Conditional Inference for Gumbel Distribution Parameters Using Generalized Progressive Hybrid Censored Data

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Abstract: - It is widely recognized that conditional inference is often as efficient as Bayesian inference based on a non-informative prior. However, it is generally less efficient than Bayesian inference that incorporates an informative prior distribution. Accordingly, the primary objective of this study is to derive conditional point estimates for the parameters of the Gumbel distribution using pivotal quantities, based on generalized progressively hybrid-censored data. These estimates are compared with Bayesian estimates obtained under various loss functions through Monte Carlo simulations. The simulation results indicate that conditional inference is highly efficient and frequently yields better estimates than its Bayesian counterparts. Finally, a real dataset pertaining to a weather phenomenon is analyzed to demonstrate the effectiveness of the proposed methods and to confirm the suitability of the model for such data.

Key-Words: - Bayes inference; Conditional inference; Informative prior; Kernel prior.

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1 Introduction

In statistical inference, conditional inference is known to be as efficient as Bayesian inference, particularly when the sample size is small or when the data are heavily censored based on the non-informative prior. However, conditional inference is less efficient than Bayesian inference based on informative prior. Therefore, the main objective of this work is to derive, for the first time in the literature, conditional point estimates for the unknown parameters that are more efficient than Bayesian point estimates based on the informative priors. To illustrate this, we applied the proposed methods on the Gumbel distribution that known as the extreme value type-I distribution (EVD-I), which is one of the lifetime distributions commonly used in analyzing meteorological data. The probability density function (pdf) and the cumulative distribution function (cdf) are given, respectively, as follows:

$$f(x;\alpha,\beta) = \alpha\beta e^{\alpha x} exp(-\beta e^{\alpha x}), \quad x > 0 \quad (1)$$

$$F(x;\alpha,\beta) = 1 - exp(-\beta e^{\alpha x}). \quad x > 0 \quad (2)$$

$\alpha, \beta > 0$are the scale parameters.

The EVD-I has been extensively employed in the analysis of meteorological phenomena, including wind gusts, drought, rainfall, and floods. For a comprehensive review of the properties and applications of this distribution, see [2], [7]. It has received substantial attention in the statistical literature, largely due to its relationship with the Weibull distribution, which is widely used in lifetime data analysis. As a result, inferential methods based on complete and censored samples have been discussed by numerous authors. Notable contributions include those by [9], [10], [23], [24], and [14], [15] who developed conditional inference procedures for the extreme value and Weibull distributions.

In contrast, the type II extreme value distribution, which referred to as the Fréchet distribution, has received comparatively less attention, despite its flexibility in modeling various types of extreme data. Early work by [12] provided parameter estimations for the Fréchet distribution using methods such as the method of moments, the method of reciprocal moments, and maximum likelihood. Subsequently, [10] derived conditional maximum likelihood estimators (MLEs) for singly censored samples. [24] derived the MLEs based on the m-th extremes from multiple samples, while [25] extended this work by deriving the MLEs from the joint distribution of the largest m extremes across several samples. [11] derived the MLEs for parameters under doubly censored samples. More recently, [7] developed Bayesian estimation and prediction methods for future observations from the EVD-I under complete and censored sampling schemes.

In the context of reliability analysis, [4], [5] proposed the generalized progressive hybrid censoring scheme (GPHCS). This scheme modifies the progressive hybrid censoring scheme by allowing the experiment to continue beyond the pre-specified time point T if the number of observed failures is below a desired threshold, thereby guaranteeing at least a specified number of failures. The GPHCS can be described as follows:

Consider n identical items are placed on a test with considering $R_1, R_2, \dots, R_m$ are the random removal units, which are fixed at the beginning of the experiment with $m < n$, where $n = m + \sum_{i=1}^{m} R_i$. The terminated time T is also fixed beforehand with the integers k and m are pre-fixed such that $k < m$. In general, at the time of the i^{th} failure, R_I units will be removed randomly from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i \in [1, m]$. Thus, we have three scenarios:

1. If the time of the m^{th} failure occurs before the time point T, then the experiment will stop at the time point $X_{m:m:n}$ and all the remaining surviving units $R_m = n - m - \sum_{j=1}^{m-1} R_j$ will be removed.

2. If the m^{th} failure does not occur before the time point T and only k failures occur after the time point T, where $X_{m:m:n} > T$. Then at the time point $X_{k:m:n}$ the experiment terminates, and all the remaining surviving units $R_k = n - k - \sum_{j=1}^{k-1} R_j$ will be removed.

3. If the m^{th} failure does not occur before the time point $T$and only D failures occur at the time point T, where $X_{k:m:n} < T < X_{m:m:n}$. Then at the time point $X_{D:m:n}$ the experiment terminates, $R_T^* = n - D - \sum_{j=1}^{D} R_j$ will be removed.

Thus, given a generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\underline{X}; \theta) = C \prod_{i=1}^{n} f(x_i) [1 - F(x_i)]^{R_i} [1 - F(T)]^{\delta R_T^*}, \quad (3)$$

where $C = \prod_{i=1}^{n} \sum_{j=i}^{m} (R_j + 1)$,

$$n = \begin{cases} m, & \delta = 0, \text{ if } X_{k:m:n} \leq X_{m:m:n} < T \\ k, & \delta = 0, \text{ if } T < X_{K:m:n} \leq X_{m:m:n}, \\ D, & \delta = 1, \text{ if } X_{k:m:n} < T < X_{m:m:n} \end{cases}$$

where $\underline{X} = (X_1, X_2, \dots, X_n)$ and R_T^* is the number of surviving units that are removed at the stopping time $T^* = \max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$.

The GPHCS has been applied for some distributions such as the Weibull distribution, see [5], the inverse Weibull distribution, see [21], the exponential distribution, see [4], the Rayleigh distribution, see [3], and the shape-scale family, see [22].

1.1 Conditional inference

This section outlines the conditional inference, which transforms the standard likelihood function into a function that depends on pivotal quantities and ancillary statistics under the generalized progressive hybrid censored data. For further details, see [13], [14], [15] and [16], [17], who utilized these pivotal quantities to construct confidence intervals for the unknown parameters based on complete and censored samples. For the extreme value type-I distribution, the likelihood function given in (3) under the generalized progressive hybrid censored data can be expressed as follows:

$$L(\underline{X}: \alpha, \beta) = C(\alpha\beta)^n \exp[\alpha \sum_{i=1}^{n} x_i]$$

$$\times \exp\{-\beta[\textstyle\sum_{i=1}^{n}(1 + R_i) e^{\alpha x_i} + \delta R_T^* e^{\alpha T}]\}. \quad (4)$$

Since

$$\beta e^{\alpha x_i} = (\beta^{\frac{\hat{\alpha}}{\alpha}} e^{\hat{\alpha} x})^{\frac{\alpha}{\hat{\alpha}}} = (\beta^{\frac{\hat{\alpha}}{\alpha}} \hat{\beta} e^{\hat{\alpha} x_i} / \hat{\beta})^{\frac{\alpha}{\hat{\alpha}}} = (Z_2 a_i(x_i))^{Z_1}$$

Thus, $Z_1 = \alpha/\hat{\alpha}$ and $Z_2 = \beta^{1/z_1}/\hat{\beta}$ are pivotal quantities and $a_i(x) = \hat{\beta} e^{\hat{\alpha} x_i}$, for $i = 1, 2, \dots, n-2$, are the ancillary statistics.

Theorem:

Let $\hat{\alpha}$ and $\hat{\beta}$ be the maximum likelihood estimators of α and β based on the generalized progressive hybrid censored sample from the EVD-I (1). Thus, the joint conditional pdf of $Z_1 = \alpha/\hat{\alpha}$and $Z_2 = \beta^{1/z_1}/\hat{\beta}$ given $\underline{A} = (a_1, a_2 \dots, a_{n-2})$can be derived in the form:

$$f(z_1, z_2 | \underline{A}) = K z_1^{n-1} z_2^{nZ_1 - 1} \prod_{i=1}^{n} a_i^{z_1}$$

$$\times \exp[-z_2^{z_1} (\textstyle\sum_{i=1}^{n} (1 + R_i) a_i^{z_1} + \delta R_T^* a_T^{Z_1})], (5)$$

where K is the normalizing constant.

Proof:

Make the change of variables from $(x_1, x_2, x_3, \dots, x_n)$ that has joint density function (4) to $(\hat{\alpha}, \hat{\beta}, a_1, a_2 \dots, a_{n-2})$. This transformation can be written as follows:

$x_i = \ln(\frac{a_i}{\hat{\beta}})/\hat{\alpha}$, $i = 1,2, \dots, n-2$,

$x_{n-1} = \ln(\frac{a_{n-1}}{\hat{\beta}})/\hat{\alpha}$ and $x_n = \ln(\frac{a_n}{\hat{\beta}})/\hat{\alpha}$ where a_n and a_{n-1} can be expressed in terms of $a_1, a_2 \dots, a_{n-2}$.

The Jacobin of this transformation can be derived as follows:

$$|J| = \left|\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(\hat{\alpha}, \hat{\beta}, a_1, a_2, \dots, a_{n-2})}\right| = \begin{bmatrix} \frac{\partial(\ln(\frac{a_1}{\hat{\beta}})/\hat{\alpha})}{\partial\hat{\alpha}} & \cdots & \frac{\partial(\ln(\frac{a_n}{\hat{\beta}})/\hat{\alpha})}{\partial\hat{\alpha}} \\ \vdots & \ddots & \vdots \\ \frac{\partial(\ln(\frac{a_1}{\hat{\beta}})/\hat{\alpha})}{\partial a_{n-2}} & \cdots & \frac{\partial(\ln(\frac{a_n}{\hat{\beta}})/\hat{\alpha})}{\partial a_{n-2}} \end{bmatrix}$$

$= \prod_{i=1}^n \frac{e^{\hat{\alpha}x_i}}{\hat{\alpha}^2\hat{\beta}} = \frac{K(A,n)}{\hat{\alpha}^{2n}\hat{\beta}^n}\prod_{i=1}^n e^{\hat{\alpha}x_i} = K(A,n)$, which is independent of Z_1 and Z_2.

Therefore, the joint pdf of $\hat{\alpha}, \hat{\beta}, \underline{A}$ can be derived as follows:

$$f(\hat{\alpha}, \hat{\beta}; \underline{A}) = C(\alpha\beta)^N \prod_{i=1}^N \left(e^{\hat{\alpha}x_i}\right)^{\hat{\alpha}/\alpha}$$

$$\times \exp\{-[\sum_{i=1}^N (1+R_i)(\beta^{\frac{\hat{\alpha}}{\alpha}}\hat{\beta}e^{\hat{\alpha}x}/\hat{\beta})^{\frac{\alpha}{\hat{\alpha}}}\}$$

$$\times \exp[\delta R_T^*(\beta^{\frac{\hat{\alpha}}{\alpha}}\hat{\beta}e^{\hat{\alpha}T}/\hat{\beta})^{\frac{\alpha}{\hat{\alpha}}}]$$

Making further the change of variables from$(\hat{\alpha}, \hat{\beta}; \underline{A})$ into $(Z_1, Z_2; \underline{A})$, the Jacobin of this transformation can be derived as follows:

$$|J| = \left|\frac{\partial(\hat{\alpha}, \hat{\beta})}{\partial(z_1, z_2)}\right| = \begin{vmatrix} -\frac{\alpha}{z_1^2} & 0 \\ * & -\frac{\beta^{1/z_1}}{Z_2^2} \end{vmatrix} = \frac{\alpha\beta^{1/z_1}}{z_1^2 Z_2^2}$$

$= \frac{\hat{\alpha}\hat{\beta}}{Z_1 Z_2} \propto \frac{1}{Z_1 Z_2}$.

Thus,

$$f(z_1, z_2; \underline{A}) = CZ_1^{n-1}Z_2^{nZ_1-1}\prod_{i=1}^n a_i^{Z_1}$$

$$\times \exp[-z_2^{z_1}(\textstyle\sum_{i=1}^n (1+R_i)\, a_i^{z_1} + \delta R_T^* a_T^{Z_1})].$$

Finally, the joint conditional distribution function of (Z_1, Z_2) given $\underline{A}$ can be derived as in (5)■

Thus, based on (5), we can derive the marginal conditional distribution for each Z_1 and Z_2 given $\underline{A}$ respectively as follows:

$$f(z_1|\underline{A}) = K\int_0^\infty z_1^{n-1}z_2^{nZ_1-1}\prod_{i=1}^n a_i^{z_1}$$

$$\times \exp[-z_2^{z_1}(\sum_{i=1}^n (1+R_i)\, a_i^{z_1} + \delta R_T^* a_T^{Z_1})]dz_2$$

$$= K\Gamma(n)z_1^{n-2}\prod_{i=1}^n a_i^{z_1}$$

$$\times [\textstyle\sum_{i=1}^n (1+R_i)\, a_i^{z_1} + \delta R_T^* a_T^{Z_1}]^{-n}. \quad (6)$$

$$f(z_2|\underline{A}) = K\int_0^\infty z_1^{n-1}z_2^{nZ_1-1}\prod_{i=1}^n a_i^{z_1}$$

$$\times \exp[-z_2^{z_1}(\textstyle\sum_{i=1}^n (1+R_i)\, a_i^{z_1} + \delta R_T^* a_T^{Z_1})]dz_1. \quad (7)$$

The conditional estimators for the pivotal quantities Z_1 and Z_2 can be derived from equations (6) and (7), respectively, and subsequently transformed fiducially to obtain estimators for the parameters α and β. [19] derived a conditional inference approach for constructing confidence intervals for the unknown parameters based on pivotal quantities. The present work extends this methodology to derive point estimators for the unknown parameters. Specifically, to obtain a point estimator for the parameter α, we integrate (6) with respect to Z_1 over the interval from t_0 to t_1, as follows:

$$F(t_1|A) - F(t_0|A) = U - W(t_0), \quad (8)$$

and $W(t_0) = K\Gamma(n)\int_0^{t_0} z_1^{n-2}\prod_{i=1}^n a_i^{z_1}$

$$\times [\sum_{i=1}^n (1+R_i)\, a_i^{z_1} + \delta R_T^* a_T^{Z_1}]^{-n} dz_1$$

and U is a uniform random number from the uniform distribution $U(0,1)$.

It is known that the second order approximation of the first derivative $\frac{dF(t|\underline{A})}{dt}$, which is defined by the centered differencing as follows:

$\frac{dF(t|\underline{A})}{dt} = \frac{F(t_1|\underline{A}) - F(t_0|\underline{A})}{t_1 - t_0} = f(t_1^*|\underline{A})$, $t_1 < t_1^* < t_0$. (9)

From (8) and (9) we can derive the conditional estimator for Z_1 by the recurrence relation:

$t_{i+1} = t_i + C[U - W(t_i)]$.(10)

The convergence of (10) is guaranteed by the condition $C < \frac{K}{f(t_1^*|\underline{A})}$, where K is the normalizing constant. The iterative process is continued for $i = 0,1,2,3, \dots$ until two consecutive numerical solutions almost the same, that is if $|t_{i+1} - t_i| < 1E-05$. Thus, we can get the successive approximation for the pivotal Z_1 from (10) and transforming fiducially for α as follows: $\alpha^* = \hat{\alpha}\, z_{i+1}$. Similarly, the estimators for β can be derived from (7).

1 Bayes inference

In this section, the Bayes estimators for the parameters α and β will be derived using the informative gamma prior and the non-parametric kernel prior, based on two different loss functions.

Firstly, the squared error loss function (SQEL), $L(g(\theta), \hat{g}(\theta)) = (g(\theta) - \hat{g}(\theta))^2$. For this loss function the Bayes estimator that minimizes the risk function is given by $\hat{g}(\theta) = E_\theta(g(\theta)|x)$.

Secondly, the compound LINEX loss function, which is defined as follows:

$$L(\Delta) = L_\delta(\Delta) + L_{-\delta}(\Delta) = e^{\delta\Delta} + e^{-\delta\Delta} - 2,\ \delta > 0.$$

It is named LINEX-based loss function, where

$$L_\delta(\Delta) = \exp[\delta\Delta] - \delta\Delta - 1,\ \Delta = \hat{g}(\theta) - g(\theta),\ \delta \neq 0,$$

is the LINEX loss function (LLF) that has been introduced, see [26]. The Bayes estimator of the parameter θ that minimizes the risk function can be derived as follows:

$$\hat{g}_L(\theta) = \frac{1}{2\delta}\log\left(\frac{E\left(e^{\delta g(\theta)}|X\right)}{E\left(e^{-\delta g(\theta)}|X\right)}\right).$$

1.1 Informative Gamma Prior

We consider the unknown parameters α and β have independent gamma prior distributions with the joint probability density function, which is given by:

$$h(\alpha, \beta) \propto \alpha^{a-1}\beta^{c-1}e^{-b\alpha - d\beta}, \qquad (11)$$

where the hyper-parameter a, b, c and d are assumed to be known non-negative real numbers and chosen to reflect the prior belief about the unknown parameters.

1.2 Informative Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $g(\alpha, \beta)$ with support on $(0, \infty)$, which is defined as follows:

$$\hat{g}(\alpha, \beta) = \frac{1}{n h_1 h_2}\sum_{i=1}^{n} K\left(\frac{\alpha - \alpha_i}{h_1}, \frac{\beta - \beta_i}{h_2}\right), \qquad (12)$$

$h_i, i = 1,2$are called the bandwidths or smoothing parameters. Based on the properties of the maximum likelihood estimates (MLEs) for the parameters, which converge in probability to the original parameters. The kernel prior has been used in [1] and [20]. Thus, using the joint priors (11) and (12) with the likelihood function of the GPHCS (3) the posterior density for the parameters α and β can be written in a unified form as follows:

$f(\alpha, \beta|\underline{X}) = Kq(\alpha, \beta)L(\underline{X}; \alpha, \beta)$, where

$$q(\alpha, \beta) = \hat{g}(\alpha, \beta)h(\alpha, \beta),$$

$$= \hat{g}_1^{p_1}(\alpha)\hat{g}_2^{p_2}(\beta)\,\alpha^{a-1}\beta^{c-1}e^{-b\alpha - d\beta}, \quad (13)$$

is the general prior distribution function with $p_1 = p_2 = 0$ for the informative gamma prior (11), and $p_1 = p_2 = 1$, $a = c = 1$, and $b = d = 0$ for the informative kernel prior (12).

Using the general prior (13) and the likelihood function of the GPHCS (3) that can be written as follows:

$$L(\alpha, \beta; x) = C(\alpha\beta)^n \exp[\alpha\sum_{i=1}^{n} x_i]$$

$$\times \exp\{-\beta[\textstyle\sum_{i=1}^{n}(1 + R_i)e^{\alpha x_i} + \delta R_T^* e^{\alpha T}]\}. \ (14)$$

Thus, the posterior density can be derived as

$$f(\alpha, \beta|\underline{X}) = C\hat{g}_1^{p_1}(\alpha)\hat{g}_2^{p_2}(\beta)\alpha^{n+a-1}\beta^{n+c-1}$$

$$\times \exp[-\alpha\left(b - \sum_{i=1}^{n} x_i\right)]$$

$$\times \exp\{-\beta[d + \textstyle\sum_{i=1}^{n}(1 + R_i)e^{\alpha x_i} + \delta R_T^* e^{\alpha T}]\}$$

3 Simulation Study

The purpose of the simulation study is to compare the performance of the estimators using the Bayesian and conditional methods with two different loss functions, through the avarage bias and the mean squared error (MSE) as given by:

$AVB = \frac{1}{L}\sum_{i=1}^{L}|\hat{\theta}_i - \theta|$and $MSE(\theta^*) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study, we chose the distribution hyper parameters of α and β as (a, b, c, d) = (5, 3, 5, 3), and the parameter values as α = (1, 2) and β = (1.5, 3). Using the above parameter values, different samples were generated from the EXVD-I distribution with sample sizes n = 20, 40, and 60 to represent small, moderate, and large sample sizes. To assess the performance of the estimates, the mean squared errors (MSEs) for each estimate were calculated using 1000 replications.

From the simulation results presented in Tables 3, 4, 5, and 6, several key observations can be made. These findings are summarized in the following main points:

1. In general, the point estimates based on the conditional method yielded the smallest MSE values compared to those based on the Bayesian method under both loss functions.
2. The estimated MSE increases as the value of α increases and decreases as the value of β increases.
3. The estimated MSE decreases when the hyper parameters of the informative priors are reduced.
4. As expected, the estimated MSE decreases with increasing the termination time T of the experiment.

5. It is evident that as the sample size increases, the estimated MSE decreases.
6. Overall, for the parameters α and β, the estimated MSEs from the conditional method are consistently lower than those from the Bayesian method.

In conclusion, the point estimates obtained using the conditional inference perform competitively and outperform those from the Bayesian method under both loss functions.

4 Real Data Applications

In this section, we analyzed two real datasets to evaluate the performance of the proposed methods for the extreme value distribution (EVD-I), which is one of the most desirable and extensively used lifetime distributions for analyzing weather phenomena such as wind gusts, droughts, rainfall, and floods. We assessed the fit of these datasets using goodness-of-fit tests, including the Kolmogorov–Smirnov (K-S), Anderson–Darling (A-D), and chi-squared (χ^2) tests, at a significance level of 0.05.

4.1 Flood Data

Consider the data given by [6], which represent the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania, over 20 four-year periods (1890–1969) as follows:

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

We found the extreme value distribution (EVD-I) to be a good fit for this dataset, as shown in Table 1 and Figure (1a). To study the behavior of the flood levels based on this dataset, we obtained estimates for the parameters that represent the shape and scale of the flood level distribution for the 20 four-year periods. We noticed that the conditional and Bayes estimates for α are greater than one, while those for β are close to zero. This indicates that the distribution is heavy-tailed and not symmetric, as seen in Figure (1b). These estimates suggest that the maximum flood levels exhibit a higher likelihood of extreme events during these periods.

4.2 Reactor Pumps Data

A real dataset for secondary nuclear pumps has been analyzed to illustrate the proposed methods. Safety is a critical aspect of nuclear energy. One of the most severe accidents in nuclear power generation is a loss-of-coolant accident (LOCA), in which the recirculating coolant in a pressurized water reactor may flash into steam. Under such conditions, the reactor coolant pumps are unable to generate the same head as they would in single-phase flow.

The secondary reactor pump is a feed water pump that draws feed water from the deaerator storage tank. The water is pressurized by a booster pump and then pushed into the steam generator via a high-pressure heater. Consequently, the main feed pump must be capable of operating at high temperatures and high pressures, as it must generate a head greater than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been specifically developed for cooling at elevated temperatures.

The following dataset represents the times between failures of the secondary reactor pumps. Classical and Bayesian estimation methods under a Type-II censoring scheme for this dataset were discussed in [24], [25]. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found that the extreme value distribution (EVD-I) provides a good fit for this dataset, as shown in Table 1 and Figure (2a). To study the reliability of these reactor pumps, we estimated the parameters representing the shape and scale of the failure times using our model to characterize the failure behavior. The conditional and Bayes estimates for α were found to be close to 1, and those for β were close to 0.5. These estimates indicate that the dataset is heavily right-skewed, implying that the failure rate decreases as time increases, as illustrated in Figure (2b). We therefore conclude that the reliability of the safety mechanisms decreases with time.

5 Conclusion

It is known that Bayesian methods based on informative priors are more efficient than most approaches in statistical inference. However, the use of pivotal quantities can lead to greater efficiency in conditional inference compared to Bayesian method based on informative priors under different loss functions. When these methods are applied to the extreme value distribution (EVD-I), they demonstrate considerable flexibility in fitting various types of extreme data.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

M. Maswadah has implemented the idea and the statistics of this work.

A. A. H. Ahmadini has carried out the simulations and revision of this work.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

APPENDIX

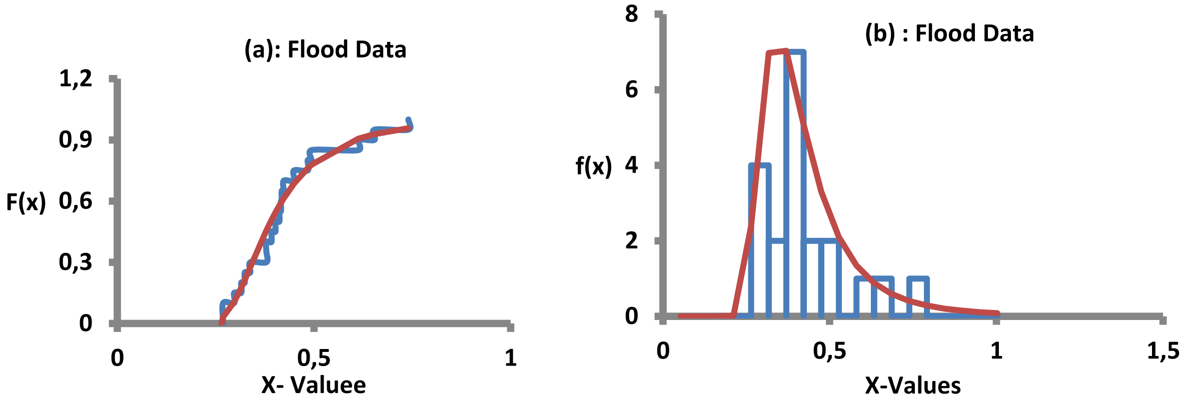

Fig. 1: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF

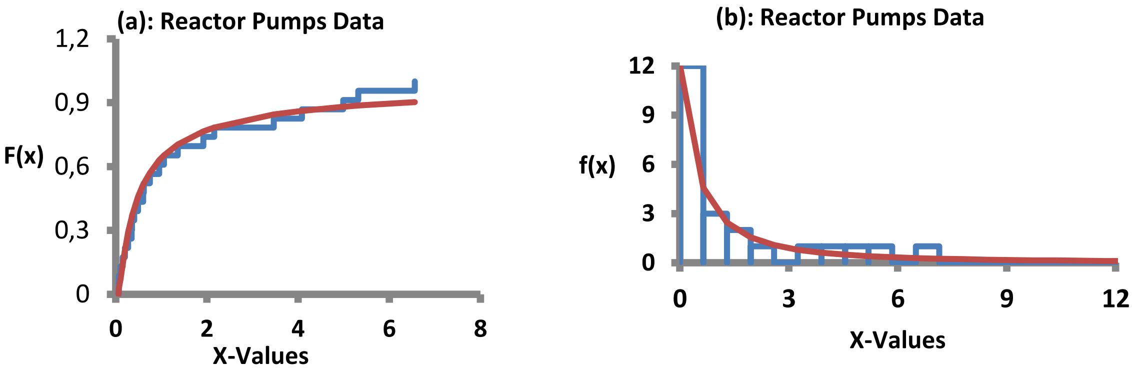

Fig. 2: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF

Table 1. The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p-values) for the inverse Weibull model. The MLE's for the parameters for these data sets have been calculated.

Data	The Tests	Calculated value	Critical value	The p-values	MLES	
					α	β
Flood N=20	**K-S**	**0.6976**	**0.8482**	**0.2138**	**4.3141**	**0.0119**
	A-D	**0.3104**	**0.7414**	**0.5899**		
	CH2	**3.5552**	**31.1109**	**0.3294**		
Reactor pumps N=23	**K-S**	**0.4741**	**0.8528**	**0.8113**	**0.7832**	**0.4463**
	A-D	**0.3443**	**0.7472**	**0.4915**		
	CH2	**10.270**	**31.5744**	**0.1223**		

Table 2. The estimate and the mean square errors (MSEs) for the parameter α and β based on the conditional and Bayes' method for the GHPCS: for $m = n/2$, and $k = 3m/4$.

Samples	T	Parameters	Conditional estimates		Bayesian estimates		MLE
			Estimate	MSE	Estimate	MSE	
Flood Data N=20	**0.2**	α	**3.86273**	**0.29580**	**2.83818**	**0.72875**	**3.56693**
		β	**0.03308**	**0.00370**	**0.08185**	**0.05248**	**0.02937**
	0.5	α	**4.40492**	**0.31333**	**2.97248**	**1.11911**	**4.09159**
		β	**0.01793**	**0.00239**	**0.06327**	**0.04773**	**0.01554**
Reactor Pumps Data N=23	**0.5**	α	**0.83827**	**0.00877**	**0.81013**	**0.01936**	**0.91589**
		β	**0.40949**	**0.03295**	**0.41393**	**0.03738**	**0.37655**
	5.0	α	**0.87217**	**0.00889**	**0.84806**	**0.01522**	**0.86328**
		β	**0.38419**	**0.03304**	**0.37928**	**0.02813**	**0.35115**

Table 3. The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the EXVD parameter α using the Conditional and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

N	m	k	α	β	Conditional Estimations	Gamma Prior		Kernel Prior	
						SQEL	LNXL	SQEL	LNXL
20	10	5	1	1.5	0.0795(0.0088)	0.3031(0.1687)	0.2296(0.0906)	0.1830(0.0572)	0.1806(0.0546)
				3	0.0840(0.0137)	0.1380(0.0310)	0.1234(0.0239)	0.1223(0.0244)	0.1257(0.0264)
			2	1.5	0.1663(0.0575)	0.6536(0.7780)	0.2866(0.1358)	0.2440(0.0937)	0.2457(0.0942)
				3	0.1609(0.0461)	0.2720(0.1241)	0.3002(0.1253)	0.1904(0.0582)	0.2037(0.0651)
		8	1	1.5	0.0832(0.0127)	0.3229(0.1881)	0.2351(0.0914)	0.1918(0.0627)	0.1837(0.0549)
				3	0.0794(0.0088)	0.1377(0.0320)	0.1215(0.0227)	0.1223(0.0249)	0.1214(0.0242)
			2	1.5	0.1520(0.0263)	0.5892(0.5937)	0.2953(0.1418)	0.2368(0.0857)	0.2511(0.0971)
				3	0.1607(0.0457)	0.2735(0.1200)	0.2924(0.1210)	0.1933(0.0573)	0.2045(0.0649)
	15	8	1	1.5	0.0798(0.0089)	0.2639(0.1229)	0.2187(0.0858)	0.1716(0.0493)	0.1729(0.0526)
				3	0.0803(0.0097)	0.1380(0.0320)	0.1262(0.0252)	0.1224(0.0243)	0.1263(0.0265)
			2	1.5	0.1514(0.0229)	0.5122(0.4734)	0.2749(0.1219)	0.2295(0.0812)	0.2344(0.0834)
				3	0.1570(0.0377)	0.2814(0.1334)	0.2887(0.1190)	0.1940(0.0593)	0.1998(0.0620)
		11	1	1.5	0.0790(0.0079)	0.2609(0.1269)	0.2168(0.0832)	0.1705(0.0500)	0.1723(0.0512)
				3	0.0803(0.0098)	0.1400(0.0323)	0.1194(0.0225)	0.1238(0.0246)	0.1243(0.0253)
			2	1.5	0.1606(0.0424)	0.5525(0.5499)	0.2863(0.1362)	0.2434(0.0910)	0.2402(0.0885)
				3	0.1534(0.0301)	0.2805(0.1305)	0.2764(0.1066)	0.1961(0.0597)	0.1935(0.0583)
40	20	10	1	1.5	0.0769(0.0059)	0.1974(0.0653)	0.1620(0.0441)	0.1372(0.0303)	0.1342(0.0294)
				3	0.0767(0.0059)	0.1037(0.0171)	0.0997(0.0155)	0.0916(0.0133)	0.0923(0.0139)
			2	1.5	0.1505(0.0227)	0.3719(0.2376)	0.2338(0.0880)	0.1910(0.0565)	0.1933(0.0591)
				3	0.1498(0.0224)	0.1995(0.0653)	0.2396(0.0810)	0.1509(0.0358)	0.1593(0.0389)
		15	1	1.5	0.0769(0.0059)	0.1886(0.0616)	0.1622(0.0453)	0.1313(0.0284)	0.1316(0.0293)
				3	0.0767(0.0059)	0.1064(0.0182)	0.0971(0.0148)	0.0942(0.0141)	0.0923(0.0137)
			2	1.5	0.1505(0.0226)	0.3854(0.2651)	0.2348(0.0916)	0.1962(0.0617)	0.1917(0.0593)
				3	0.1498(0.0224)	0.2128(0.0731)	0.2354(0.0788)	0.1612(0.0403)	0.1586(0.0393)
	30	15	1	1.5	0.0772(0.0060)	0.1631(0.0456)	0.1453(0.0362)	0.1230(0.0246)	0.1220(0.0246)
				3	0.0767(0.0059)	0.1057(0.0181)	0.0930(0.0134)	0.0941(0.0142)	0.0891(0.0126)
			2	1.5	0.1509(0.0228)	0.3779(0.2376)	0.2336(0.0897)	0.1943(0.0595)	0.1925(0.0589)
				3	0.1498(0.0224)	0.2005(0.0649)	0.2233(0.0721)	0.1507(0.0363)	0.1571(0.0381)
		23	1	1.5	0.0773(0.0060)	0.1637(0.0456)	0.1526(0.0403)	0.1265(0.0261)	0.1316(0.0286)
				3	0.0768(0.0059)	0.1002(0.0165)	0.0978(0.0144)	0.0892(0.0129)	0.0937(0.0138)
			2	1.5	0.1523(0.0232)	0.3216(0.1761)	0.2321(0.0876)	0.1882(0.0555)	0.1952(0.0598)
				3	0.1501(0.0225)	0.2089(0.0714)	0.2103(0.0656)	0.1547(0.0380)	0.1557(0.0377)
60	30	15	1	1.5	0.0768(0.0059)	0.1518(0.0390)	0.1322(0.0300)	0.1102(0.0197)	0.1114(0.0205)
				3	0.0767(0.0059)	0.0865(0.0120)	0.0889(0.0116)	0.0756(0.0091)	0.0792(0.0097)
			2	1.5	0.1504(0.0226)	0.3038(0.1583)	0.2089(0.0713)	0.1726(0.0478)	0.1718(0.0473)
				3	0.1498(0.0224)	0.1827(0.0545)	0.1986(0.0570)	0.1386(0.0306)	0.1304(0.0269)
		23	1	1.5	0.0768(0.0059)	0.1467(0.0363)	0.1325(0.0286)	0.1074(0.0187)	0.1089(0.0190)
				3	0.0767(0.0059)	0.0869(0.0118)	0.0844(0.0108)	0.0764(0.0089)	0.0746(0.0089)
			2	1.5	0.1504(0.0226)	0.3059(0.1618)	0.2159(0.0751)	0.1772(0.0495)	0.1773(0.0491)
				3	0.1498(0.0224)	0.1743(0.0490)	0.2007(0.0573)	0.1338(0.0279)	0.1336(0.0277)
	45	23	1	1.5	0.0773(0.0060)	0.1242(0.0269)	0.1214(0.0244)	0.1020(0.0171)	0.1058(0.0183)
				3	0.0767(0.0059)	0.0892(0.0127)	0.0822(0.0106)	0.0782(0.0096)	0.0769(0.0094)
			2	1.5	0.1518(0.0230)	0.2550(0.1085)	0.1953(0.0610)	0.1674(0.0434)	0.1650(0.0429)
				3	0.1499(0.0225)	0.1715(0.0487)	0.1825(0.0493)	0.1326(0.0280)	0.1361(0.0283)
		34	1	1.5	0.0773(0.0060)	0.1218(0.0250)	0.1189(0.0240)	0.1018(0.0169)	0.1058(0.0182)
				3	0.0767(0.0059)	0.0889(0.0128)	0.0816(0.0102)	0.0779(0.0096)	0.0757(0.0090)
			2	1.5	0.1518(0.0231)	0.2615(0.1179)	0.1990(0.0652)	0.1710(0.0460)	0.1669(0.0448)
				3	0.1499(0.0225)	0.1801(0.0510)	0.1794(0.0474)	0.1368(0.0290)	0.1337(0.0279)

Table 4. The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the EXVD parameter α using th Conditional and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and $\delta = 2$ for LINEX loss function.

N	M	k	α	β	Conditional Estimations	Gamma Prior		Kernel Prior	
						SQEL	LNXL	SQEL	LNXL
20	10	5	1	1.5	0.0786(0.0078)	0.2944(0.1587)	0.2296(0.0906)	0.1791(0.0548)	0.1806(0.0546)
				3	0.0776(0.0069)	0.1395(0.0310)	0.1234(0.0239)	0.1241(0.0244)	0.1257(0.0264)
			2	1.5	0.1611(0.0459)	0.5682(0.5852)	0.2866(0.1358)	0.2293(0.0835)	0.2457(0.0942)
				3	0.1588(0.0418)	0.2805(0.1282)	0.3002(0.1253)	0.1997(0.0613)	0.2037(0.0651)
		8	1	1.5	0.0795(0.0088)	0.2867(0.1487)	0.2351(0.0914)	0.1748(0.0521)	0.1837(0.0549)
				3	0.0831(0.0127)	0.1399(0.0310)	0.1215(0.0227)	0.1248(0.0245)	0.1214(0.0242)
			2	1.5	0.1668(0.0580)	0.6025(0.6710)	0.2953(0.1418)	0.2433(0.0920)	0.2511(0.0971)
				3	0.1570(0.0379)	0.2753(0.1218)	0.2924(0.1210)	0.1944(0.0579)	0.2045(0.0649)
	15	8	1	1.5	0.0804(0.0089)	0.2250(0.0939)	0.1920(0.0627)	0.1673(0.0476)	0.1576(0.0415)
				3	0.0831(0.0126)	0.1515(0.0390)	0.1189(0.0226)	0.1319(0.0284)	0.1213(0.0238)
			2	1.5	0.1591(0.0347)	0.4455(0.3564)	0.2686(0.1180)	0.2280(0.0816)	0.2238(0.0790)
				3	0.1541(0.0303)	0.2870(0.1330)	0.2466(0.0887)	0.1979(0.0591)	0.1963(0.0598)
		11	1	1.5	0.0830(0.0116)	0.2132(0.0834)	0.1903(0.0646)	0.1595(0.0426)	0.1570(0.0424)
				3	0.0804(0.0095)	0.1423(0.0346)	0.1294(0.0274)	0.1247(0.0255)	0.1320(0.0288)
			2	1.5	0.1625(0.0420)	0.4271(0.3483)	0.2632(0.1130)	0.2214(0.0773)	0.2198(0.0748)
				3	0.1631(0.0491)	0.2965(0.1500)	0.2434(0.0881)	0.2003(0.0618)	0.1946(0.0588)
40	20	10	1	1.5	0.0769(0.0059)	0.2023(0.0709)	0.1640(0.0452)	0.1384(0.0318)	0.1319(0.0293)
				3	0.0767(0.0059)	0.1067(0.0184)	0.0976(0.0147)	0.0947(0.0144)	0.0905(0.0130)
			2	1.5	0.1505(0.0226)	0.3804(0.2444)	0.2366(0.0889)	0.1992(0.0616)	0.1963(0.0596)
				3	0.1498(0.0224)	0.2127(0.0700)	0.2376(0.0796)	0.1614(0.0396)	0.1597(0.0401)
		15	1	1.5	0.0768(0.0059)	0.1927(0.0630)	0.1666(0.0455)	0.1324(0.0286)	0.1351(0.0296)
				3	0.0767(0.0059)	0.1025(0.0171)	0.0979(0.0150)	0.0908(0.0132)	0.0908(0.0137)
			2	1.5	0.1505(0.0227)	0.3949(0.2724)	0.2426(0.0967)	0.2001(0.0633)	0.2021(0.0647)
				3	0.1498(0.0224)	0.2055(0.0672)	0.2393(0.0815)	0.1540(0.0372)	0.1551(0.0379)
	30	15	1	1.5	0.0778(0.0061)	0.1350(0.0327)	0.1222(0.0252)	0.1155(0.0228)	0.1077(0.0191)
				3	0.0769(0.0059)	0.0982(0.0161)	0.0922(0.0136)	0.0869(0.0124)	0.0901(0.0132)
			2	1.5	0.1541(0.0238)	0.2712(0.1310)	0.2099(0.0724)	0.1798(0.0514)	0.1797(0.0510)
				3	0.1507(0.0227)	0.2130(0.0715)	0.1827(0.0505)	0.1584(0.0377)	0.1484(0.0346)
		23	1	1.5	0.0778(0.0061)	0.1370(0.0327)	0.1259(0.0274)	0.1174(0.0227)	0.1113(0.0210)
				3	0.0769(0.0059)	0.1059(0.0188)	0.0923(0.0137)	0.0945(0.0146)	0.0909(0.0133)
			2	1.5	0.1546(0.0239)	0.2589(0.1194)	0.2099(0.0728)	0.1724(0.0484)	0.1797(0.0517)
				3	0.1507(0.0227)	0.1952(0.0602)	0.1907(0.0553)	0.1455(0.0328)	0.1543(0.0375)
60	30	15	1	1.5	0.0768(0.0059)	0.1450(0.0362)	0.1316(0.0298)	0.1085(0.0191)	0.1102(0.0201)
				3	0.0767(0.0059)	0.0865(0.0121)	0.0850(0.0111)	0.0761(0.0093)	0.0763(0.0093)
			2	1.5	0.1504(0.0226)	0.2937(0.1478)	0.2102(0.0698)	0.1683(0.0446)	0.1722(0.0461)
				3	0.1498(0.0224)	0.1862(0.0548)	0.2022(0.0589)	0.1413(0.0309)	0.1371(0.0292)
		23	1	1.5	0.0769(0.0059)	0.1451(0.0371)	0.1319(0.0294)	0.1071(0.0191)	0.1103(0.0200)
				3	0.0767(0.0059)	0.0939(0.0138)	0.0842(0.0107)	0.0823(0.0104)	0.0751(0.0088)
			2	1.5	0.1504(0.0226)	0.3031(0.1574)	0.2154(0.0757)	0.1724(0.0473)	0.1781(0.0499)
				3	0.1498(0.0224)	0.1838(0.0518)	0.2026(0.0602)	0.1409(0.0297)	0.1341(0.0281)
	45	23	1	1.5	0.0779(0.0061)	0.0970(0.0163)	0.0992(0.0169)	0.0873(0.0128)	0.0909(0.0138)
				3	0.0769(0.0059)	0.0839(0.0114)	0.0778(0.0096)	0.0740(0.0087)	0.0745(0.0088)
			2	1.5	0.1544(0.0238)	0.2110(0.0772)	0.1692(0.0467)	0.1588(0.0404)	0.1464(0.0344)
				3	0.1507(0.0227)	0.1719(0.0464)	0.1632(0.0420)	0.1318(0.0265)	0.1361(0.0293)
		34	1	1.5	0.0778(0.0061)	0.1048(0.0184)	0.1005(0.0167)	0.0938(0.0143)	0.0921(0.0137)
				3	0.0769(0.0059)	0.0884(0.0125)	0.0798(0.0101)	0.0791(0.0098)	0.0766(0.0094)
			2	1.5	0.1544(0.0238)	0.2115(0.0752)	0.1777(0.0501)	0.1569(0.0389)	0.1550(0.0376)
				3	0.1507(0.0227)	0.1763(0.0490)	0.1627(0.0405)	0.1353(0.0280)	0.1350(0.0288)

Table 5. The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the EXVD parameter β using the Conditional and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

N	M	k	α	β	Conditional Estimations	Gamma Prior		Kernel Prior	
						SQEL	LNXL	SQEL	LNXL
20	10	5	1	1.5	0.1321(0.0174)	0.7275(0.7573)	0.4021(0.2328)	0.2572(0.0942)	0.2612(0.0973)
				3	0.1752(0.0524)	0.2789(0.1657)	0.6927(0.5288)	0.1537(0.0488)	0.1657(0.0535)
			2	1.5	0.1426(0.0239)	0.7685(0.8105)	0.3133(0.1470)	0.2396(0.0834)	0.2504(0.0909)
				3	0.1878(0.0353)	0.2874(0.1719)	0.8333(0.7443)	0.1755(0.0598)	0.1779(0.0592)
		8	1	1.5	0.1361(0.0237)	0.7383(0.7856)	0.4262(0.2624)	0.2696(0.1026)	0.2734(0.1064)
				3	0.1669(0.0279)	0.2527(0.1279)	0.6811(0.5151)	0.1453(0.0407)	0.1628(0.0516)
			2	1.5	0.1396(0.0195)	0.7095(0.7336)	0.2961(0.1287)	0.2426(0.0846)	0.2393(0.0818)
				3	0.1930(0.0514)	0.2820(0.1680)	0.8259(0.7285)	0.1791(0.0646)	0.1713(0.0540)
	15	8	1	1.5	0.1319(0.0174)	0.5634(0.5272)	0.3682(0.2148)	0.2383(0.0852)	0.2534(0.0953)
				3	0.1751(0.0525)	0.2521(0.1186)	0.6269(0.4460)	0.1526(0.0438)	0.1529(0.0463)
			2	1.5	0.1393(0.0194)	0.5458(0.4963)	0.3025(0.1434)	0.2161(0.0717)	0.2449(0.0886)
				3	0.1983(0.0697)	0.2676(0.1343)	0.7827(0.6666)	0.1759(0.0577)	0.1737(0.0570)
		11	1	1.5	0.1318(0.0174)	0.5903(0.5708)	0.3740(0.2259)	0.2537(0.0938)	0.2550(0.0986)
				3	0.1724(0.0449)	0.2700(0.1401)	0.6015(0.4247)	0.1651(0.0525)	0.1607(0.0549)
			2	1.5	0.1420(0.0236)	0.5871(0.5650)	0.3039(0.1427)	0.2317(0.0802)	0.2442(0.0866)
				3	0.1897(0.0433)	0.2620(0.1340)	0.7373(0.6031)	0.1712(0.0574)	0.1750(0.0592)
40	20	10	1	1.5	0.1321(0.0175)	0.4781(0.4020)	0.3211(0.1733)	0.2190(0.0725)	0.2190(0.0726)
				3	0.1671(0.0279)	0.2648(0.1235)	0.5015(0.3212)	0.1556(0.0454)	0.1578(0.0459)
			2	1.5	0.1396(0.0195)	0.4695(0.3905)	0.2580(0.1109)	0.1999(0.0599)	0.1984(0.0603)
				3	0.1872(0.0351)	0.2646(0.1131)	0.6198(0.4497)	0.1623(0.0456)	0.1606(0.0463)
		15	1	1.5	0.1321(0.0175)	0.4590(0.3831)	0.3222(0.1782)	0.2114(0.0688)	0.2160(0.0724)
				3	0.1672(0.0280)	0.2837(0.1374)	0.4892(0.3114)	0.1692(0.0511)	0.1558(0.0457)
			2	1.5	0.1395(0.0195)	0.4797(0.4038)	0.2690(0.1208)	0.1974(0.0600)	0.2058(0.0647)
				3	0.1873(0.0351)	0.2780(0.1323)	0.6336(0.4697)	0.1692(0.0519)	0.1693(0.0510)
	30	15	1	1.5	0.1318(0.0174)	0.3978(0.2907)	0.2905(0.1471)	0.2111(0.0677)	0.2109(0.0692)
				3	0.1669(0.0279)	0.2722(0.1249)	0.4242(0.2511)	0.1603(0.0470)	0.1595(0.0448)
			2	1.5	0.1395(0.0195)	0.4643(0.3734)	0.2388(0.0971)	0.2059(0.0630)	0.1913(0.0565)
				3	0.1869(0.0350)	0.2721(0.1183)	0.5280(0.3593)	0.1646(0.0471)	0.1734(0.0526)
		23	1	1.5	0.1318(0.0174)	0.3607(0.2461)	0.2744(0.1324)	0.2025(0.0636)	0.2018(0.0644)
				3	0.1668(0.0278)	0.2803(0.1246)	0.4049(0.2373)	0.1702(0.0499)	0.1647(0.0473)
			2	1.5	0.1391(0.0193)	0.3623(0.2563)	0.2425(0.1029)	0.1936(0.0592)	0.1956(0.0605)
				3	0.1866(0.0348)	0.2779(0.1222)	0.5040(0.3346)	0.1736(0.0518)	0.1767(0.0531)
60	30	15	1	1.5	0.1321(0.0174)	0.3779(0.2663)	0.2718(0.1300)	0.1937(0.0578)	0.1900(0.0568)
				3	0.1672(0.0279)	0.2781(0.1227)	0.4014(0.2332)	0.1602(0.0452)	0.1584(0.0435)
			2	1.5	0.1396(0.0195)	0.3674(0.2595)	0.2308(0.0934)	0.1792(0.0507)	0.1792(0.0506)
				3	0.1872(0.0351)	0.2914(0.1314)	0.4980(0.3221)	0.1677(0.0485)	0.1571(0.0444)
		23	1	1.5	0.1320(0.0174)	0.3708(0.2621)	0.2741(0.1296)	0.1913(0.0577)	0.1914(0.0567)
				3	0.1670(0.0279)	0.2762(0.1210)	0.4058(0.2354)	0.1571(0.0435)	0.1597(0.0433)
			2	1.5	0.1396(0.0195)	0.3707(0.2657)	0.2458(0.1074)	0.1793(0.0510)	0.1880(0.0564)
				3	0.1873(0.0351)	0.2866(0.1262)	0.5170(0.3451)	0.1684(0.0479)	0.1667(0.0496)
	45	23	1	1.5	0.1317(0.0173)	0.2862(0.1579)	0.2221(0.0866)	0.1807(0.0521)	0.1733(0.0477)
				3	0.1669(0.0278)	0.2939(0.1261)	0.3527(0.1907)	0.1699(0.0474)	0.1734(0.0510)
			2	1.5	0.1391(0.0194)	0.2873(0.1567)	0.2092(0.0760)	0.1719(0.0459)	0.1711(0.0467)
				3	0.1867(0.0349)	0.2914(0.1258)	0.4185(0.2479)	0.1740(0.0495)	0.1742(0.0504)
		34	1	1.5	0.1317(0.0173)	0.2629(0.1288)	0.2207(0.0880)	0.1697(0.0456)	0.1721(0.0478)
				3	0.1669(0.0278)	0.2937(0.1300)	0.3416(0.1801)	0.1719(0.0501)	0.1667(0.0463)
			2	1.5	0.1392(0.0194)	0.2951(0.1651)	0.2105(0.0763)	0.1759(0.0481)	0.1721(0.0464)
				3	0.1868(0.0349)	0.2987(0.1291)	0.4037(0.2351)	0.1755(0.0508)	0.1720(0.0490)

Table 6. The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the EXVD parameter β using the Conditional and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and $\delta = 2$ for LINEX loss function.

N	m	k	α	β	Conditional Estimation	Gamma Prior		Kernel Prior	
						SQEL	LNXL	SQEL	LNXL
20	10	5	1	1.5	0.1321(0.0174)	0.7019(0.7200)	0.4021(0.2328)	0.2536(0.0917)	0.2612(0.0973)
				3	0.1672(0.0280)	0.2786(0.1565)	0.6927(0.5288)	0.1598(0.0506)	0.1657(0.0535)
			2	1.5	0.1434(0.0258)	0.7030(0.7238)	0.3133(0.1470)	0.2410(0.0852)	0.2504(0.0909)
				3	0.1926(0.0517)	0.2673(0.1517)	0.8333(0.7443)	0.1708(0.0565)	0.1779(0.0592)
		8	1	1.5	0.1320(0.0174)	0.6942(0.7105)	0.4262(0.2624)	0.2542(0.0919)	0.2734(0.1064)
				3	0.1671(0.0280)	0.2729(0.1478)	0.6811(0.5151)	0.1569(0.0478)	0.1628(0.0516)
			2	1.5	0.1451(0.0281)	0.6984(0.7098)	0.2961(0.1287)	0.2368(0.0812)	0.2393(0.0818)
				3	0.1874(0.0352)	0.2696(0.1502)	0.8259(0.7285)	0.1708(0.0565)	0.1713(0.0540)
	15	8	1	1.5	0.1315(0.0173)	0.4454(0.3706)	0.3170(0.1701)	0.2316(0.0832)	0.2386(0.0873)
				3	0.1746(0.0531)	0.2746(0.1261)	0.4903(0.3147)	0.1710(0.0525)	0.1686(0.0516)
			2	1.5	0.1386(0.0192)	0.4436(0.3628)	0.2655(0.1230)	0.2218(0.0766)	0.2201(0.0768)
				3	0.1886(0.0435)	0.2952(0.1489)	0.5901(0.4198)	0.1916(0.0681)	0.1771(0.0576)
		11	1	1.5	0.1329(0.0195)	0.4709(0.3987)	0.3333(0.1887)	0.2448(0.0922)	0.2466(0.0933)
				3	0.1712(0.0416)	0.2787(0.1313)	0.4863(0.3079)	0.1752(0.0545)	0.1639(0.0497)
			2	1.5	0.1397(0.0213)	0.4458(0.3581)	0.2770(0.1294)	0.2178(0.0735)	0.2291(0.0810)
				3	0.1886(0.0428)	0.2845(0.1369)	0.5996(0.4324)	0.1886(0.0629)	0.1810(0.0597)
40	20	10	1	1.5	0.1321(0.0174)	0.4868(0.4147)	0.3235(0.1759)	0.2175(0.0711)	0.2150(0.0718)
				3	0.1671(0.0279)	0.2730(0.1212)	0.5104(0.3390)	0.1600(0.0440)	0.1695(0.0529)
			2	1.5	0.1394(0.0194)	0.4796(0.3990)	0.2460(0.0999)	0.2005(0.0615)	0.1921(0.0560)
				3	0.1874(0.0351)	0.2615(0.1204)	0.6203(0.4558)	0.1665(0.0503)	0.1684(0.0491)
		15	1	1.5	0.1320(0.0174)	0.4932(0.4190)	0.3309(0.1826)	0.2165(0.0719)	0.2215(0.0741)
				3	0.1671(0.0279)	0.2766(0.1291)	0.5042(0.3271)	0.1641(0.0475)	0.1607(0.0483)
			2	1.5	0.1395(0.0195)	0.4997(0.4352)	0.2583(0.1068)	0.2097(0.0659)	0.2014(0.0606)
				3	0.1872(0.0351)	0.2858(0.1420)	0.6203(0.4548)	0.1743(0.0555)	0.1642(0.0489)
	30	15	1	1.5	0.1315(0.0173)	0.2715(0.1383)	0.2369(0.0993)	0.1785(0.0509)	0.1886(0.0567)
				3	0.1661(0.0276)	0.2891(0.1240)	0.3487(0.1868)	0.1803(0.0523)	0.1760(0.0491)
			2	1.5	0.1383(0.0191)	0.2811(0.1505)	0.2132(0.0817)	0.1773(0.0496)	0.1785(0.0521)
				3	0.1854(0.0344)	0.2968(0.1286)	0.4067(0.2365)	0.1835(0.0557)	0.1803(0.0522)
		23	1	1.5	0.1314(0.0173)	0.2832(0.1501)	0.2135(0.0827)	0.1862(0.0553)	0.1731(0.0495)
				3	0.1659(0.0275)	0.2877(0.1227)	0.3421(0.1782)	0.1774(0.0508)	0.1748(0.0492)
			2	1.5	0.1383(0.0191)	0.2648(0.1350)	0.2036(0.0703)	0.1721(0.0479)	0.1733(0.0475)
				3	0.1848(0.0342)	0.3044(0.1330)	0.3952(0.2296)	0.1879(0.0559)	0.1831(0.0538)
60	30	15	1	1.5	0.1320(0.0174)	0.3661(0.2636)	0.2705(0.1295)	0.1896(0.0581)	0.1904(0.0566)
				3	0.1671(0.0279)	0.2888(0.1285)	0.4048(0.2380)	0.1689(0.0474)	0.1615(0.0448)
			2	1.5	0.1395(0.0195)	0.3722(0.2573)	0.2337(0.0960)	0.1810(0.0507)	0.1794(0.0508)
				3	0.1872(0.0351)	0.2815(0.1277)	0.5138(0.3404)	0.1659(0.0477)	0.1665(0.0473)
		23	1	1.5	0.1321(0.0175)	0.3661(0.2459)	0.2806(0.1365)	0.1941(0.0568)	0.1967(0.0595)
				3	0.1670(0.0279)	0.2766(0.1231)	0.4083(0.2399)	0.1626(0.0456)	0.1579(0.0443)
			2	1.5	0.1396(0.0195)	0.3838(0.2684)	0.2390(0.1008)	0.1859(0.0534)	0.1827(0.0529)
				3	0.1872(0.0351)	0.2752(0.1197)	0.5123(0.3435)	0.1654(0.0457)	0.1702(0.0491)
	45	23	1	1.5	0.1314(0.0173)	0.1943(0.0669)	0.1783(0.0524)	0.1453(0.0338)	0.1503(0.0354)
				3	0.1661(0.0276)	0.2876(0.1183)	0.2863(0.1324)	0.1762(0.0478)	0.1739(0.0487)
			2	1.5	0.1383(0.0191)	0.2017(0.0736)	0.1783(0.0557)	0.1460(0.0345)	0.1538(0.0386)
				3	0.1853(0.0343)	0.2981(0.1269)	0.3208(0.1534)	0.1826(0.0530)	0.1693(0.0450)
		34	1	1.5	0.1314(0.0173)	0.2027(0.0772)	0.1830(0.0566)	0.1494(0.0365)	0.1537(0.0372)
				3	0.1661(0.0276)	0.2869(0.1168)	0.2995(0.1424)	0.1740(0.0470)	0.1817(0.0527)
			2	1.5	0.1383(0.0191)	0.2085(0.0756)	0.1756(0.0517)	0.1510(0.0358)	0.1523(0.0367)
				3	0.1853(0.0344)	0.2828(0.1151)	0.3316(0.1690)	0.1733(0.0479)	0.1843(0.0535)