Analytical approach to a three species food chain model by applying Homotopy perturbation method

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Abstract— In this paper, a mathematical model is proposed and analyzed to study the effect of toxicant in a three species food chain system with "food limited" growth of prey population. The mathematical model is formulated using the system of non-linear ordinary differential equation. In the model, there are seven state variables, viz, prey population density, intermediate predator population density, top predator population density, toxicant concentration in the environment, toxicant concentration in the prey population, toxicant concentration in the intermediate predator population, toxicant concentration in the top predator population. Approximate analytical solutions for all above seven state variables are presented using Homotopy Perturbation Method (HPM) for all possible values of parameter. In addition, in this work the numerical simulation of the problem is also presented using MATLAB program to investigate the dynamic of the system. The same information also presented in graphical form which makes it easier to understand. A good agreement found between the analytical and the numerical solution.

Keywords— Analytical solution, Concentration, Food chain model, Homotopy perturbation method, Non-linear system of differential equations.

I. Introduction

Many species are exposed to various kinds of stresses including toxicants which are affecting their growth rate, carrying capacity and their resources. The effects of toxicants on ecological communities including three-species food chain system are very complex dynamical systems to be undertaken for mathematical study. Three species food chain system have received much attention from many applied mathematical and ecologists in recent years [1-3]. Previously, some research has been done on tri- trophic food-chain systems including toxicant effects on the survival or extinction of species in the system [4, 5]. It has been observed that toxicants have very pronounced effects on the species if the availability of the resources is limited.

In order to use and regulate toxic substance wisely, we must asses the risk of the populations exposed to toxicant. The problem of estimating qualitatively the effect of a toxicant on a population by mathematical models is a relatively new field that began only in the early 1980s. For a general class of a single population models with toxicant stress, Ma et al. [6] obtained a survival threshold distinguishing between persistence in the mean end extinction of a single population under the hypothesis that the capacity of the environment is large relative to the population biomass, and that the exogenous input of toxicant into the environment is bounded. The threshold of the survival for a system of two species in a polluted environment was studied by Huaping and Ma [7]. Population toxicant coupling has been applied in several contexts including Lotka-Volterra and chemostat like environments, resulting in ordinary, integro-differential and stochastic models. All these studies rely on the hypothesis of a complete spatially homogenous environment.

In recent years, several investigators have studied the effects of toxicants on a single species population. In particular, Zhan Li et al. [8] and Hallam et al. [9] studied the effect of a toxicant present in the environment on a single species population by assuming that its growth rate density decreases linearly with the concentration but the corresponding carrying capacity does not depend upon the concentration of toxicant present in the environment. However, Freedaman and shukla [10] proposed models to study the effects of a toxicant on single species and predator-prey systems by assuming that the intrinsic growth rate of the species decreases as the uptake concentration of the toxicant increases while its carrying capacity decreases with the environmental concentration of the toxicant..

In recent decades, several investigators have studied the system of two biological species in a polluted environment; In particular, Huaping and Ma Zhien [11] proposed a mathematical model to study the effect of a toxicant on population of two competing species and derived persistence-extinction criteria for each population. Shukla et al. [12] have also studied the survival of two competing species in a polluted environment and showed that the usual competitive outcomes may be altered in the presence of a toxicant Hsu et al. [13] proposed and analyzed a model to study the interaction between two species competing for a resource in the presence of an inhibitor or a toxicant that affects one of the competitors but is removed by the other. To our knowledge, almost no studies have been conducted to investigate the effect of toxicant on a three species food chain systems with ''food limited'' growth of prey population and therefore, in the paper, a mathematical model is proposed to study the effects of toxicants on a three species food –chain system with ''food limited'' growth of prey population. The present model may be suited for the food chain system comprising of Spider-_Mouse _Snake and also for the food chain system consisting of Spider-Lizard-Hawk.

II. Mathematical Model

Consider a three species food chain system under the stress of a toxicant considering "food-limited" growth of population. The model is formulated with the help of following system of ordinary differential equation [14],

$$\frac{dx}{dy} = xr(U)\left(\frac{K(T)-x}{K(T)+r_0cx}\right) - a_1xy \tag{1}$$

$$\frac{dy}{dx} = \beta_1a_1xy - a_2yz - \beta_{11}Vy - d_1y - b_1y^2 \tag{2}$$

$$\frac{dz}{dt} = \beta_2a_2yz - \beta_{22}Wz - d_2z - c_3z^2 \tag{3}$$

$$\frac{dT}{dt} = Q_0 - \delta_0T - \alpha_1xT \tag{4}$$

$$\frac{dU}{dt} = \alpha_1xT - \delta_1U - \beta_3(U)a_1xy \tag{5}$$

$$\frac{dV}{dt} = \beta_3(U)a_1xy - \delta_2V - \beta_4(V)a_2yz \tag{6}$$

$$\frac{dW}{dt} = \beta_4(V)a_2yz - \delta_3W \tag{7}$$

with the initial conditions

$$\begin{aligned} &x(0)>0, y(0)>0, z(0)>0, T(0)>0, \\ &U(0)\geq 0, V(0)\geq 0, W(0)\geq 0 \end{aligned} \tag{8}$$

III. Approximate Analytical Solution of the System of Nonlinear Eqns. (1)-(8) Using Homotopy Perturbation Method (HPM)

Recently, HPM is often working to solve a number of systematic problems. In addition, several groups established the efficient and suitability of the HPM for solving non-linear differential equation problems. Recently, there are so many authors have applied the HPM to various non-linear problems and demonstrated the efficiency of the HPM for handling non-linear structures and solving various physics and engineering problems [15-18]. Using HPM (refer Appendix A), we obtain the approximate solutions of the Eqns. (1)-(7) with the initial condition Eqn. (8) as follows:

$$\begin{aligned} x(t) = {}&[L - \frac{r_1LA}{\delta_1} + \frac{L^2}{k_0} - \frac{r_1AL^2}{k_0(r_0-\delta_1)} + \frac{r_0k_1PL^2}{k_0^{\,2}(r_0+\delta_0)} \\ &- \frac{r_1k_1APL^2}{k_0^{\,2}(r_0-\delta_1+\delta_0)} + \frac{r_0cL^2}{k_0} - \frac{r_0r_1cAL^2}{k_0(r_0-\delta_1)} + \frac{r^2ck_1PL^2}{k_0^{\,2}(r_0+\delta_0)} \\ &- \frac{r_1r_0cAk_1PL^2}{k_0^{\,2}(r_0-\delta_1+\delta_0)} - \frac{r_0cL^3}{2k_0^{\,2}} + \frac{r_0r_1cAL^3A}{k_0^{\,2}(2r_0-\delta_1)} \\ &- \frac{2k_1r_0^{\,2}cL^3P}{k_0^{\,3}(2r_0+\delta_0)} - \frac{aLM}{d_1} + \frac{2r_0r_1ck_1AL^3P}{k_0^{\,3}(2r_0-\delta_1+\delta_0)}]e^{r_0t} \\ &+ \left[\frac{r_0r_1cAL^2}{k_0(r_0-\delta_1)} + \frac{r_1AL^2}{k_0(r_0-\delta_1)}\right]e^{(2r_0-\delta_1)t} + \frac{r_1LA}{\delta_1}e^{(r_0-\delta_1)t} \\ &- \left[\frac{r^2ck_1PL^2}{k_0^{\,2}(r_0+\delta_0)} + \frac{r_0k_1PL^2}{k_0^{\,2}(r_0+\delta_0)}\right]e^{(2r_0+\delta_0)t} \\ &+ \left[\frac{r_1r_0cAk_1PL^2}{k_0^{\,2}(r_0-\delta_1+\delta_0)} + \frac{r_1k_1APL^2}{k_0^{\,2}(r_0-\delta_1+\delta_0)}\right]e^{(2r_0-\delta_1+\delta_0)t} \\ &- \frac{r_0cL^2}{k_0}e^{2r_0t} + \frac{r_0cL^3}{2k_0^{\,2}}e^{3r_0t} - \frac{r_0r_1cAL^3A}{k_0^{\,2}(2r_0-\delta_1)}e^{(3r_0-\delta_1)t} \\ &+ \frac{2k_1r_0^{\,2}cL^3P}{k_0^{\,3}(2r_0+\delta_0)}e^{(3r_0+\delta_0)t} + \frac{aLM}{d_1}e^{(r_0-d_1)t} \\ &- \frac{2r_0r_1ck_1AL^3P}{k_0^{\,3}(2r_0-\delta_1+\delta_0)}e^{(3r_0-\delta_1+\delta_0)t} - \frac{L^2}{k_0}e^{2r_0t} \end{aligned} \tag{9}$$

$$\begin{aligned} y(t) = {}&[M \frac{-\beta_1\alpha_1LM}{r_0} - \frac{a_2MN}{d_2} - \frac{\beta_{11}BM}{\delta_2} - \frac{b_1M^2}{d_1}]e^{-d_1t} \\ &+ \frac{\beta_1\alpha_1LM}{r_0}e^{(r_0-d_1)t} + \frac{a_2MN}{d_2}e^{-(d_1+d_2)t} \\ &+ \frac{\beta_{11}BM}{\delta_2}e^{-(\delta_2+d_1)t} + \frac{b_1M^2}{d_1}e^{-2d_1t} \end{aligned} \tag{10}$$

$$\begin{aligned} z(t) = {}&[N + \frac{\beta_2a_2MN}{d_1} - \frac{\beta_{22}DN}{\delta_3} - \frac{c_3N^2}{d_2}]e^{-d_2t} \\ &- \frac{\beta_2a_2MN}{d_1}e^{-(d_1+d_2)t} + \frac{\beta_{22}DN}{\delta_3}e^{-(\delta_3+d_2)t} + \frac{c_3N^2}{d_2}e^{-2d_2t} \end{aligned} \tag{11}$$

$$\begin{aligned} T(t) = {}&[P + \frac{Q_0}{\delta_0} + \frac{\alpha_1LP}{r_0}]e^{\delta_0t} \\ &- \frac{Q_0}{\delta_0} - \frac{\alpha_1LP}{r_0}e^{(r_0+\delta_0)t} \end{aligned} \tag{12}$$

$$\begin{aligned} U(t) = {}&[A + \frac{-\alpha_1LP}{r_0+\delta_0+\delta_1} + \frac{a_1a_3ALM}{r_0-d_1}]e^{-\delta_1t} \\ &+ \frac{\alpha_1LP}{r_0+\delta_0+\delta_1}e^{(r_0+\delta_0)t} - \frac{a_1a_3ALM}{r_0-d_1}e^{(-\delta_1+r_0-d_1)t} \end{aligned} \tag{13}$$

$$V(t)=[B-\frac{a_1a_3ALM}{r_0-d_1-\delta_1+\delta_2}-\frac{a_2a_4BMN}{d_1+d_2}]e^{-\delta_2 t}+\frac{a_1a_3ALM}{r_0-d_1-\delta_1+\delta_2}e^{(-\delta_1+r_0-d_1)t}+\frac{a_2a_4BMN}{d_1+d_2}e^{-(d_1+d_2+\delta_2)t} \quad (14)$$

$$W(t)=[D+\frac{a_2a_4BMN}{d_1+d_2+\delta_2-\delta_3}]e^{-\delta_3 t}+\frac{a_2a_4BMN}{\delta_3-d_1-d_2-\delta_2}e^{-(\delta_2+d_1+d_2)t} \quad (15)$$

IV. Numerical Solution

In this section, we expressed the dynamical behavior of a three species food chain system with "foot-limited" growth of prey population with toxicant numerically to help the interpretation of our mathematical findings. In order to investigate accuracy of the HPM solution with finite number of terms, the system of differential equation were solved numerically for various values of L, M, N, P, A, B and D and all other given parameters by using MATLAB software. The figures illustrate the models for the given sets of parameters and the graphs have been plotted by using MATLAB software. The MATLAB program is also given in Appendix B.

V. Results and Discussion

Equations (9)-(15) represent the analytical expressions of seven state populations for all values of parameters. To compare our mathematical finding with numerical solution we are considering the following two sets of parameters referred from [14].

(i) $r_0=5.00;\ r_1=11.0;\ k_0=14.2;\ k_1=3.0;\ c_3=0.02;\ \alpha_1=1.31;\ b_1=1.0231;\ \beta_1=0.4;\ \beta_{11}=1.01;\ \beta_2=0.6;\ \beta_{22}=1.15;\ a_1=3.22;\ a_2=0.9913;\ a_3=2.865;\ a_4=4.21;\ \delta_0=5.82;\ \delta_1=2.0;\ \delta_2=1.9890;\ \delta_3=2.9;\ Q_0=1.988;\ d_1=0.35;\ d_2=0.5;\ c=5.58;$

(ii) $r_0=5.66;\ r_1=11.0;\ k_0=16.2;\ k_1=3.0;\ c=4.25;\ c_3=1.02;\ \alpha_1=2.12;\ b_1=1.231;\ \beta_1=1.2;\ \beta_{11}=1.1;\ \beta_2=1.6;\ \beta_{22}=1.15;\ a_1=3.22;\ a_2=2.13;\ a_3=2.865;\ a_4=4.21;\ \delta_0=7.52;\ \delta_1=2.5;\ \delta_2=3.99;\ \delta_3=1.3;\ Q_0=2.988;\ d_1=1.45;\ d_2=1.49;$

In the following figures, we present the series of normalized population profile for a prey, intermediate predator, top predator population, toxin in the environment, toxin in the prey, toxin in the intermediate predator and toxin in the top predator for various values of parameters [14]. All series solutions satisfy the initial conditions. We used the first set of parameter to draw Figures (1)-(3). From these figures, it is inferred that the maximum and minimum values of prey, intermediate predator and top predator affected by the increasing values of parameters. When the dimensionless time t increases, all the populations are increase. We used the second set of parameter to draw Figures (4)-(6). From these figures, it is inferred that when the dimensionless time t increases, the prey population x, the intermediate predator population y, toxin in the environment T, toxin in prey U and toxin in intermediate predator V are increase and top predator population z and toxin in *top* predator W remain same.

From Figure (7), it is inferred that the values of the intrinsic growth rate of prey r_0 increases, the prey population x also increases. From Figures (8) and (9), it is inferred that the values of the conversion coefficient β_1 and the death rates of intermediate predator d_1 increase; the intermediate predator population y also increases. From Figure (10), it is inferred that the values of the death rate of top predator population d_2 increases, the top predator population z decreases. From Figure (11), it is inferred that the values of the rate of introduction of toxicant into the environment Q_0 increases, the toxin in the environment T decreases. From Figure (12), it is inferred that the values of the rate of toxicants in the population δ_1 increases, the toxin in prey populations U *is* decreases. From Figure (13), it is inferred that the values of the rate of toxicants in the population δ_2 increases, the toxin in intermediate predator population V decreases. From Figure (14), it is inferred that the values of the rate of toxicants in the population δ_3 increases, the toxin in top predator population W decreases.

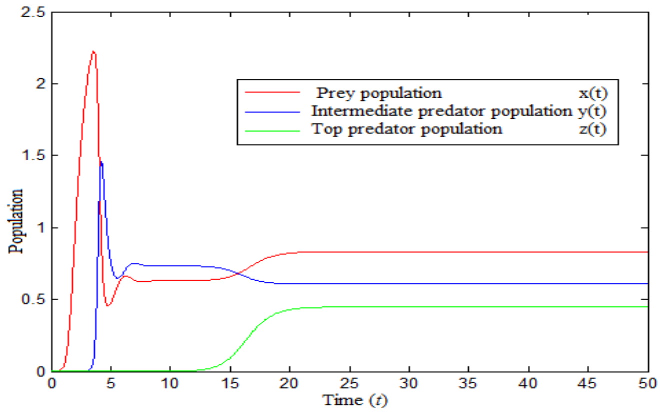

Fig. (1)

Fig.(1) Profiles of the populations of the prey x, intermediate predator y and top predator z were computed using equations (9)-(11). Populations versus time t for the values of parameters $L=M=N=P=A=B=D=0.0001$ with the first set of parameters.

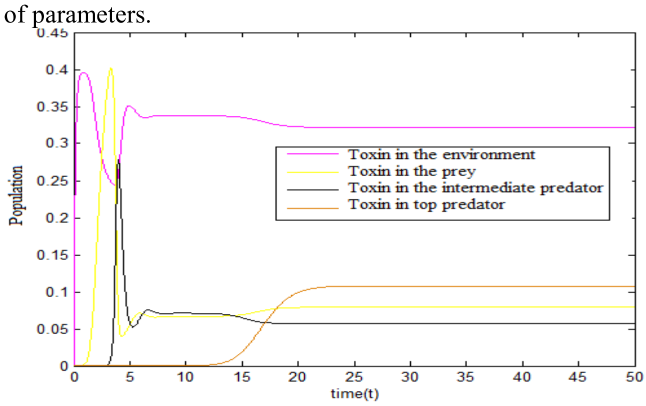

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Fig. (2)

Fig.(2) Profiles of the populations of the toxicant in the environment *T*, toxicant in the prey *U*, toxicant in the intermediate predator *V* and toxicant in the top predator W were computed using equations (12)-(15). Populations versus time *t* for the values of parameters *L* = *M* =*N*=*P*=*A*=*B*=*D*= 0.0001 with the first set of parameters.

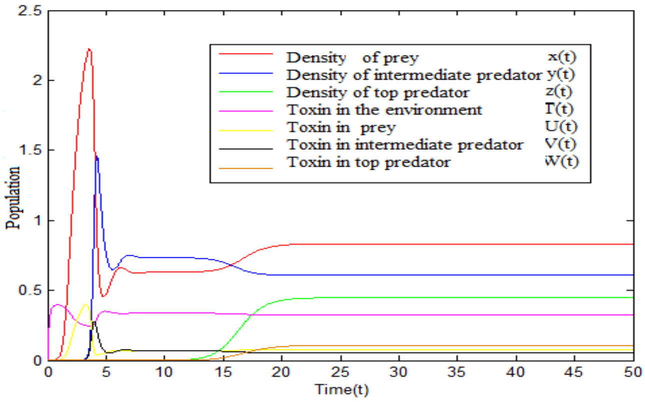

Fig. (3)

Fig.(3) Profiles of the populations of all seven state variables were computed using equations (9)-(15). Populations versus time *t* for the values of parameters *L*= *M* =*N*=*P*=*A*=*B*=*D*= 0.0001 with the first set of parameters.

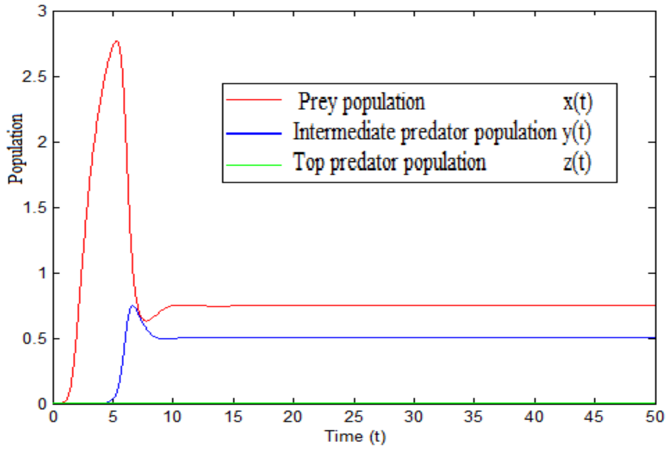

Fig. (4)

Fig.(4) Profiles of the populations of the prey *x*, intermediate predator *y* and top predator *z* were computed using equations (9)-(11). Populations versus time *t* for the values of parameters *L* = *M* =*N*=*P*=*A*=*B*=*D*= 0.0001 with the second set of parameters.

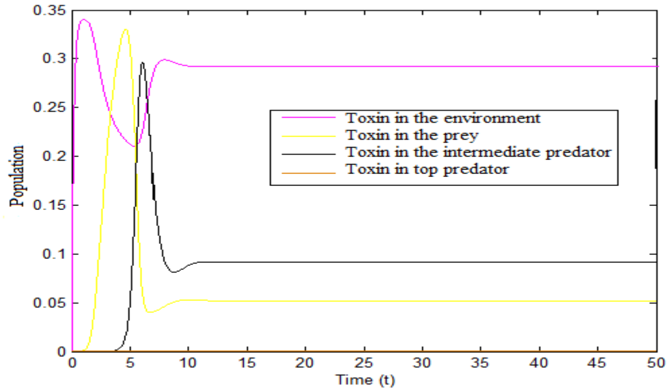

Fig. (5)

Fig.(5) Profiles of the populations of the toxicant in the environment *T*, toxicant in the prey *U*, toxicant in the intermediate predator *V* and toxicant in the top predator *W* were computed using equations (12)-(15). Populations versus time t for the values of parameters *L* = *M* =*N*=*P*=*A*=*B*=*D*= 0.0001 with the second set of parameters.

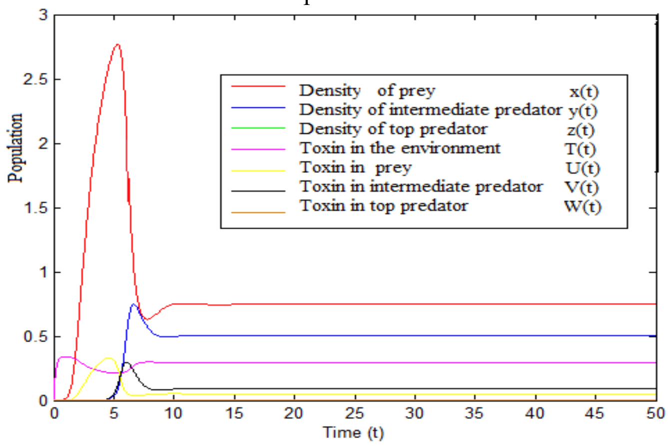

Fig.(6)

Fig.(6) Profiles of the populations of all seven state variables were computed using equations (9)-(15). Populations versus time *t* for the values of parameters *L* = *M* =*N*=*P*=*A*=*B*=*D*= 0.0001 with the second set of parameters.

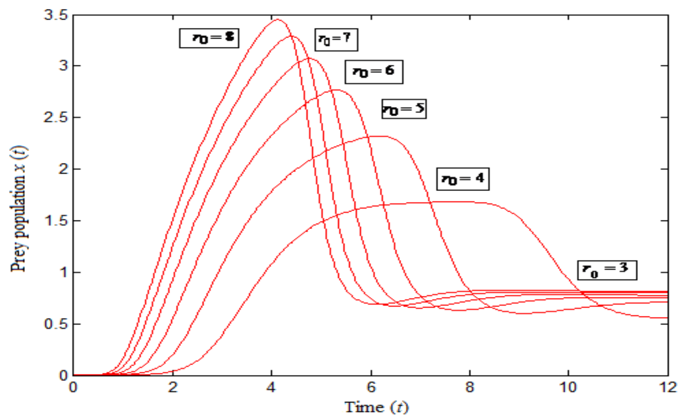

Fig.(7)

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Fig.(7) Profiles of the normalized prey populations were computed using Eqn. (9) versus dimensionless time t for various values of the intrinsic growth rate of prey r_0.

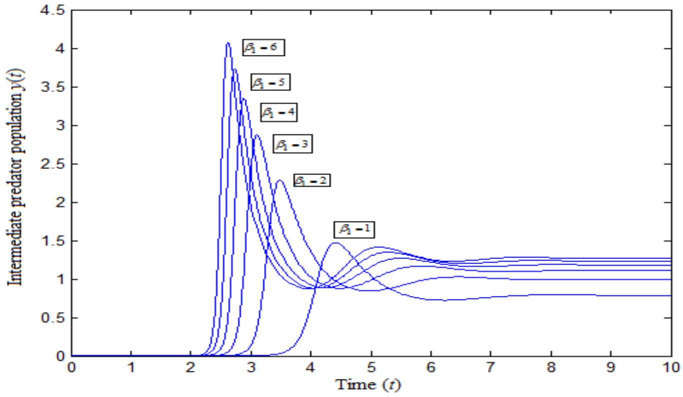


Fig.(8)

Fig.(8) Profiles of the normalized intermediate predator populations were computed using Eqn. (10) versus dimensionless time t for various values of the conversion coefficients β_1.

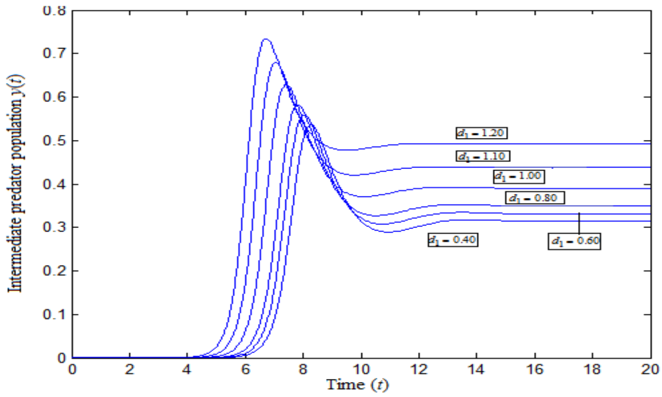


Fig.(9)

Fig.(9) Profiles of the normalized intermediate predator populations were computed using Eqn. (10) versus dimensionless time t for various values of the death rates of intermediate predator d_1.

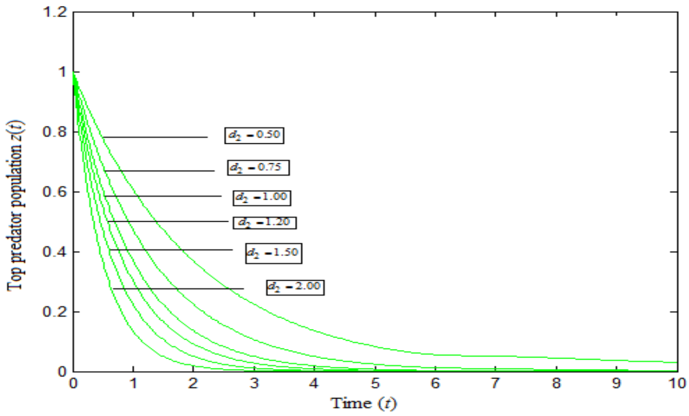


Fig.(10)

Fig.(10) Profiles of the normalized top predator populations were computed using equation (11) versus dimensionless time t for various values of the death rate of top predator population d_2.

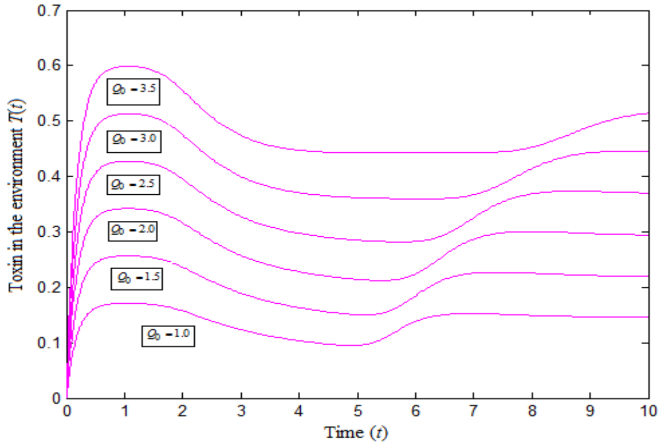
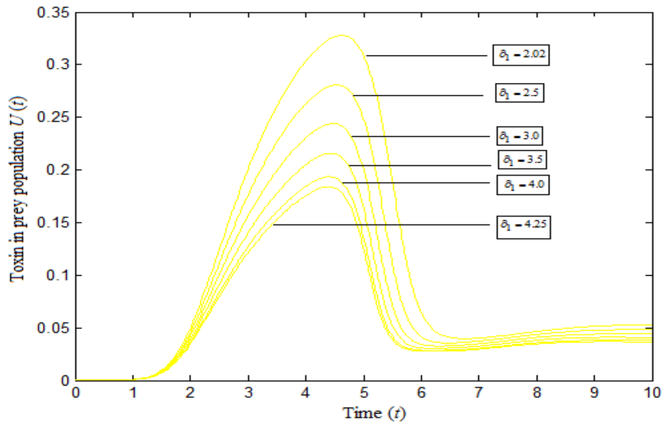


Fig.(11)

Fig.(11)) Profiles of the normalized toxicant in the environment were computed using equation (12) versus dimensionless time t for various values of the rate of introduction of toxicant into the environment Q_0.



Fig(12)

Fig.(12) Profiles of the normalized toxicant in prey populations were computed using equation (13) versus dimensionless time t for various values of the rate of toxicants in the population δ_1 .

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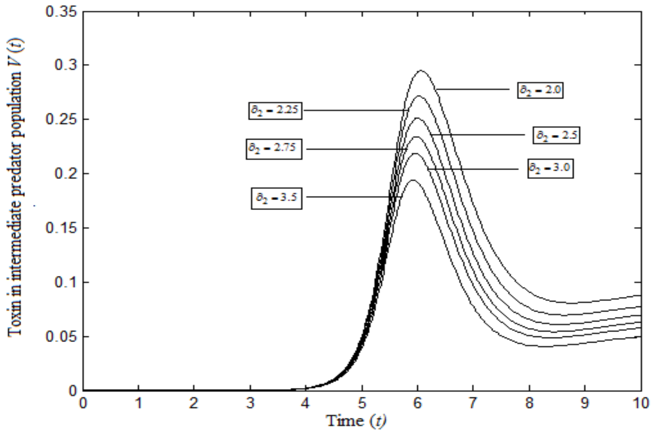
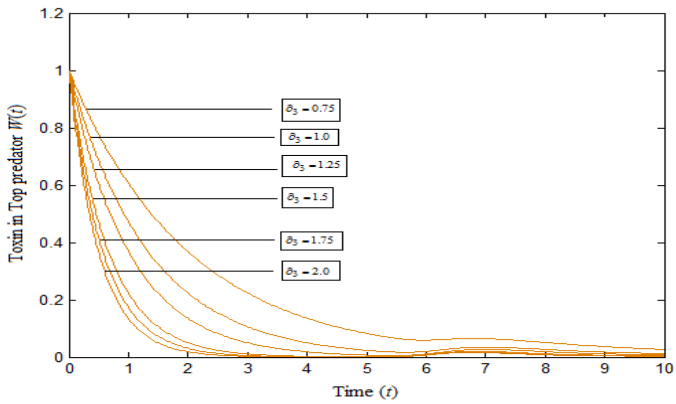


Fig(13)

Fig.(13) Profiles of the normalized toxicant in intermediate predator populations were computed using equation (14) versus dimensionless time t for various values of the rate of toxicants in the population δ_2.



Fig(14)

Fig.(14) Profiles of the normalized toxicant in top predator populations were computed using equation (15) versus dimensionless time *t* for various values of the rate of toxicants in the population δ_3.

VI. Conclusion

There is a goal that we aimed in this work. That is to employ the hopeful homotopy perturbation method to solve non-linear equations arising in the mathematical model of effect of toxicant in a three species food chain system with "food limited" growth of prey population problem. From this we achieved good results in predicting the solutions. All the analytical results are compared with the numerical solutions. A good agreement with the available numerical results is notified. The analytical result is a powerful tool for analyzing the model. Also the analytical result derived in this paper is useful for a better understanding and optimization of the biological system.

Appendix A

Approximate analytical solutions of the system of Eqns. (9)-(15) using Homotopy perturbation method [19, 20]. To find the solutions of equations (1)-(7) with the initial condition (8), we first constructed a Homotopy as follows,

$$\begin{aligned}[1-p]\left[\frac{dx}{dt}-xr_0\right]+p\Big[&\frac{dx}{dt}-xr_0+xr_1U+\frac{x^2r_0}{k_0}-\frac{x^2r_1U}{k_0}\\ &+\frac{x^2r_0k_1T}{k_0^2}-\frac{x^2r_1k_1UT}{k_0^2}+\frac{x^2r_0^2c}{k_0}-\frac{x^2r_0r_1cU}{k_0}\\ &+\frac{x^2r_0^2ck_1T}{k_0^2}-\frac{x^2r_0r_1cUk_1T}{k_0^2}-\frac{x^3r_0^2c}{k_0^2}+\frac{x^3r_0cr_1U}{k_0^2}\\ &-\frac{2k_1Tx^3r_0c}{k_0^3}+\frac{2x^3r_0cr_1Uk_1T}{k_0^3}+a_1xy\Big]=0\end{aligned} \tag{A1}$$

$$[1-p]\left[\frac{dy}{dt}+d_1y\right]+p\Big[\frac{dy}{dt}+d_1y-\beta_1\alpha_1xy+a_2yz+\beta_{11}Vy+b_1y^2\Big]=0 \tag{A-2}$$

$$[1-p]\left[\frac{dz}{dt}+d_2z\right]+p\Big[\frac{dz}{dt}+d_2z-\beta_2a_2yz+\beta_{22}Wz+c_3z^2\Big]=0 \tag{A-3}$$

$$[1-p]\left[\frac{dT}{dt}-\delta_0T\right]+p\Big[\frac{dT}{dt}-\delta_0T-Q_0+\alpha_1xT\Big]=0 \tag{A-4}$$

$$[1-p]\left[\frac{dU}{dt}+\delta_1U\right]+p\Big[\frac{dU}{dt}-\delta_1U-\alpha_1xT+a_1a_3Uxy\Big]=0 \tag{A-5}$$

$$[1-p]\left[\frac{dV}{dt}+\delta_2V\right]+p\Big[\frac{dV}{dt}+\delta_2V-a_1a_3Uxy+a_2a_4Vyz\Big]=0 \tag{A-6}$$

$$[1-p]\left[\frac{dW}{dt}+\delta_3W\right]+p\Big[\frac{dW}{dt}+\delta_3W-a_2a_4Vyz\Big]=0 \tag{A-7}$$

with the initial approximations are as follows,

$$x_0=L,\ y_0=M,\ z_0=N,\ T_0=P,\quad U_0=A,\ V_0=B,\ W_0=D \qquad \text{When } t=0 \tag{A-8}$$

$$x=x_0+px_1+p^2x_2+\ldots\ldots \tag{A-9}$$

$$y=y_0+py_1+p^2y_2+\ldots.. \tag{A-10}$$

$$z=z_0+pz_1+p^2z_2+\ldots\ldots \tag{A-11}$$

$$T=T_0+pT_1+p^2T_2+\ldots.. \tag{A-12}$$

$$U=U_0+pU_1+p^2U_2+\ldots\ldots \tag{A-13}$$

$$V=V_0+pV_1+p^2V_2+\ldots. \tag{A-14}$$

$$W=W_0+pW_1+p^2W_2+\ldots.. \tag{A-15}$$

Substituting Eqns. (A-9) - (A-15) into Eqns. (A-1) - (A-7) and comparing the coefficients of like powers

$$p^0 : \frac{dx_0}{dt} - x_0 r_0 = 0 \qquad \text{(A-16)}$$

$$p^1 : \frac{dx_1}{dt} - x_1 r_0 + r_1 U_0 x_0 + \frac{r_0 x_0^2}{k_0} - \frac{r_1 U_0 x_0^2}{k_0} + \frac{r_0 k_1 T_0 x_0^2}{k_0^2} - \frac{r_1 U_0 x_0^2 k_1 T_0}{k_0^2} + \frac{r_0^2 c x_0^2}{k_0} - \frac{r_0 r_1 c U_0 x_0^2}{k_0} + \frac{r_0^2 c k_1 T_0 x_0^2}{k_0^2} - \frac{r_0 r_1 c k_1 T_0 x_0^2 U_0}{k_0^2} \frac{r_0^2 c x_0^3}{k_0^2} + \frac{r_0 c r_1 x_0^2 U_0}{k_0^2} - \frac{2 k_1 r_0^2 c x_0^3 T_0}{k_0^3} + \frac{2 r_0 c r_1 k_1 x_0^3 U_0 T_0}{k_0^3} + a_1 x_0 y_0 = 0 \qquad \text{(A-17)}$$

$$p^0 : \frac{dy_0}{dt} + d_1 y_0 = 0 \qquad \text{(A-18)}$$

$$p^1 : \frac{dy_1}{dt} + d_1 y_0 - \beta_1 \alpha_1 x_0 y_0 + a_2 y_0 z_0 + \beta_{11} V_0 y_0 + b_1 y_0^2 = 0 \qquad \text{(A-19)}$$

$$p^0 : \frac{dz_0}{dt} + d_2 z_0 = 0 \qquad \text{(A-20)}$$

$$p^1 : \frac{dz_1}{dt} + d_2 z_0 - \beta_2 a_2 z_0 y_0 + \beta_{22} W_0 z_0 + c_3 z_0^2 = 0 \qquad \text{(A-21)}$$

$$p^0 : \frac{dT_0}{dt} - \delta_0 T_0 = 0 \qquad \text{(A-22)}$$

$$p^1 : \frac{dT_1}{dt} - \delta_0 T_1 - Q_0 + \alpha_1 x_0 T_0 = 0 \qquad \text{(A-23)}$$

$$p^0 : \frac{dU_0}{dt} + \delta_1 U_0 = 0 \qquad \text{(A-24)}$$

$$p^1 : \frac{dU_1}{dt} + \delta_1 U_1 - \alpha_1 x_0 T_0 + a_1 a_3 U_0 x_0 y_0 = 0 \qquad \text{(A-25)}$$

$$p^0 : \frac{dV_0}{dt} + \delta_2 V_0 = 0 \qquad \text{(A-26)}$$

$$p^1 : \frac{dV_1}{dt} + \delta_2 V_1 - a_1 a_3 U_0 x_0 y_0 + a_2 a_4 V_0 y_0 z_0 = 0 \qquad \text{(A-27)}$$

$$p^0 : \frac{dW_0}{dt} + \delta_3 W_0 = 0 \qquad \text{(A-28)}$$

$$p^1 : \frac{dW_1}{dt} + \delta_3 W_1 - a_2 a_4 V_0 y_0 z_0 = 0 \qquad \text{(A-29)}$$

$$x_0 = L e^{r_0 t} \qquad \text{(A-30)}$$

$$x_1(t) = [-\frac{r_1 LA}{\delta_1} + \frac{L^2}{k_0} - \frac{r_1 AL^2}{k_0(r_0-\delta_1)} + \frac{r_0 k_1 PL^2}{k_0^2(r_0-\delta_1)} - \frac{r_1 k_1 APL^2}{k_0^2(r_0-\delta_1+\delta_0)} + \frac{r_0 cL^2}{k_0} - \frac{r_0 r_1 cAL^2}{k_0(r_0-\delta_1)} + \frac{r^2 c k_1 PL^2}{k_0^2(r_0+\delta_0)} - \frac{r_1 r_0 cA k_1 PL^2}{k_0^2(r_0-\delta_1+\delta_0)} - \frac{r_0 cL^3}{2k_0^2} + \frac{r_0 r_1 cAL^3 A}{k_0^2(2r_0-\delta_1)} - \frac{aLM}{d_1} - \frac{2k_1 r_0^2 cL^3 P}{k_0^3(2r_0+\delta_0)} + \frac{2r_0 r_1 c k_1 AL^3 P}{k_0^3(2r_0-\delta_1+\delta_0)}]e^{r_0 t} + \frac{r_1 LA}{\delta_1} e^{(r_0-\delta_1)t} - \frac{L^2}{k_0} e^{2r_0 t} + \frac{r_1 AL^2}{k_0(r_0-\delta_1)} e^{(2r_0-\delta_1)t} - \frac{r_0 cL^2}{k_0} e^{2r_0 t} - \frac{r_0 k_1 PL^2}{k_0^2(r_0+\delta_0)} e^{(2r_0+\delta_0)t} + \frac{r_1 k_1 APL^2}{k_0^2(r_0-\delta_1+\delta_0)} e^{(2r_0-\delta_1+\delta_0)t} + \frac{r_0 r_1 cAL^2}{k_0(r_0-\delta_1)} e^{(2r_0-\delta_1)t} - \frac{r^2 c k_1 PL^2}{k_0^2(r_0+\delta_0)} e^{(2r_0+\delta_0)t} + \frac{aLM}{d_1} e^{(r_0-d_1)t} + \frac{r_0 cL^3}{2k_0^2} e^{3r_0 t} + \frac{r_1 r_0 cA k_1 PL^2}{k_0^2(r_0-\delta_1+\delta_0)} e^{(2r_0+\delta_0-\delta_1)t} - \frac{r_0 r_1 cAL^3 A}{k_0^2(2r_0-\delta_1)} e^{(3r_0-\delta_1)t} + \frac{2k_1 r_0^2 cL^3 P}{k_0^3(2r_0+\delta_0)} e^{(3r_0+\delta_0)t} - \frac{2r_0 r_1 c k_1 AL^3 P}{k_0^3(2r_0-\delta_1+\delta_0)} e^{(3r_0-\delta_1+\delta_0)t} \qquad \text{(A-31)}$$

$$y_0 = M e^{-d_1 t} \qquad \text{(A-32)}$$

$$y_1(t) = -[\frac{\beta_1 \alpha_1 LM}{r_0} + \frac{a_2 MN}{d_2} + \frac{\beta_{11} BM}{\delta_2} + \frac{b_1 M}{d_1}]e^{-d_1 t} + \frac{\beta_1 \alpha_1 LM}{r_0} e^{(r_0-d_1)t} + \frac{a_2 MN}{d_2} e^{-(d_1+d_2)t} + \frac{\beta_{11} BM}{\delta_2} e^{-(\delta_2+d_1)t} + \frac{b_1 M^2}{d_1} e^{-2d_1 t} \qquad \text{(A-33)}$$

$$z_0 = N e^{-d_2 t} \qquad \text{(A-34)}$$

$$z_1(t) = [\frac{\beta_2 a_2 MN}{d_1} - \frac{\beta_{22} DN}{\delta_3} - \frac{c_3 N^2}{d_2}]e^{-d_2 t} - \frac{\beta_2 a_2 MN}{d_1} e^{-(d_1+d_2)t} + \frac{c_3 N^2}{d_2} e^{-2d_2 t} + \frac{\beta_{22} DN}{\delta_3} e^{-(\delta_3+d_2)t} \qquad \text{(A-35)}$$

$$T_0 = P e^{\delta_0 t} \qquad \text{(A-36)}$$

$$T_1(t) = [\frac{Q_0}{\delta_0} + \frac{\alpha_1 LP}{r_0}]e^{\delta_0 t} - \frac{Q_0}{\delta_0} - \frac{\alpha_1 LP}{r_0} e^{(r_0+\delta_0)t} \qquad \text{(A-37)}$$

$$U_0 = A e^{-\delta_1 t} \qquad \text{(A-38)}$$

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$$U_1(t)=[\frac{-\alpha_1 LP}{r_0+\delta_0+\delta_1}+\frac{a_1a_3ALM}{r_0-d_1}]e^{-\delta_1 t} + \frac{\alpha_1 LP}{r_0+\delta_0+\delta_1}e^{(r_0+\delta_0)t}-\frac{a_1a_3ALM}{r_0-d_1}e^{(-\delta_1+r_0-d_1)t} \tag{A-39}$$

$$V_0 = Be^{-\delta_2 t} \tag{A-40}$$

$$V_1(t)=[-\frac{a_1a_3ALM}{r_0-d_1-\delta_1+\delta_2}-\frac{a_2a_4BMN}{d_1+d_2}]e^{-\delta_2 t} + \frac{a_1a_3ALM}{r_0-d_1-\delta_1+\delta_2}e^{(-\delta_1+r_0-d_1)t}+\frac{a_2a_4BMN}{d_1+d_2}e^{-(d_1+d_2+\delta_2)t} \tag{A-41}$$

$$W_0 = De^{-\delta_3 t} \tag{A-42}$$

$$W_1(t)=[\frac{a_2a_4BMN}{d_1+d_2+\delta_2-\delta_3}]e^{-\delta_3 t} + \frac{a_2a_4BMN}{\delta_3-d_1-d_2-\delta_2}e^{-(\delta_2+d_1+d_2)t} \tag{A-43}$$

$$x(t) = \lim_{p\to 1} x(t) = x_0 + x_1 + ... \tag{A-44}$$

$$y(t) = \lim_{p\to 1} y(t) = y_0 + y_1 + ... \tag{A-45}$$

$$z(t) = \lim_{p\to 1} z(t) = z_0 + z_1 + ... \tag{A-46}$$

$$T(t) = \lim_{p\to 1} T(t) = T_0 + T_1 + ... \tag{A-47}$$

$$U(t) = \lim_{p\to 1} U(t) = U_0 + U_1 + ... \tag{A-48}$$

$$V(t) = \lim_{p\to 1} V(t) = V_0 + V_1 + ... \tag{A-49}$$

$$W(t) = \lim_{p\to 1} W(t) = W_0 + W_1 + ... \tag{A-50}$$

After putting Eqns. (A-30) and (A-31) into Eqns. (A-44), (A-32) and (A-33) into Eqns. (A-45), (A-34) and (A-35) into Eqns. (A-46), (A-36) and (A-37) into Eqns. (A-47), (A-38) and (A-39) into Eqns. (A-48), (A-40) and (A-41) into Eqns. (A-49), (A-42) and (A-43) into Eqns. (A-50). We can obtain the final results which can be described in Eqns. (9-15).

APPENDIX B

MATLAB Program to find the numerical solution of the Eqns. (1)- (8)

```
function viji2
options= odeset('RelTol',1e-6,'Stats','on');
%initial conditions
Xo = [0.0001;0.0001;0.0001;0.0001;0.0001;0.0001;0.0001];
tspan = [0,20];
tic
[t,X] = ode45(@TestFunction,tspan,Xo,options);
toc
figure
hold on
plot(t, X(:,1), 'red')
plot(t, X(:,2), 'black')
plot(t, X(:,3), 'green')
plot(t, X(:,4), 'rose')
plot(t, X(:,5), 'yellow')
plot(t, X(:,6), 'black')
plot(t, X(:,7), 'm')
legend('x1','x2','x3','x4','x5','x6','x7')
ylabel('x - Population')
xlabel('t - Time')
return
function [dx_dt]= TestFunction(~,x)
r0=5.66; r1=11.0; k0=16.2; k1=3.0; c=4.25; c3=1.02;
alpha1=2.12; b1=1.231; beta1=1.2; beta11=1.1; beta2=1.6;
beta22=1.15; a1=3.22; a2=2.13; a3=2.865; a4=4.21;
delta0=7.52; delta1=2.5; delta2=3.99; delta3=1.3; Q0=2.988;
d1=1.45; d2=1.49;
dx_dt(1)=(x(1)*(r0-(r1*x(5))))*(((k0-(k1*x(4)))-x(1))/((k0-
(k1*x(4)))+(r0*c*x(1))))-(a1*x(1)*x(2));
dx_dt(2)=(beta1*a1*x(1)*x(2))-(a2*x(2)*x(3))-(beta11* x(6)*
x(2)) -(d1*x(2))-(b1*(x(2))^2);
dx_dt(3)=(beta2*a2*x(2)*x(3))-(beta22*x(7)*x(3))-(d2*x(3))-
(c3*(x(3))^2);
dx_dt(4) = Q0-( delta0*x(4))-(alpha1*x(1)*x(4));
dx_dt(5)=(alpha1*x(1)*x(4))-(delta1*x(5))-
(a3*x(5)*a1*x(1)*x(2));
dx_dt(6) =(a3*x(5)*a1*x(1)*x(2))-(delta2*x(6))-
(a4*x(6)*a2*x(2)*x(3));
dx_dt(7) =(a4*x(6)*a2*x(2)*x(3))-(delta3*x(7));
dx_dt = dx_dt';
return
```

APPENDIX C

NOMENCLATURE

SYMBOL	**MEANING**
x	prey population of density
y	intermediate predator population of density
z	top predator population of density
T	toxicant concentration in the environment
U	toxicant concentration in the prey population
V	toxicant concentration in the intermediate predator population
W	Toxicant concentration in the top predator population
$K(T)$	carrying capacity of prey
d_1	death rates of intermediate predator
d_2	death rates of top predator
$r(U)$	specific growth rate of prey population

β_1 and β_2	conversion coefficients
$\beta_3(U)$ and $\beta_4(U)$	toxicant transfer function
Q_0	toxicant rate of introduction into the environment
$\delta_0, \delta_1, \delta_2, \delta_3$	rate of toxicants in the environment as well as in the populations
β_{11} and β_{22}	death rates of predators due to organismal toxicant concentration
k_0	natural carrying capacity
k_1	rate of decrease carrying capacity
r_0	intrinsic growth rate of prey
r_1	growth rate of prey population
α_1	depletion rate of toxicant in the environment
c, a_1, a_2, a_3, a_4	positive constants
$\beta_3(U)$	$a_3 U$
$\beta_4(V)$	$a_4 V$
$r(U)$	$r_0 - r_1 U$
$K(T)$	$k_0 - k_1 T$

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