HYPERBOLIC EQUATIONS BY THE REAL PROCEDURE

To derive the invariants of the system (1), the infinitesimal equivalence transformation map is determined in (Tsaousi and Sophocleous, 2010). The generators associated with these infinitesimal transformations are then applied to obtain invariants of (1). This section presents the derivation of the invariants associated with the subclass (2) of the system of hyperbolic equations (1) by the real infinitesimal method.

The system of two hyperbolic PDEs (2) is obtainable from a hyperbolic PDE with two independent variables (3), when its dependent variable is considered as a complex function of two real independent variables. Both the equations of such systems satisfy the CR equations. The group of equivalence transformations associated with (2) is obtained through a generator

$$\begin{aligned}\mathbf{Z} &= \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_u + \eta^2\partial_v + \eta^1_t\partial_{u_t} + \eta^1_x\partial_{u_x} \\ &+ \eta^2_t\partial_{v_t} + \eta^2_x\partial_{v_x} + \eta^1_{tx}\partial_{u_{tx}} + \eta^2_{tx}\partial_{v_{tx}} + \mu^{11}\partial_{\alpha_1} \\ &+ \mu^{12}\partial_{\alpha_2} + \mu^{21}\partial_{\beta_1} + \mu^{22}\partial_{\beta_2} + \mu^{31}\partial_{\gamma_1} + \mu^{32}\partial_{\gamma_2},\end{aligned} \tag{16}$$

where ξ^κ, η^κ and $\mu^{1\kappa}, \mu^{2\kappa}$, $\mu^{3\kappa}$, for $\kappa = 1, 2$, are functions of (t, x, u, v) and $(t, x, u, v, \alpha_\kappa, \beta_\kappa, \gamma_\kappa)$, respectively. The first extension coefficients η^κ_t, η^κ_x, are obtainable from

$$\begin{aligned}\eta^1_t &= D_t(\eta^1) - u_tD_t(\xi^1) - u_xD_t(\xi^2), \\ \eta^2_t &= D_t(\eta^2) - v_tD_t(\xi^1) - v_xD_t(\xi^2), \\ \eta^1_x &= D_x(\eta^1) - u_tD_x(\xi^1) - u_xD_x(\xi^2), \\ \eta^2_x &= D_x(\eta^2) - v_tD_x(\xi^1) - v_xD_x(\xi^2),\end{aligned} \tag{17}$$

with D_t, and D_x, as total derivatives with respect to t and x. Applying $\mathbf{Z}$, in (16) on both the equations of the system (2) results in a system of linear PDEs when the coefficients of ξ^κ, η^κ, and all their partial derivatives are equated to zero. Solving it with MAPLE leads to

$$\xi_1 = F_1(t), \quad \xi_2 = F_2(x), \tag{18}$$

$$\begin{aligned}\eta^1 &= F_3(t, x)u + F_4(t, x)v, \\ \eta^2 &= F_3(t, x)v - F_4(t, x)u,\end{aligned} \tag{19}$$

$$\begin{aligned}\mu^{11} &= -F_{3,x} - \alpha_1F_{2,x}, \\ \mu^{12} &= F_{4,x} - \alpha_2F_{2,x}, \\ \mu^{21} &= -F_{3,t} - \beta_1F_{1,t}, \\ \mu^{22} &= F_{4,t} - \beta_2F_{1,t}, \\ \mu^{31} &= -F_{3,tx} - \alpha_1F_{3,t} - \alpha_2F_{4,t} - \beta_1F_{3,x} - \beta_2F_{4,x} - \gamma_1(F_{1,t} + F_{2,x}), \\ \mu^{32} &= F_{4,tx} + \alpha_1F_{4,t} - \alpha_2F_{3,t} + \beta_1F_{4,x} - \beta_2F_{3,x} - \gamma_2(F_{1,t} + F_{2,x}).\end{aligned} \tag{20}$$

Invariants of the system (2) can be derived using a generator of the form

$$\mathbf{Z} = \xi^1\partial_t + \xi^2\partial_x + \mu^{11}\partial_{\alpha_1} + \mu^{12}\partial_{\alpha_2} + \mu^{21}\partial_{\beta_1} + \mu^{22}\partial_{\beta_2} + \mu^{31}\partial_{\gamma_1} + \mu^{32}\partial_{\gamma_2}, \tag{21}$$

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where ξ^κ, and $\mu^{1,\kappa}$, $\mu^{2,\kappa}$, $\mu^{3,\kappa}$, are as given in (18) and (20) respectively. The first-order semi-invariants (as zeroth-order do not exist) obtained are

$$\begin{aligned}
\overset{r}{h}_1 &= \alpha_{1,t} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1,\\
\overset{r}{h}_2 &= \alpha_{2,t} + \alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_2,\\
\overset{r}{k}_1 &= \beta_{1,x} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1,\\
\overset{r}{k}_2 &= \beta_{2,x} + \alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_2.
\end{aligned} \tag{22}$$

These are associated with the system (2) due to a change of (only) the dependent variables (19) and are derived by employing the generator of the form

$$\begin{aligned}
\mathbf{Z}_D = &-F_{3,x}\partial_{\alpha_1} + F_{4,x}\partial_{\alpha_2} - F_{3,t}\partial_{\beta_1} + F_{4,t}\partial_{\beta_2}\\
&- (F_{3,tx} + \alpha_1 F_{3,t} + \alpha_2 F_{4,t} + \beta_1 F_{3,x} + \beta_2 F_{4,x})\partial_{\gamma_1}\\
&+ (F_{4,tx} + \alpha_1 F_{4,t} - \alpha_2 F_{3,t} + \beta_1 F_{4,x} - \beta_2 F_{3,x})\partial_{\gamma_2}.
\end{aligned} \tag{23}$$

It is extracted from (21) and by considering $F_1(t) = F_2(x) = 0$, and $F_3(t,x)$, $F_4(t,x)$, as arbitrary functions of their arguments. By plugging the above generator in the invariance criterion

$$\mathbf{Z}_D^{[1]} J(\alpha_\kappa, \beta_\kappa, \gamma_\kappa, \alpha_{\kappa,t}, \beta_{\kappa,t}, \gamma_{\kappa,t}, \alpha_{\kappa,x}, \beta_{\kappa,x}, \gamma_{\kappa,x}) = 0, \tag{24}$$

leads to a system of linear PDEs when the coefficients of $F_3(t,x)$, $F_3(t,x)$, and all their partial derivatives are equated to zero. On solving the obtained system of PDEs, one arrives at (22). Transforming only the independent variables, i.e. keeping $F_1(t)$, $F_2(x)$, as arbitrary functions of their arguments and $F_3(t,x) = F_4(t,x) = 0$, leads to the infinitesimal generator

$$\begin{aligned}
\mathbf{Z}_I = &F_1(t)\partial_t + F_2(x)\partial_x - \alpha_1 F_{2,x}\partial_{\alpha_1} - \alpha_2 F_{2,x}\partial_{\alpha_2}\\
&- \beta_1 F_{1,t}\partial_{\beta_1} - \beta_2 F_{1,t}\partial_{\beta_2} - \gamma_1(F_{1,t} + F_{2,x})\partial_{\gamma_1}\\
&- \gamma_2(F_{1,t} + F_{2,x})\partial_{\gamma_2}.
\end{aligned} \tag{25}$$

Applying it on $J(\alpha_\kappa, \beta_\kappa, \gamma_\kappa)$ results in the following zeroth-order invariants

$$\overset{r}{I}_1 = \frac{\alpha_2}{\alpha_1},\quad \overset{r}{I}_2 = \frac{\beta_2}{\beta_1},\quad \overset{r}{I}_3 = \frac{\gamma_1}{\alpha_1\beta_1},\quad \overset{r}{I}_4 = \frac{\gamma_2}{\alpha_1\beta_1}. \tag{26}$$

Further, the first-order invariants are arrived at when the first extension of the generator (25) acts on $J(\alpha_\kappa, \beta_\kappa, \gamma_\kappa, \alpha_{\kappa,t}, \beta_{\kappa,t}, \gamma_{\kappa,t}, \alpha_{\kappa,x}, \beta_{\kappa,x}, \gamma_{\kappa,x})$, which gives the following quantities

$$\begin{aligned}
&\overset{r}{I}_5 = \frac{\alpha_{1,t}}{\alpha_1\beta_1},\quad \overset{r}{I}_6 = \frac{\alpha_{2,t}}{\alpha_1\beta_1},\quad \overset{r}{I}_7 = \frac{\beta_{1,x}}{\alpha_1\beta_1},\quad \overset{r}{I}_8 = \frac{\beta_{2,x}}{\alpha_1\beta_1},\\
&\overset{r}{I}_9 = \frac{\beta_1\beta_{2,t} - \beta_2\beta_{1,t}}{\beta_1^3},\quad \overset{r}{I}_{10} = \frac{\beta_1\gamma_{1,t} - \gamma_1\beta_{1,t}}{\alpha_1\beta_1^3},\\
&\overset{r}{I}_{11} = \frac{\beta_1\gamma_{2,t} - \gamma_2\beta_{1,t}}{\alpha_1\beta_1^3},\quad \overset{r}{I}_{12} = \frac{\alpha_1\alpha_{2,x} - \alpha_2\alpha_{1,x}}{\alpha_1^3},\\
&\overset{r}{I}_{13} = \frac{\alpha_1\gamma_{1,x} - \gamma_1\alpha_{1,x}}{\alpha_1^3\beta_1},\quad \overset{r}{I}_{14} = \frac{\alpha_1\gamma_{2,x} - \gamma_2\alpha_{1,x}}{\alpha_1^3\beta_1},
\end{aligned} \tag{27}$$

including the four zeroth-order invariants (26). The joint invariants of the system (2)

$$\overset{r}{J}_1 = \frac{\overset{r}{h}_2}{\overset{r}{h}_1},\quad \overset{r}{J}_2 = \frac{\overset{r}{k}_1}{\overset{r}{h}_1},\quad \overset{r}{J}_3 = \frac{\overset{r}{k}_2}{\overset{r}{h}_1}, \tag{28}$$

are found in (Tsaousi and Sophocleous, 2010) by solving the PDE

$$\overset{r}{h}_1\partial_{\overset{r}{h}_1} + \overset{r}{h}_2\partial_{\overset{r}{h}_2} + \overset{r}{k}_1\partial_{\overset{r}{k}_1} + \overset{r}{k}_2\partial_{\overset{r}{k}_2} = 0. \tag{29}$$

This equation arises by transforming the first extension of the generator (25) to the space of invariants, $\overset{r}{h}_\kappa$, $\overset{r}{k}_\kappa$ and applying it on $J(\overset{r}{h}_\kappa, \overset{r}{k}_\kappa)$.

IV. INVARIANTS OF A SYSTEM OF TWO HYPERBOLIC EQUATIONS

Semi-invariants associated with the system of two hyperbolic equations (2) obtained from a scalar linear hyperbolic equation (3) are derived in this section by complex methods. A few of the invariants presented here have already been presented earlier (Mahomed *et al.*, 2011). Here we demonstrate the complete complex procedure involved in deriving them. The generator of the form (6) associated with the equation (3), written in terms of z_1 and z_2, becomes complex due to the presence of the complex dependent variable and the complex coefficients split (6) into two operators

$$\begin{aligned}
\mathbf{X}_1 = &\eta_{1,z_2}\partial_{\alpha_1} + \eta_{2,z_2}\partial_{\alpha_2} + \eta_{1,z_1}\partial_{\beta_1} + \eta_{2,z_1}\partial_{\beta_2}\\
&+ (\eta_{1,z_1z_2} + \alpha_1\eta_{1,z_1} - \alpha_2\eta_{2,z_1} + \beta_1\eta_{1,z_2}\\
&- \beta_2\eta_{2,z_2})\partial_{\gamma_1} + (\eta_{2,z_1z_2} + \alpha_2\eta_{1,z_1} + \alpha_1\eta_{2,z_1}\\
&+ \beta_2\eta_{1,z_2} + \beta_1\eta_{2,z_2})\partial_{\gamma_2},
\end{aligned} \tag{30}$$

$$\begin{aligned}
\mathbf{X}_2 = &\eta_{2,z_2}\partial_{\alpha_1} - \eta_{1,z_2}\partial_{\alpha_2} + \eta_{2,z_1}\partial_{\beta_1} - \eta_{1,z_1}\partial_{\beta_2}\\
&+ (\eta_{2,z_1z_2} + \alpha_2\eta_{1,z_1} + \alpha_1\eta_{2,z_1} + \beta_2\eta_{1,z_2}\\
&+ \beta_1\eta_{2,z_2})\partial_{\gamma_1} - (\eta_{1,z_1z_2} + \alpha_1\eta_{1,z_1} - \alpha_2\eta_{2,z_1}\\
&+ \beta_1\eta_{1,z_2} - \beta_2\eta_{2,z_2})\partial_{\gamma_2}.
\end{aligned} \tag{31}$$

We find four first-order semi-invariants

$$\begin{aligned}
h_1 &= \alpha_{1,z_1} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1,\\
h_2 &= \alpha_{2,z_1} + \alpha_2\beta_1 + \alpha_1\beta_2 - \gamma_2,\\
k_1 &= \beta_{1,z_2} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1,\\
k_2 &= \beta_{2,z_2} + \alpha_2\beta_1 + \alpha_1\beta_2 - \gamma_2,
\end{aligned} \tag{32}$$

associated with the system (2) on using the pair of operators (30) and (31). These are exactly the same as represented by $\overset{r}{h}_\kappa$, $\overset{r}{k}_\kappa$ in (22). Therefore, in this case the real and complex procedures lead to the same semi-invariants of the system (2). Notice that all the four semi-invariants (32) are readable from the the first-order semi-invariants associated with the base complex hyperbolic linear equation (3) and satisfy

$$\begin{aligned}
\mathbf{X}_1^{[1]} h_1 \big|_{h_1=0} &= \mathbf{X}_2^{[1]} h_2 \big|_{h_2=0} = 0,\\
\mathbf{X}_1^{[1]} k_1 \big|_{k_1=0} &= \mathbf{X}_2^{[1]} k_2 \big|_{k_2=0} = 0.
\end{aligned} \tag{33}$$

The linear combination $\mathbf{X}_3$ of both the operators $\mathbf{X}_1$ and $\mathbf{X}_2$ results in the following relations

$$\begin{aligned}
\mathbf{X}_3^{[1]} h_1 \big|_{h_1=0} &= \mathbf{X}_3^{[1]} h_2 \big|_{h_2=0} = 0\\
\mathbf{X}_3^{[1]} k_1 \big|_{k_1=0} &= \mathbf{X}_3^{[1]} k_2 \big|_{k_2=0} = 0.
\end{aligned} \tag{34}$$

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It was seen that the complex invariants of the base complex hyperbolic equation (3) split into two real invariants of the system (2). Further, we observed an agreement between the real and complex approaches in the case of invariants that are derived under transformations of only the dependent variables for systems (2). In order to show that this will always be the case let us start with the following invariance criterion

$$\mathbf{Z}J(\alpha, \beta, \gamma, \alpha_\rho, \beta_\rho, \gamma_\rho) = 0, \quad \rho \in \{t, x\},$$

where $\mathbf{Z}$, in general, reads as

$$\mathbf{Z} = \mu_1 \partial\alpha + \mu_2 \partial\beta + \mu_3 \partial\gamma,$$

and μ's are functions of t, x, w, α, β, and γ, that is used to derive the zeroth-order invariants of the complex base equation (3). Here considering $\mathbf{Z} = \mathbf{X}_1 + \iota\mathbf{X}_2$, $\mu_1 = \mu_{11} + \iota\mu_{12}$, $\mu_2 = \mu_{21} + \iota\mu_{22}$, $\mu_3 = \mu_{31} + \iota\mu_{32}$ and $J = J_1 + \iota J_2$ as well as splitting the above invariance criterion we obtain

$$\mathbf{X}_1 J_1 - \mathbf{X}_2 J_2 = 0, \quad \mathbf{X}_2 J_1 + \mathbf{X}_1 J_2 = 0,$$

which expands to

$$\begin{aligned}
&\mu_{11}(J_{1,\alpha_1} + J_{2,\alpha_2}) + \mu_{12}(J_{1,\alpha_2} - J_{2,\alpha_1}) \\
&+\mu_{21}(J_{1,\beta_1} + J_{2,\beta_2}) + \mu_{22}(J_{1,\beta_2} - J_{2,\beta_1}) \\
&+\mu_{31}(J_{1,\gamma_1} + J_{2,\gamma_2}) + \mu_{32}(J_{1,\gamma_2} - J_{2,\gamma_1}) = 0, \\
&\mu_{11}(J_{2,\alpha_1} - J_{1,\alpha_2}) + \mu_{12}(J_{1,\alpha_1} + J_{2,\alpha_2}) \\
&+\mu_{21}(J_{2,\beta_1} - J_{1,\beta_2}) + \mu_{22}(J_{1,\beta_1} + J_{2,\beta_2}) \\
&+\mu_{31}(J_{2,\gamma_1} - J_{1,\gamma_2}) + \mu_{32}(J_{1,\gamma_1} + J_{2,\gamma_2}) = 0,
\end{aligned} \tag{35}$$

respectively. On using the CR-equations $J_{1,\alpha_1} = J_{2,\alpha_2}$, $J_{1,\alpha_2} = -J_{2,\alpha_1}$, $J_{1,\beta_1} = J_{2,\beta_2}$, $J_{1,\beta_2} = -J_{2,\beta_1}$, and $J_{1,\gamma_1} = J_{2,\gamma_2}$, $J_{1,\gamma_2} = -J_{2,\gamma_1}$, the above equations take the form

$$\begin{aligned}
&\mu_{11} J_{1,\alpha_1} + \mu_{12} J_{1,\alpha_2} + \mu_{21} J_{1,\beta_1} + \mu_{22} J_{1,\beta_2} \\
&+\mu_{31} J_{1,\gamma_1} + \mu_{32} J_{1,\gamma_2} = 0, \\
&\mu_{11} J_{2,\alpha_1} + \mu_{12} J_{2,\alpha_2} + \mu_{21} J_{2,\beta_1} + \mu_{22} J_{2,\beta_2} \\
&+\mu_{31} J_{2,\gamma_1} + \mu_{32} J_{2,\gamma_2} = 0.
\end{aligned} \tag{36}$$

Both these equations are invariance criteria for a system of two hyperbolic equations (2) to derive its zeroth-order invariants under transformation of only the dependent variables, i.e.,

$$\mathbf{X} J_1 = 0, \ \ \mathbf{X} J_2 = 0,$$

where

$$\begin{aligned}
\mathbf{X} = {} & \mu_{11}\partial_{\alpha_1} + \mu_{12}\partial_{\alpha_2} + \mu_{21}\partial_{\beta_1} + \mu_{22}\partial_{\beta_2} \\
&+\mu_{31}\partial_{\gamma_1} + \mu_{32}\partial_{\gamma_2}.
\end{aligned} \tag{37}$$

In the above discussion we used the invariance criterion that yields invariants of order zero associated with base scalar complex equation (3) due to an invertible transformation of the dependent variable. The two real parts of this complex invariance criterion are shown as equivalent with the help of the CR-equations, to the invariance criteria that lead to zeroth-order invariants of systems (2) under transformations of only the dependent variables. This result could be extended to prove equivalence of the invariance criteria (of equation (3) and corresponding systems of the form (2)) of an arbitrary order q, with the help of the CR-equations.

The semi-invariants of the system (2) under transformation of only the independent variables are

$$\begin{aligned}
I_1^c &= \frac{(\alpha_1\gamma_1+\alpha_2\gamma_2)\beta_1+(\alpha_1\gamma_2-\alpha_2\gamma_1)\beta_2}{(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2)}, \\
I_2^c &= \frac{(\alpha_1\gamma_2-\alpha_2\gamma_1)\beta_1-(\alpha_1\gamma_1+\alpha_2\gamma_2)\beta_2}{(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2)}, \\
I_3^c &= \frac{(\alpha_1\beta_1-\alpha_2\beta_2)\alpha_{1,t}+(\alpha_2\beta_1+\alpha_1\beta_2)\alpha_{2,t}}{\alpha_{1,t}^2+\alpha_{2,t}^2}, \\
I_4^c &= \frac{(\alpha_2\beta_1+\alpha_1\beta_2)\alpha_{1,t}-(\alpha_1\beta_1-\alpha_2\beta_2)\alpha_{2,t}}{\alpha_{1,t}^2+\alpha_{2,t}^2}, \\
I_5^c &= \frac{\alpha_{1,t}\beta_{1,x}+\alpha_{2,t}\beta_{2,x}}{\alpha_{1,t}^2+\alpha_{2,t}^2}, \quad I_6^c = \frac{\alpha_{1,t}\beta_{2,x}-\alpha_{2,t}\beta_{1,x}}{\alpha_{1,t}^2+\alpha_{2,t}^2}, \\
I_7^c &= \frac{\alpha_{1,t}\gamma_1+\alpha_{2,t}\gamma_2}{\alpha_{1,t}^2+\alpha_{2,t}^2}, \quad I_8^c = \frac{\alpha_{1,t}\gamma_2-\alpha_{2,t}\gamma_1}{\alpha_{1,t}^2+\alpha_{2,t}^2}, \\
I_9^c &= \frac{(\alpha_{1,t}^2-\alpha_{2,t}^2)(\alpha_1\beta_1\Omega_1-\alpha_2\beta_1\Omega_2+\alpha_2\beta_2\Omega_1+\alpha_1\beta_2\Omega_2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)} \\
&+\frac{2\alpha_{1,t}\alpha_{2,t}(\alpha_2\beta_1\Omega_2+\alpha_1\beta_1\Omega_2-\alpha_1\beta_2\Omega_1+\alpha_2\beta_2\Omega_2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)}, \\
I_{10}^c &= \frac{(\alpha_{1,t}^2-\alpha_{2,t}^2)(\alpha_2\beta_1\Omega_1+\alpha_1\beta_1\Omega_2-\alpha_1\beta_2\Omega_1+\alpha_2\beta_2\Omega_2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)} \\
&-\frac{2\alpha_{1,t}\alpha_{2,t}(\alpha_1\beta_1\Omega_1-\alpha_2\beta_1\Omega_2+\alpha_2\beta_2\Omega_1+\alpha_1\beta_2\Omega_2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)}, \\
I_{11}^c &= \frac{(\alpha_1^2-\alpha_2^2)(\Omega_3\alpha_{1,t}+\Omega_4\alpha_{2,t})}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2} + \frac{2\alpha_1\alpha_2(\Omega_4\alpha_{1,t}-\Omega_3\alpha_{2,t})}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2}, \\
I_{12}^c &= \frac{(\alpha_1^2-\alpha_2^2)(\Omega_4\alpha_{1,t}-\Omega_3\alpha_{2,t})}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2} - \frac{2\alpha_1\alpha_2(\Omega_3\alpha_{1,t}+\Omega_4\alpha_{2,t})}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2},
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
\Omega_1 &= \beta_1\gamma_{1,t} - \beta_2\gamma_{2,t} - \gamma_1\beta_{1,t} + \gamma_2\beta_{2,t}, \\
\Omega_2 &= \beta_2\gamma_{1,t} + \beta_1\gamma_{2,t} - \gamma_2\beta_{1,t} - \gamma_1\beta_{2,t}, \\
\Omega_3 &= \alpha_1\gamma_{1,x} - \alpha_2\gamma_{2,x} - \gamma_1\alpha_{1,x} + \gamma_2\alpha_{2,x}, \\
\Omega_4 &= \alpha_2\gamma_{1,x} + \alpha_1\gamma_{2,x} - \gamma_2\alpha_{1,x} - \gamma_1\alpha_{2,x}.
\end{aligned} \tag{39}$$

We notice that $I_1^c, I_2^c, \ldots, I_{12}^c$ constitute an alternate invariance criterion for system (2) that is different from $I_1^r, I_2^r, \ldots, I_{14}^r$ derived earlier using the real symmetry method. The correspondence of these semi-invariants of independent variables with the system (2) is established due to the following operators

$$\begin{aligned}
\mathbf{X}_1 = {} & 2\xi_1\partial_t + 2\xi_2\partial_x - \alpha_1\xi_{2,x}\partial_{\alpha_1} - \alpha_2\xi_{2,x}\partial_{\alpha_2} - \beta_1\xi_{1,t}\partial_{\beta_1} \\
& - \beta_2\xi_{1,t}\partial_{\beta_2} - \gamma_1(\xi_{1,t} + \xi_{2,x})\partial_{\gamma_1} - \gamma_2(\xi_{1,t} + \xi_{2,x})\partial_{\gamma_2},
\end{aligned} \tag{40}$$

$$\begin{aligned}
\mathbf{X}_2 = {} & -\alpha_2\xi_{2,x}\partial_{\alpha_1} + \alpha_1\xi_{2,x}\partial_{\alpha_2} - \beta_2\xi_{1,t}\partial_{\beta_1} + \beta_1\xi_{1,t}\partial_{\beta_2} \\
& - \gamma_2(\xi_{1,t} + \xi_{2,x})\partial_{\gamma_1} + \gamma_1(\xi_{1,t} + \xi_{2,x})\partial_{\gamma_2},
\end{aligned} \tag{41}$$

which are the real and imaginary parts of the complex generator (10). Using these operators it is observed that

$$\begin{aligned}
&\mathbf{X}_1^{[1]} I_1^c\,|_{I_1^c=0} = \mathbf{X}_2^{[1]} I_2^c\,|_{I_2^c=0} = \mathbf{X}_1^{[1]} I_3^c\,|_{I_3^c=0} = 0,\\
&\mathbf{X}_2^{[1]} I_4^c\,|_{I_3^c=I_4^c=0} = \mathbf{X}_1^{[1]} I_5^c\,|_{I_5^c=0} = \mathbf{X}_2^{[1]} I_6^c\,|_{I_5^c=I_6^c=0} = 0,\\
&\mathbf{X}_1^{[1]} I_7^c\,|_{I_7^c=0} = \mathbf{X}_2^{[1]} I_8^c\,|_{I_7^c=I_8^c=0} = \mathbf{X}_1^{[1]} I_9^c\,|_{I_9^c=0} = 0\\
&\mathbf{X}_2^{[1]} I_{10}^c\,|_{I_9^c=I_{10}^c=0} = \mathbf{X}_1^{[1]} I_{11}^c\,|_{I_{11}^c=0} = \mathbf{X}_2^{[1]} I_{12}^c\,|_{I_{11}^c=I_{12}^c=0} = 0.
\end{aligned} \tag{42}$$

It is seen that the above invariants are the real parts of the complex invariants (11) and (12). Similarly, the linear combination of both $\mathbf{X}_1$ and $\mathbf{X}_2$, if denoted by $\mathbf{X}_3$, satisfy the relations

$$\begin{aligned}
&\mathbf{X}_3^{[1]} I_1^c\,|_{I_1^c=0} = \mathbf{X}_3^{[1]} I_2^c\,|_{I_2^c=0} = \mathbf{X}_3^{[1]} I_3^c\,|_{I_3^c=0} = 0,\\
&\mathbf{X}_3^{[1]} I_4^c\,|_{I_3^c=I_4^c=0} = \mathbf{X}_3^{[1]} I_5^c\,|_{I_5^c=0} = \mathbf{X}_3^{[1]} I_6^c\,|_{I_5^c=I_6^c=0} = 0,\\
&\mathbf{X}_3^{[1]} I_7^c\,|_{I_7^c=I_8^c=0} = \mathbf{X}_3^{[1]} I_8^c\,|_{I_7^c=I_8^c=0} = \mathbf{X}_3^{[1]} I_9^c\,|_{I_9^c=I_{10}^c=0} = 0\\
&\mathbf{X}_3^{[1]} I_{10}^c\,|_{I_9^c=I_{10}^c=0} = \mathbf{X}_3^{[1]} I_{11}^c\,|_{I_{11}^c=I_{12}^c=0} = 0\\
&\mathbf{X}_3^{[1]} I_{12}^c\,|_{I_{11}^c=I_{12}^c=0} = 0.
\end{aligned} \tag{43}$$

These invariants also satisfy the invariance criterion

$$\mathbf{Z}_I^{[1]} I_\delta^c\,|_{I_\delta^c=0} = 0, \quad \delta = 1, 2, \ldots, 12, \tag{44}$$

where $\mathbf{Z}_I^{[1]}$, is the first extension of the generator (25). This implies that semi-invariants derived under transformations of the independent variables using the complex approach also satisfy the real invariance criteria. For derivation of joint invariants of the coupled system of two hyperbolic equations (2), the operators (40) and (41) need to be transformed to the space of invariants h_κ, k_κ. The same procedure was adopted in (Johnpillai *et al.*, 2002) before using the generator (10) in determining the joint invariants of the scalar linear hyperbolic equation. The complex generator was transformed to h and k, i.e., to the space of the semi-invariants associated with the hyperbolic equation under a change of the dependent variables. The procedure to transform (40) and (41) to (h_κ, k_κ)-space starts by splitting (13) when $\mathbf{Z}(h)$ and $\mathbf{Z}(k)$ are taken as complex, i.e. $\mathbf{Z}(h) = \mathbf{Z}(h)_1 + \iota\mathbf{Z}(h)_2$ and $\mathbf{Z}(k) = \mathbf{Z}(k)_1 + \iota\mathbf{Z}(k)_2$. The real and imaginary parts of (13) are

$$\begin{aligned}
\mathbf{X}_1 &= \tfrac{1}{2}[\mathbf{Z}(h)_1\partial_{h_1} + \mathbf{Z}(h)_2\partial_{h_2} + \mathbf{Z}(k)_1\partial_{k_1} + \mathbf{Z}(k)_2\partial_{k_2}],\\
\mathbf{X}_2 &= \tfrac{1}{2}[\mathbf{Z}(h)_2\partial_{h_1} - \mathbf{Z}(h)_1\partial_{h_2} + \mathbf{Z}(k)_2\partial_{k_1} - \mathbf{Z}(k)_1\partial_{k_2}],
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
\mathbf{Z}(h)_1 &= \mathbf{X}_1 h_1 - \mathbf{X}_2 h_2 = -(\xi_{1,t} + \xi_{2,x})h_1,\\
\mathbf{Z}(h)_2 &= \mathbf{X}_2 h_1 + \mathbf{X}_1 h_2 = -(\xi_{1,t} + \xi_{2,x})h_2,\\
\mathbf{Z}(k)_1 &= \mathbf{X}_1 k_1 - \mathbf{X}_2 k_2 = -(\xi_{1,t} + \xi_{2,x})k_1,\\
\mathbf{Z}(k)_2 &= \mathbf{X}_2 k_1 + \mathbf{X}_1 k_2 = -(\xi_{1,t} + \xi_{2,x})k_2.
\end{aligned} \tag{46}$$

Using (46) in (45) gives the following two operators

$$\begin{aligned}
\mathbf{X}_1 &= -\tfrac{(\xi_{1,t}+\xi_{2,x})}{2}[h_1\partial_{h_1} + h_2\partial_{h_2} + k_1\partial_{k_1} + k_2\partial_{k_2}],\\
\mathbf{X}_2 &= -\tfrac{(\xi_{1,t}+\xi_{2,x})}{2}[h_2\partial_{h_1} - h_1\partial_{h_2} + k_2\partial_{k_1} - k_1\partial_{k_2}],
\end{aligned} \tag{47}$$

that are the real and imaginary parts of the complex generator (14). These operators are utilized to deduce the following joint invariants for the system of two linear hyperbolic equations, viz. (2)

$$\begin{aligned}
J_{11} &= \tfrac{h_1k_1+h_2k_2}{k_1^2+k_2^2},\\
J_{12} &= \tfrac{h_2k_1-h_1k_2}{k_1^2+k_2^2},\\
J_{13} &= \tfrac{\mu_1\nu_1}{\mu_1^2+\mu_2^2}\omega_1 + \tfrac{\mu_2\nu_2}{\mu_1^2+\mu_2^2}\omega_1 + \tfrac{\mu_2\nu_1}{\mu_1^2+\mu_2^2}\omega_2 + \tfrac{\mu_1\nu_2}{\mu_1^2+\mu_2^2}\omega_2,\\
J_{14} &= \tfrac{-\mu_2\nu_1}{\mu_1^2+\mu_2^2}\omega_1 + \tfrac{\mu_1\nu_2}{\mu_1^2+\mu_2^2}\omega_1 + \tfrac{\mu_1\nu_1}{\mu_1^2+\mu_2^2}\omega_2 + \tfrac{\mu_2\nu_2}{\mu_1^2+\mu_2^2}\omega_2,\\
J_{15} &= \tfrac{\mu_3\nu_3}{\mu_3^2+\mu_4^2} + \tfrac{\mu_4\nu_4}{\mu_3^2+\mu_4^2},\\
J_{16} &= \tfrac{\mu_4\nu_3}{\mu_3^2+\mu_4^2} + \tfrac{\mu_3\nu_4}{\mu_3^2+\mu_4^2},\\
J_{17} &= \tfrac{k_1\mu_5+k_2\mu_6}{\mu_5^2+\mu_6^2}\omega_3 - \tfrac{k_2\mu_5-k_1\mu_6}{\mu_5^2+\mu_6^2}\omega_4,\\
J_{18} &= \tfrac{k_2\mu_5-k_1\mu_6}{\mu_5^2+\mu_6^2}\omega_3 + \tfrac{k_1\mu_5+k_2\mu_6}{\mu_5^2+\mu_6^2}\omega_4,\\
J_{19} &= \tfrac{\mu_7\nu_5+\mu_8\nu_6}{\mu_7^2+\mu_8^2}\omega_5 + \tfrac{\mu_8\nu_5-\mu_7\nu_6}{\mu_7^2+\mu_8^2}\omega_6,\\
J_{20} &= \tfrac{\mu_7\nu_6-\mu_8\nu_5}{\mu_7^2+\mu_8^2}\omega_5 + \tfrac{\mu_7\nu_5+\mu_8\nu_6}{\mu_7^2+\mu_8^2}\omega_6,
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
\mu_1 &= h_1^5 - 10h_1^3h_2^2 + 5h_1h_2^4,\\
\mu_2 &= 5h_1^4h_2 - 10h_1^2h_2^3 + h_2^5,\\
\mu_3 &= h_1^3 - 3h_1h_2^2,\\
\mu_4 &= 3h_1^2h_2 - h_2^3,\\
\mu_5 &= -6h_1^2h_2^2 + h_1^4 + h_2^4,\\
\mu_6 &= 4h_1^3h_2 - 4h_1h_2^3,\\
\mu_7 &= h_1^9 - 36h_1^7h_2^2 + 126h_1^5h_2^4 - 84h_1^3h_2^6 + 9h_1h_2^8,\\
\mu_8 &= 9h_1^8h_2 - 84h_1^6h_2^3 + 126h_1^4h_2^5 - 36h_1^2h_2^7 + h_2^9,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\nu_1 &= h_1k_{1,t} - h_2k_{2,t} - k_1h_{1,t} + k_2h_{2,t},\\
\nu_2 &= h_2k_{1,t} + h_1k_{2,t} - k_2h_{1,t} - k_1h_{2,t},\\
\nu_3 &= k_1h_{1,tx} - k_2h_{2,tx} + h_1k_{1,tx} - h_2k_{2,tx} - h_{1,t}k_{1,x}\\
&\quad + h_{2,t}k_{2,x} - h_{1,x}k_{1,t} + h_{2,x}k_{2,t},\\
\nu_4 &= k_2h_{1,tx} + k_1h_{2,tx} + h_2k_{1,tx} + h_1k_{2,tx} - h_{2,t}k_{1,x}\\
&\quad - h_{1,t}k_{2,x} - h_{2,x}k_{1,t} - h_{1,x}k_{2,t},\\
\nu_5 &= k_2^2h_{2,x}^2 + 2h_1k_{2,x}k_1h_{2,x} - 2h_2k_{2,x}k_2h_{2,x}\\
&\quad + 2h_2k_{1,x}k_1h_{2,x} - 4k_1h_{1,x}k_2h_{2,x} - k_1^2h_{2,x}^2\\
&\quad + h_2^2k_{2,x}^2 + 2h_2k_{1,x}k_2h_{1,x} - 2h_1k_{1,x}k_1h_{1,x}\\
&\quad + 2h_2k_{2,x}k_1h_{1,x} + k_1^2h_{1,x}^2 + h_1^2k_{1,x}^2\\
&\quad + 2h_1k_{2,x}k_2h_{1,x} - h_2^2k_{1,x}^2 - 4h_1k_{1,x}h_2k_{2,x}\\
&\quad - h_1^2k_{2,x}^2 - k_2^2h_{1,x}^2 + 2h_1k_{1,x}k_2h_{2,x},\\
\nu_6 &= -2k_2h_{2,x}^2k_1 - 2h_1k_{1,x}k_1h_{2,x} - 2k_1h_{1,x}h_1k_{2,x}\\
&\quad + 2h_2k_{2,x}k_2h_{1,x} - 2k_2^2h_{2,x}h_{1,x} + 2h_2k_{2,x}k_1h_{2,x}\\
&\quad + 2k_2h_{2,x}h_2k_{1,x} - 2h_2^2k_{2,x}k_{1,x} + 2k_2h_{2,x}h_1k_{2,x}\\
&\quad + 2h_1k_{1,x}^2h_2 - 2h_1k_{1,x}k_2h_{1,x} - 2h_2k_{2,x}^2h_1\\
&\quad + 2h_1^2k_{1,x}k_{2,x} + 2k_1h_{1,x}^2k_2 - 2k_1h_{1,x}h_2k_{1,x}\\
&\quad + 2k_1^2h_{1,x}h_{2,x},
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\omega_1 &= h_1k_{1,x} - h_2k_{2,x} - k_1h_{1,x} + k_2h_{2,x},\\
\omega_2 &= h_2k_{1,x} + h_1k_{2,x} - k_2h_{1,x} - k_1h_{2,x},\\
\omega_3 &= h_1h_{1,tx} - h_2h_{2,tx} - h_{1,t}h_{1,x} + h_{2,t}h_{2,x},\\
\omega_4 &= h_2h_{1,tx} + h_1h_{2,tx} - h_{2,t}h_{1,x} - h_{1,t}h_{2,x},\\
\omega_5 &= (h_1k_1 - h_2k_2)h_{1,tt} - (h_2k_1 + h_1k_2)h_{2,tt}\\
&\quad - (h_1^2 - h_2^2)k_{1,tt} + 2h_1h_2k_{2,tt} - 3k_1(h_{1,t}^2\\
&\quad - h_{2,t}^2) + 6k_2h_{1,t}h_{2,t} + (3h_1h_{1,t} - 3h_2h_{2,t})k_{1,t}\\
&\quad - (3h_2h_{1,t} + 3h_1h_{2,t})k_{2,t},\\
\omega_6 &= (h_2k_1 + h_1k_2)h_{1,tt} + (h_1k_1 - h_2k_2)h_{2,tt}\\
&\quad + (-h_1^2 + h_2^2)k_{2,tt} - 2h_1h_2k_{1,tt} - 3k_2(h_{1,t}^2\\
&\quad - h_{2,t}^2) - 6k_1h_{1,t}h_{2,t} + (3h_2h_{1,t} + 3h_1h_{2,t})k_{1,t}\\
&\quad + (3h_1h_{1,t} - 3h_2h_{2,t})k_{2,t}.
\end{aligned} \tag{51}$$

These invariants are also the real parts of the complex joint invariants (15). A comparison of (28) and (48) shows that these appear to be two different invariance criteria developed for system (2) using the real and complex symmetry methods. Detailed calculations of invariants presented in this section and the previous one are contained in (Aslam, 2014).

The invariants provided by the real approach under transformations of only the independent variables are found different from those obtained by the complex procedure. In order to show that this is the case, we consider the invariance criterion $\mathbf{Z}J(\alpha,\beta,\gamma,\alpha_\rho,\beta_\rho,\gamma_\rho)$=0, for $\rho \in \{t,x\}$, where

$$\mathbf{Z} = \xi_1\partial_t + \xi_2\partial_x + \mu_1\partial\alpha + \mu_2\partial\beta + \mu_3\partial\gamma.$$

Inserting it into the given invariance criterion and splitting it (by considering $\xi_1 = \xi_{11} + \iota\xi_{12}$, $\xi_2 = \xi_{21} + \iota\xi_{22}$ and all those used in the previous result) into two real parts and using the CR-equations we find

$$\begin{aligned}
&2(\xi_{11}J_{1,t} + \xi_{21}J_{1,x} - \xi_{12}J_{2,t} - \xi_{22}J_{2,x}) + \mu_{11}J_{1,\alpha_1}\\
&+\mu_{12}J_{1,\alpha_2} + \mu_{21}J_{1,\beta_1} + \mu_{22}J_{1,\beta_2} + \mu_{31}J_{1,\gamma_1}\\
&+\mu_{32}J_{1,\gamma_2} = 0,\\
&2(\xi_{11}J_{2,t} + \xi_{21}J_{2,x} + \xi_{12}J_{1,t} + \xi_{22}J_{1,x}) + \mu_{11}J_{2,\alpha_1}\\
&+\mu_{12}J_{2,\alpha_2} + \mu_{21}J_{2,\beta_1} + \mu_{22}J_{2,\beta_2} + \mu_{31}J_{2,\gamma_1}\\
&+\mu_{32}J_{2,\gamma_2} = 0.
\end{aligned} \tag{52}$$

These equations are different from the invariance criteria associated with a system of two hyperbolic equations (2) that provide its invariants under transformation of only the independent variables, i.e.,

$$\mathbf{X}J_1 = 0, \ \ \mathbf{X}J_2 = 0,$$

where

$$\begin{aligned}
\mathbf{X} &= \xi_1\partial_t + \xi_2\partial_x + \mu_{11}\partial_{\alpha_1} + \mu_{12}\partial_{\alpha_2} + \mu_{21}\partial_{\beta_1} + \mu_{22}\partial_{\beta_2}\\
&\quad + \mu_{31}\partial_{\gamma_1} + \mu_{32}\partial_{\gamma_2}.
\end{aligned} \tag{53}$$

Indeed, considering ξ_1, and ξ_2, as real functions one arrives at the same conclusion. A similar argument can be presented for the joint invariants.

V. APPLICATIONS

In this section a few examples of systems of hyperbolic equations are provided to illustrate the invariance criteria developed.

1. The system of two hyperbolic PDEs

$$\begin{aligned}
&u_{tx} + \left(a_1 - \tfrac{1}{x}\right)u_t - a_2v_t + \left(b_1 + \tfrac{2}{t}\right)u_x - b_2v_x\\
&\quad + \left(c_1 - \tfrac{b_1}{x} + 2\tfrac{a_1}{t} - \tfrac{2}{tx}\right)u - \left(c_2 - \tfrac{b_2}{x} + 2\tfrac{a_2}{t}\right)v = 0,\\
&v_{tx} + a_2u_t + \left(a_1 - \tfrac{1}{x}\right)v_t + b_2u_x + \left(b_1 + \tfrac{2}{t}\right)v_x\\
&\quad + \left(c_2 - \tfrac{b_2}{x} + 2\tfrac{a_2}{t}\right)u + \left(c_1 - \tfrac{b_1}{x} + 2\tfrac{a_1}{t} - \tfrac{2}{tx}\right)v = 0,
\end{aligned} \tag{54}$$

corresponds to the complex hyperbolic equation in two independent variables

$$\begin{aligned}
&w_{tx} + \left(a - \tfrac{1}{x}\right)w_t + \left(b + \tfrac{2}{t}\right)w_x\\
&\quad + \left(c - \tfrac{b}{x} + 2\tfrac{a}{t} - \tfrac{2}{tx}\right)w = 0,
\end{aligned} \tag{55}$$

where a is the complex constant $a_1 + \iota a_2$. The complex transformation of the dependent variable $w = (x/t^2)\bar{w}$ maps the above equation to

$$\bar{w}_{tx} + a\bar{w}_t + b\bar{w}_x + c\bar{w} = 0. \tag{56}$$

Transformation of the complex hyperbolic equations (55) and (56) into each other is guaranteed as the associated semi-invariants $h = ab - c = k$ agree for both of them. The system of hyperbolic equations (54) is transformable to

$$\begin{aligned}
\bar{u}_{tx} + a_1\bar{u}_t - a_2\bar{v}_t + b_1\bar{u}_x - b_2\bar{v}_x + c_1\bar{u} - c_2\bar{v} = 0,\\
\bar{v}_{tx} + a_2\bar{u}_t + a_1\bar{v}_t + b_2\bar{u}_x + b_1\bar{v}_x + c_2\bar{u} + c_1\bar{v} = 0,
\end{aligned} \tag{57}$$

under $u = (x/t^2)\bar{u}$, $v = (x/t^2)\bar{v}$, that is obtained by splitting the complex transformation used to map the complex equations (55) and (56) into each other. Semi-invariants associated with (54) and (57) are

$$\begin{aligned}
h_1 &= a_1b_1 - a_2b_2 - c_1 = k_1,\\
h_2 &= a_1b_2 + a_2b_1 - c_2 = k_2,
\end{aligned} \tag{58}$$

which implies that both the systems are mappable to each other.

2. An uncoupled system of PDEs

$$\begin{aligned}
u_{z_1z_2} + 2az_1^2u_{z_1} + 2bz_1u_{z_2} + 4cz_1u = 0,\\
v_{z_1z_2} + 2az_1^2v_{z_1} + 2bz_1v_{z_2} + 4cz_1v = 0,
\end{aligned} \tag{59}$$

is transformable to

$$\begin{aligned}
u_{tx} + atu_t + bu_x + cu = 0,\\
v_{tx} + atv_t + bv_x + cv = 0,
\end{aligned} \tag{60}$$

via an invertible change of the independent variables

$$z_1 = \sqrt{t}, \ \ z_2 = \frac{1}{2}(x-1). \tag{61}$$

These are the invertible maps that reduce the base complex hyperbolic equation

$$w_{z_1z_2}+2az_1^2w_{z_1}+2bz_1w_{z_2}+4cz_1w=0, \quad (62)$$

with the semi-invariants

$$I_1=\frac{c}{abz_1^2},\ I_2=bz_1^2,\ I_4=\frac{c}{a},\ I_3=I_5=I_6=0, \quad (63)$$

to the simple linear form $w_{tx}+atw_t+bw_x+cw=0$, with the semi-invariants

$$I_1=\frac{c}{abt},\ I_2=bt,\ I_4=\frac{c}{a},\ I_3=I_5=I_6=0. \quad (64)$$

Note that the semi-invariants (63) and (64) are the same under (61). The complex hyperbolic equation (62) does not only yield an uncoupled system of hyperbolic equations (59) but gives a coupled system

$$\begin{aligned}&u_{z_1z_2}+2a_1z_1^2u_{z_1}-2a_2z_1^2v_{z_1}+2b_1z_1u_{z_2}-2b_2z_1v_{z_2}\\&\quad+4c_1z_1u-4c_2z_1v=0,\\&v_{z_1z_2}+2a_2z_1^2u_{z_1}+2a_1z_1^2v_{z_1}+2b_2z_1u_{z_2}+2b_1z_1v_{z_2}\\&\quad+4c_2z_1u+4c_1z_1v=0,\end{aligned} \quad (65)$$

when we consider $a=a_1+\iota a_2$, $b=b_1+\iota b_2$ and $c=c_1+\iota c_2$ in (62). This system of two hyperbolic equations can be mapped to

$$\begin{aligned}&u_{tx}+a_1tu_t-a_2tv_t+b_1u_x-b_2v_x+c_1u-c_2v=0,\\&v_{tx}+a_2tu_t+a_1tv_t+b_2u_x+b_1v_x+c_2u+c_1v=0,\end{aligned} \quad (66)$$

under the transformations (61) that are already used to map the base complex equation to its canonical form. Semi-invariants derived under the transformations of independent variables using real as well as complex approach agree for systems (59), (60) and (65), (66), respectively.

3. Consider the uncoupled system of two hyperbolic PDEs

$$\begin{aligned}u_{tx}+\frac{\lambda}{2}(u_t+u_x)=0,\\ v_{tx}+\frac{\lambda}{2}(v_t+v_x)=0,\end{aligned} \quad (67)$$

for which $h_1=k_1=\frac{\lambda^2}{4}$ and $h_2=k_2=0$. This implies that

$$J_{11}=1,\ \ J_{12}=\cdots=J_{22}=0. \quad (68)$$

The system (67) is transformable to another system with the same invariants as given in (68), where $h_1=k_1=-1$, $h_2=k_2=0$. The transformed system reads as

$$\begin{aligned}\overline{u}_{z_1z_2}+\overline{u}=0,\\ \overline{v}_{z_1z_2}+\overline{v}=0.\end{aligned} \quad (69)$$

In (68) we show an agreement between those joint invariants that are derived using the complex method (48). The same is true for real joint invariants (28), i.e., they agree for both the systems given above. The correspondence between the systems (67) and (69) is established through

$$t=\tfrac{2}{\lambda}z_1,\ x=-\tfrac{2}{\lambda}z_2,\ u=\tfrac{\overline{u}}{\exp(z_1-z_2)},\ v=\tfrac{\overline{v}}{\exp(z_1-z_2)}. \quad (70)$$

These transformations are obtainable from

$$t=\tfrac{2}{\lambda}z_1,\ \ x=-\tfrac{2}{\lambda}z_2,\ \ w=\tfrac{\overline{w}}{\exp(z_1-z_2)}, \quad (71)$$

with $\overline{w}=\overline{u}+\iota\overline{v}$, and $w=u+\iota v$. The complex transformations map the complex scalar PDE

$$w_{tx}+\frac{\lambda}{2}(w_t+w_x)=0, \quad (72)$$

with $h=k=\frac{\lambda^2}{4}$ and $J_1=1$, to the equation

$$\overline{w}_{z_1z_2}+\overline{w}=0, \quad (73)$$

for which $h=k=-1$ and $J_1=1$. Notice here that the substitution $\lambda=\lambda_1+\iota\lambda_2$, in the equation (72) results in a coupled system of two hyperbolic PDEs but it can not be transformed by the complex method. The reason is the complex transformations (71) for which the two independent variables split into four which adds extra dimensions. Therefore, the complex procedure does not apply for that case.

4. The complex transformations

$$z_1=\frac{1}{t},\ \ z_2=2x,\ \ \overline{w}=\frac{w}{x}, \quad (74)$$

map the Lie canonical form

$$\overline{w}_{z_1z_2}+\alpha z_2^2\overline{w}_{z_2}+2\overline{w}=0, \quad (75)$$

to

$$w_{tx}-\frac{1}{x}w_t-4\frac{\alpha x^2}{t^2}w_x+\frac{4}{t^2}(\alpha x-1)w=0. \quad (76)$$

The invariant quantities associated with both the scalar Lie canonical form and the hyperbolic equations agree. Inserting $w=u+\iota v$, in the equation (76) while keeping α, *a real constant* yields an uncoupled system of two PDEs

$$\begin{aligned}u_{tx}-\frac{1}{x}u_t-4\frac{\alpha x^2}{t^2}u_x+\frac{4(\alpha x-1)}{t^2}u=0,\\ v_{tx}-\frac{1}{x}v_t-4\frac{\alpha x^2}{t^2}v_x+\frac{4(\alpha x-1)}{t^2}v=0.\end{aligned} \quad (77)$$

The system (77) is transformable to another system of the form

$$\overline{u}_{z_1z_2}+\alpha x^2\overline{u}_{z_2}+2\overline{u}=0,$$

$$\overline{v}_{z_1z_2} + \alpha x^2 \overline{v}_{z_2} + 2\overline{v} = 0, \tag{78}$$

under a change of the dependent and independent variables

$$z_1 = \frac{1}{t},\ \ z_2 = 2x,\ \ \overline{u} = \frac{u}{x},\ \ \overline{v} = \frac{v}{x}. \tag{79}$$

These transformations are the real and imaginary parts of the complex transformations (74) and the transformed system is obtained by splitting the Lie canonical form (75) into real and imaginary parts. Joint invariants (48) for both the systems (77) and (78) are

$$h_1 = \frac{2}{t^2},\ k_1 = \frac{2(1-\alpha x)}{t^2},\ h_2 = 0 = k_2,\ J_{11} = \frac{-1}{\alpha x - 1}, \tag{80}$$

and

$$h_1 = -2,\ k_1 = 2(\alpha x - 1),\ h_2 = 0 = k_2,\ J_{11} = \frac{1}{1-\alpha x}, \tag{81}$$

respectively, while all others given in (48) are zero. A coupled system

$$\begin{aligned} u_{tx} - \tfrac{1}{x}u_t - \tfrac{4\alpha_1 x^2}{t^2}u_x + \tfrac{4\alpha_2 x^2}{t^2}v_x \\ + \tfrac{4(\alpha_1 x - 1)}{t^2}u - \tfrac{4\alpha_2 x}{t^2}v = 0, \\ v_{tx} - \tfrac{1}{x}v_t - \tfrac{4\alpha_2 x^2}{t^2}u_x - \tfrac{4\alpha_1 x^2}{t^2}v_x \\ + \tfrac{4\alpha_2 x}{t^2}u + \tfrac{4(\alpha_1 x - 1)}{t^2}v = 0, \end{aligned} \tag{82}$$

with the invariants

$$\begin{aligned} &h_1 = \tfrac{2}{t^2},\ k_1 = \tfrac{2(1-\alpha_1 x)}{t^2},\ h_2 = 0,\ k_2 = \tfrac{-2\alpha_2 x}{t^2}, \\ &J_{11} = \tfrac{1-\alpha_1 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2},\ J_{12} = \tfrac{\alpha_2 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2}, \\ &J_{13} = J_{14} = \ldots = J_{22} = 0, \end{aligned} \tag{83}$$

is obtainable from the complex scalar PDE (76) when α is also complex, i.e., $\alpha = \alpha_1 + \iota\alpha_2$. Employing the transformations (79) on (65) one arrives at a coupled system

$$\begin{aligned} \overline{u}_{z_1z_2} + \alpha_1 z_2^2 \overline{u}_{z_2} - \alpha_2 z_2^2 \overline{v}_{z_2} + 2\overline{u} = 0, \\ \overline{v}_{z_1z_2} + \alpha_2 z_2^2 \overline{u}_{z_2} + \alpha_1 z_2^2 \overline{v}_{z_2} + 2\overline{v} = 0, \end{aligned} \tag{84}$$

which is the real analogue of the complex transformed equation (75) and satisfies the invariance criteria, where

$$\begin{aligned} &h_1 = -2,\ k_1 = 2(\alpha_1 x - 1),\ h_2 = 0,\ k_2 = 2\alpha_2 x, \\ &J_{11} = \tfrac{1-\alpha_1 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2},\ J_{12} = \tfrac{\alpha_2 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2}, \\ &J_{13} = J_{14} = \ldots = J_{20} = 0. \end{aligned} \tag{85}$$

VI. CONCLUSION

Semi-invariants of hyperbolic PDEs in two independent variables have been obtained by transforming the dependent or independent variables. Further, the infinitesimal approach has been used to derive the joint invariants for linear hyperbolic equations. Semi-invariants of the hyperbolic PDEs under transformations of only the dependent variable have been extended to systems of such equations by CSA. Here using the real and complex approaches we derive invariants of the system of two linear hyperbolic equations under transformation of: (a) only dependent variables (re-derivation); (b) only independent variables; (c) both the dependent and independent variables.

In the case of transformation of only the dependent variables of the system of two linear hyperbolic equations (2), an agreement between associated semi-invariants derived by the real and complex procedures is shown. However, semi-invariants under transformations of the independent variables, of this class of systems obtained by real symmetry analysis appear different from those provided by the complex procedure. Furthermore, the joint invariants of this system of hyperbolic equations obtained by both methods are also disparate. Indeed the complex symmetry approach is seen to reveal different invariants from those provided by real symmetry analysis. This is seen mostly in the case where the real method yields only 3 invariants, see (28), while the complex method yields 10 invariants, in (48).

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