On the Relation Between Binary Palindromes with Three Alternate Blocks and Fermat Numbers

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Abstract: We study positive integers whose binary representation is a palindrome of the form $(1\ldots1)(0\ldots0)(1\ldots1)$, where the number of ones is twice the integer $a \geq 1$, and the number of zeros is $b \geq 0$. We give an elementary algebraic description of all such integers, derive a factorization formula, and deduce consequences for primality. In particular we show that except for the case $a = 1$ (which yields the numbers $2^m + 1$ with $m = b + 1$), every such integer is composite by a simple factorization; consequently the only possible primes in this family are Fermat-type numbers $2^m + 1$, and thus (by a classical necessary condition) their exponents m must be powers of two. Several illustrative examples are given and implications for search strategies are discussed.

Key-Words: binary palindrome, repunit, Fermat number, factorization, primality, computer algebra system Maple

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"Symmetry is what we see at a glance; based on it, the mind guesses the missing."

— Hermann Weyl (1885–1955)

1 Introduction

The study of Fermat numbers remains a classical yet active area of research in number theory, continuing to attract attention due to their deep connections with primality, factorization, and computational aspects of arithmetic. Despite their simple definition, many fundamental questions concerning Fermat numbers are still open, and ongoing research continues to produce new results and perspectives. A comprehensive treatment of Fermat numbers and related topics can be found in the classical monograph [1], which remains an important reference.

In addition to this foundational work, Fermat numbers have been the subject of numerous contemporary research articles, including for example [2], [3], [4], [5], [6], [7], [8], and [9]. These papers address various aspects of Fermat numbers, ranging from theoretical properties to computational investigations and applications.

Palindromic numbers also continue to be of considerable interest to mathematicians. In recent years, a number of papers have explored different aspects of palindromic structures, their arithmetic properties, and their occurrence in various numeral systems; see, for instance, [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], and [25]. These works demonstrate that palindromic numbers provide a rich source of problems connecting combinatorial, algebraic, and computational methods.

Classical literature in number theory also remains highly relevant. The well-known book [26] continues to be widely used and influential; its most recent edition has been revised and typeset using modern typographical standards. Other important and frequently cited monographs in number theory include, for example, [27], [28], [29], and [30], which provide further background and context for the topics studied in this paper.

Among online resources relevant to the topic of this paper, an important role is played by The On-Line Encyclopedia of Integer Sequences (OEIS), which provides a valuable source of examples, data, and references. In particular, several sequences related to Fermat numbers, binary palindromes, and associated notions are recorded in the OEIS; see, for example, [31], [32], [33], [34], [35], [36], and [37]. These entries illustrate the breadth of current interest in the arithmetic properties of numbers defined by simple digit patterns.

The aim of this paper is to clarify the precise algebraic structure of this class of binary palindromes and to explain why their apparent similarity to Fermat numbers leads to strong restrictions on their possible primality.

Binary palindromes, that is, integers whose base–2 representation reads the same forwards and backwards, form a natural and well-studied class

of integers. Their simple combinatorial definition contrasts with often subtle arithmetic behavior, making them an attractive object of study in number theory. In this contribution we focus on a specific subclass of binary palindromes that begin and end with equal-length runs of consecutive ones, separated by an arbitrary block of zeros in the middle. Concretely, for integers $a \geq 1$ and $b \geq 0$ we consider the binary string

$$\underbrace{1\ldots1}_{a}\ \underbrace{0\ldots0}_{b}\ \underbrace{1\ldots1}_{a},$$

and the corresponding positive integer n.

The main questions addressed in this paper are the following:

- What is a convenient closed-form expression for such integers n?
- What factorization properties do these numbers possess?
- Under what conditions can such an integer be prime?

We provide complete answers to these questions using elementary algebraic manipulations and standard facts from elementary number theory, leading to a transparent description of the structure and arithmetic nature of this family of binary palindromes.

2 Notation and basic identity

For an integer $a \geq 1$ we write

$$A_a := 2^a - 1,$$

that is, the positive integer whose binary expansion consists of a consecutive ones. In the binary palindrome $1^a0^b1^a$, the left block of a ones corresponds to the integer A_a shifted left by $(a+b)$ binary positions, while the right block corresponds to A_a occupying the lowest a bits.

Consequently, the integer n whose binary representation is $1^a0^b1^a$ can be written as

$$n = A_a \cdot 2^{a+b} + A_a = A_a(2^{a+b}+1).$$

Equivalently, we obtain the explicit factorization

$$n = (2^a - 1)(2^{a+b}+1). \tag{1}$$

This identity follows immediately from the binary decomposition and provides a complete algebraic description of the integers under consideration. It will serve as the central tool for all subsequent arguments concerning factorization and primality.

3 Consequences: Factorization and primality

The simple algebraic identity from the previous section already contains all the information needed to determine the factorization and primality of the numbers under consideration.

Theorem 1. Let $a \geq 1$ and $b \geq 0$ be integers, and let n be the integer whose binary representation is $1^a0^b1^a$. Then

$$n = (2^a - 1)(2^{a+b}+1).$$

In particular:

1. If $a \geq 2$, then n is composite.
2. If $a = 1$, then $n = 2^{b+1} + 1$. In this case n may be prime, and these numbers are precisely the integers of the form $2^m + 1$ with $m = b + 1$.

Proof. The identity (1) was established in Section 2. If $a \geq 2$, then $2^a - 1 \geq 3$ and $a + b \geq a \geq 2$, which implies

$$2^{a+b} + 1 \geq 2^2 + 1 = 5.$$

Hence both factors in (1) exceed 1, and therefore n is composite.

If $a = 1$, then $2^a - 1 = 1$, and (1) reduces to

$$n = 2^{b+1} + 1.$$

Thus the subfamily with $a = 1$ coincides exactly with the Fermat-type numbers $2^m + 1$, where the exponent $m = b + 1$. □

Thus the only chance to obtain a prime from the binary palindromes $1^a0^b1^a$ is the subfamily with $a = 1$.

4 Primality condition for the $a = 1$ family

It is classical and elementary that a necessary condition for a number of the form $2^m + 1$ to be prime is that the exponent m be a power of two. For completeness and to keep the exposition self-contained, we briefly recall the standard proof.

Proposition 1. If $2^m + 1$ is prime, then m is a power of two.

Proof. Suppose that m has an odd prime divisor $r > 1$, so that $m = rt$ for some integer $t \geq 1$. Then

$$2^m + 1 = (2^t)^r + 1.$$

Since r is odd, the well-known polynomial identity

$$x^r + 1 = (x+1)(x^{r-1} - x^{r-2} + \cdots - x + 1)$$

implies that $2^t + 1$ is a nontrivial divisor of $2^m + 1$. Hence $2^m + 1$ cannot be prime in this case.

Therefore, if 2^m+1 is prime, no odd prime divisor of m can exist, and m must be a power of two. □

Combining Proposition 1 with Theorem 1, we obtain the following consequence.

Corollary 1. A binary palindrome of the form $1^a0^b1^a$ can be prime only in the case $a = 1$ and $b + 1 = 2^t$ for some integer $t \geq 0$, that is,

$$n = 2^{2^t} + 1,$$

a Fermat number.

The known Fermat primes correspond to the values $t = 0, 1, 2, 3, 4$, yielding

$$n \in \{3,\ 5,\ 17,\ 257,\ 65537\}.$$

For $t \geq 5$, the numbers $2^{2^t} + 1$ are known to be composite by extensive numerical computations; however, the question whether infinitely many Fermat primes exist remains open.

Remark 1. Fermat numbers have played a central role in the historical development of number theory, serving as early examples of integers defined by a simple explicit formula whose arithmetic properties are unexpectedly subtle. Despite their elementary appearance, the question of the existence of infinitely many Fermat primes remains open and is one of the classical unsolved problems in number theory.

5 Examples and remark

Example 1. The case $a = 1$, $b = 1$ yields the binary palindrome 101, and

$$n = (2^1 - 1)(2^{1+1} + 1) = 1 \cdot (2^2 + 1) = 5,$$

which is prime and coincides with the Fermat number $F_1 = 5$.

Example 2. The case $a = 2$, $b = 2$ yields the binary palindrome 110011, and

$$n = (2^2 - 1)(2^{2+2} + 1) = 3 \cdot 17 = 51,$$

which is composite, as predicted by Theorem 1.

Example 3. The case $a = 3$, $b = 1$ yields the binary palindrome 1110111, and

$$n = (2^3 - 1)(2^{3+1} + 1) = 7 \cdot 17 = 119,$$

again composite.

Example 4. The case $a = 4$, $b = 5$ yields the binary palindrome 1111000001111, and

$$n = (2^4 - 1)(2^{4+5} + 1) = 15 \cdot 513 = 7695,$$

once more composite.

Example 5. The case $a = 1$, $b = 7$ yields the binary palindrome 100000001, and

$$n = (2^1 - 1)(2^{1+7} + 1) = 1 \cdot 257 = 257,$$

which is prime and coincides with the Fermat number $F_3 = 257$.

Remark 2. (Borderline case.) The case $b = 0$ corresponds to the binary string 1^a1^a (that is, $2a$ consecutive ones). Although this case formally lies on the boundary of the definition, the preceding formulas still apply:

$$n = (2^a - 1)(2^a + 1) = 2^{2a} - 1,$$

which is composite for all $a \geq 2$ by the classical factorization of $x^2 - 1$ with $x = 2^a$. For the exceptional case $a = 1$, $b = 0$, one obtains $n = 3$.

6 Computational comment

The factorization formula (1) immediately reduces any computational search for prime binary palindromes of the form $1^a0^b1^a$ to the exceptional subfamily $a = 1$. Consequently, only numbers of the form $2^m + 1$ need to be tested for primality.

In practice, it suffices to consider exponents m that are powers of two, since this condition is necessary for primality. Even under this strong restriction, efficient primality testing for large values of m remains computationally demanding and typically relies on advanced methods such as ECPP (elliptic curve primality proving) or related algorithms.

Known Fermat primes are extremely rare. To date, the only exponents $m = 2^t$ for which $2^m + 1$ is prime correspond to $t = 0, 1, 2, 3, 4$. For all larger values of t that have been examined, extensive numerical evidence shows that the corresponding Fermat numbers are composite.

This illustrates how a simple structural identity can drastically reduce the computational complexity of an otherwise naive search for prime candidates.

7 Maple script

We use the following simple Maple script to generate numbers of the form $2^{k+1} + 1$, which correspond exactly to binary palindromes with $a = 1$, up to a given bound N:

```
generateNumbers := proc(N)
local k, num, L;
L := [];
for k from 0 while 2^(k+1)+1 <= N do
num := 2^(k+1)+1;
L := [op(L), num];
od;
return L;
end proc;

# Example usage:
generateNumbers(100000000);
```

This script (procedure `generateNumbers`) generates the following numbers less than 100,000,000 that are palindromes in binary representation:

3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, 1048577, 2097153, 4194305, 8388609, 16777217, 33554433, 67108865

8 Conclusion

We have provided a complete and elementary structural description of positive integers whose binary representation is the palindrome $1^a0^b1^a$. The identity

$$n = (2^a - 1)(2^{a+b} + 1)$$

immediately implies that all such integers are composite for $a \geq 2$. The only potential primes in this family arise in the exceptional case $a = 1$, which yields the Fermat-type numbers $2^m + 1$ with $m = b + 1$. In this case, the classical necessary condition for the primality of Fermat numbers forces the exponent m to be a power of two as a necessary condition. Consequently, apart from the known small Fermat primes, no new prime numbers occur within the family of binary palindromes of the form $1^a0^b1^a$.

Area of Further Development

The results of this paper invite several natural extensions. Binary palindromes with more than three alternating blocks represent an obvious next step, where the simplicity of the present factorization is no longer guaranteed. It would also be of interest to study analogous palindromic constructions in other numeral bases and to compare their arithmetic and primality properties. From a computational perspective, the structural results obtained here may help to optimize search strategies for primes in families of integers defined by digit patterns.

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