On Properties Between G-matrices and Hyper G-matrices

HASAN KELEŞ[1], FRANK J. HALL[2]

[1]Department of Mathematics
Karadeniz Technical University
Campus of Kanuni, Ortahisar, 61080 Trabzon
TÜRKİYE

[2]Department of Mathematics and Statistics
Georgia State University, Georgia Atlanta
USA.

Abstract: This paper is about *G*-matrices, escort matrix, lower-upper triangular matrix, Companion matrix, diagonal matrix and Hyper *G*-matrix pairs formed by them. And also this paper has further study on Hyper *G*-matrix pairs of matrices. In particular, structured matrices and escort elements are emphasized. Connections are made with *G*-matrices.

Key-Words: *G*-matrix, Hyper *G*-matrix pair, structured matrix, escort element.
2020 Mathematics Subject Classification: 15A23, 15A20, 15A30, 47C99, 53C10.

Received: May 8, 2025. Revised: July 4, 2025. Accepted: August 11, 2025. Published: December 31, 2025.

1 Introduction

Saunders and Schneider obtained definitions and properties on diagonal similarity and diagonal equivalence of matrices [1]. Fiedler and Hall gave the definition of *G*-matrix in 2012 [2]. Golshan, Armandnejad and Hall obtained some results for *G*-matrices. Their studies about this subject have contributed new properties and characteristics [3]. Fiedler and Markham have studied *G*-matrices in depth [4]. Golshan, Armandnejad and Hall made new contributions to this subject by studying two classes of nth order *G*-matrices with finite intersection [5]. Hall, Li, Parnass, and Rozložník conclude that for $n \leq 4$, all $n \times n$ full sign patterns allow a *G*-matrix along with a J-orthogonal matrix [6]. Cheon, Curtis, Kim have obtained characterization results for some special matrix classes related to G-matrices [7]. Some relations between the division of matrices and this subject are given by Keleş and with the help of these properties, G-matrices are reached [9]. Hyper G-matrices are defined using the multiplication properties of matrices [8, 10, 11].

The analysis of structured matrices and their spectral properties has been extended in various directions. For example, Grammont and Largillier examined ϵ-spectra and stability radii [16], while Cullum and Ruehli applied pseudospectra analysis to nonlinear eigenvalue problems and systems with delays [17]. Ahmad and Mehrmann studied backward errors and pseudospectra for structured nonlinear eigenvalue problems [18], and Tisseur and Higham developed structured pseudospectra for polynomial eigenvalue problems [19]. Lancaster and Psarrakos further investigated the pseudospectra of matrix polynomials [20]. These approaches provide powerful tools for understanding the sensitivity of eigenvalue problems under perturbations. Vander Meulen and Vanderwoerd studied polynomial roots using intercyclic companion matrices [12].

Classical matrix theory and computational aspects are covered in several fundamental works [13, 14, 21, 22, 23, 24, 25]. Golub and Van Loan gave comprehensive treatments of matrix computations [21], while Meyer [22], Lancaster and Tismenetsky [23], Axler [24], and Strang [25] presented detailed accounts of matrix analysis and linear algebra. Wilkinson's monograph on the algebraic eigenvalue problem remains a cornerstone [26], and Stewart's work on error bounds for subspaces associated with eigenvalue problems is widely cited [27]. Higham discussed accuracy and stability in numerical algorithms [28], and Bernstein compiled an extensive collection of matrix facts and formulas [29]. Gantmacher's theory of matrices provides a rigorous foundation [30]. Important algorithmic contributions include Paige's work on large sparse eigenvalue problems [31], Davis and Kahan's perturbation theory [32], Lanczos's iteration method [33], and

Arnoldi's minimized iteration principle [34]. Gallopoulos, Philippe, and Sameh addressed the computation of matrix pseudospectra [35], and Angelova and Petkov performed componentwise perturbation analysis of the singular value decomposition [36]. Watkins reviewed GR and QR algorithms for the matrix eigenvalue problem [37].

Motlaghian, Armandnejad, and Hall provided additional insights on certain classes of G-matrices [38]. Recent applications of matrix analysis appear in various fields. Murrja et al. compared matrix analysis with multifactorial linear regression for risk perception in farming [39]. Spectral clustering using transition matrices was studied by Vaziri, Jamshidi, and Motallebi [40]. Inequalities involving C-spectral norms of operator matrices were examined by Rezagholi and Hosseini [41], while Soleymani and Salemi gave conditions for linear preservers of multivariate group majorization on matrices [42]. Inverse eigenvalue problems for distance matrices were tackled by Nazari, Nezami, and Bayat [43]. Special structured matrices have also received attention: Ilkhanizadeh Manesh investigated L-rays of Toeplitz matrices [44], Talebi and Aminizadeh discussed Sonnenschein summability matrices [45], and Beheshti, Fathi, and Zangiabadi studied interval tensors [46]. Applied studies include Melnikov and Melnikova's use of paired correlation algorithms for distance matrices between DNA chains [47], Porto's analysis of conductance matrices for fast processing [48], and Karkour, Tajani, and Khattabi's generation of consistent pairwise comparison matrices in AHP using particle swarm optimization [49]. Further contributions on matrix transforms and special classes include the work of Aasma and Natarajan on matrix transforms into α-absolutely convergent sequences [50], Erişir and Yıldırım on spinor matrices of real quaternions [51], and Kuloğlu, Eser, and Özkan on r-circulant matrices associated with k-Fermat and k-Mersenne numbers [52]. Circulant and related structured matrices have been extensively studied; for example, Gao, Jiang, and Gong derived determinants and inverses of skew circulant matrices with Fibonacci and Lucas numbers [53], Garcia-Planas and Dominguez-Garcia presented a general approach for computing residues of partial-fraction expansions of transfer matrices [54], Jiang, Shen, and Li gave spectral decompositions of certain tridiagonal matrices [55], and Zhou and Jiang estimated spectral norms of circulant-type matrices involving well-known numbers [56]. Pesaranghader and Muthaiyah employed co-occurrence matrices for semantic similarity [57]. Li and Wu compared preconditioners for L-matrices [58]. Kririm and Hmamed designed robust H_∞ filtering via LMIs and polynomial matrices [59]. Zatopek applied transformation matrices beyond deriving equations of motion [60]. Liu and Jie constructed deterministic compressed sensing matrices using singular linear spaces over finite fields [61]. Kantalo investigated determinants and permanents of Hessenberg matrices with Perrin's bivariate complex polynomials [62]. Several authors focused on explicit determinants of circulant-type matrices: Shen, Jiang, and Li treated RFMLR and RLMFL circulant matrices [63]; Jiang, Li, and Shen examined r-circulant and left r-circulant matrices involving famous numbers [64]; and Jiang provided efficient algorithms for finding minimal polynomials and inverses of level-k FLS $(r_1, \ldots, r_k)$-circulant matrices [65].

Families of Artin-Schreier curves with Cartier-Manin matrix of constant rank were studied by Farnell and Pries [15]. We remark that a different matrix with the same name G-matrix is used by biologists and genetic scientists.

The set of all $n \times n$ real matrices is denoted by

$$\mathbb{M}_n(\mathbb{R}) = \{[a_{ij}]_n | a_{ij} \in \mathbb{R}\}.$$

For $A \in \mathbb{M}_n(\mathbb{R})$, the inverse of the tranpose of A is denoted by A^{-T}. The set of diagonal matrices is

$$\mathbb{D}_n(\mathbb{R}) = \{[d_{ij}]_n | d_{ii} \neq 0, \textit{if } i \neq j, d_{ij} = 0\}.$$

2 Preliminaries

A nonsingular matrix $A \in \mathbb{M}_n(\mathbb{R})$ is called a G-matrix if there exist nonsingular matrices $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$ such that $A^{-T} = D_1 A D_2$. These matrices form a rich class and were first studied by Fiedler and Hall in [2]. Some properties of these matrices are as follows:

All orthogonal matrices are G matrices.

The nonsingular matrices $A \in \mathbb{D}_n(\mathbb{R})$ are G-matrices.

Any n positive real numbers are singular values and eigenvalues of a diagonal G-matrix D.

If A is a G-matrix, both A^T and A^{-1} are G-matrices.

If A is an $n \times n$ G-matrix and D is an $n \times n$ nonsingular diagonal matrix, then both AD and DA are G-matrices.

If A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then both AP and PA are G-matrices.

A pair of matrices (A, B) is said to form a Hyper G-matrix pair, if there exist two nonsingular matrices $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$ such that $A^{-T} = D_1 B D_2$; $A \bowtie B$ is the notation for a Hyper G-matrix pair. Similar to a G-class of matrices [3],we have the Hyper G-class

$$HG(D_1, D_2) = \{(A, B) | A^{-T} = D_1 B D_2\} \text{ in [10], [11].} \quad (1)$$

It is easy to see the nice symmetric property that $(A, B) \in HG(D_1, D_2)$ if and only if $(B, A) \in HG(D_1, D_2)$.

Note that if A is a G-matrix, then clearly (A, A) is a Hyper G-matrix pair.

We now mention the following properties. If (A, B) forms a Hyper G-matrix pair, where $A, B \in \mathbb{M}_n(\mathbb{R})$, then it is easy to show that the following are also Hyper G-matrix pairs:

(i) (A^T, B^T)

(ii) (A^{-1}, B^{-1})

(iii) (P^TAP, P^TBP), where P is an $n \times n$ permutation matrix

(iv) $(D^{-1}AD, D^{-1}BD)$, where $D \in \mathbb{D}_n(\mathbb{R})$.

For any two $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$, we can write the equation $A^{-T} = D_1BD_2$ as

$$A^{-1} = D_2B^TD_1, \tag{2}$$

which says that A^{-1} and B^T are diagonally equivalent. Moreover, by properties of G-matrices, we can see that A is a G-matrix if and only if B is a G-matrix.

It is of course easy to find Hyper G-matrix pairs of matrices. Just form the diagonal equivalence D_1BD_2, call that product A^{-T}, and then find A. We of course would like meaningful Hyper G-matrix pairs (A, B). A purpose of this paper is in fact to find such meaningful pairs of matrices.

The next result and the following two lemmas will be utilized in Section 3.
Recall that $A, B \in \mathbb{M}_n(\mathbb{R})$ are "diagonally equivalent" when $D_1BD_2 = A$ for some diagonal matrices $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$. And note that for $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$, the product $D_1D_2 \in \mathbb{D}_n(\mathbb{R})$.

Suppose $A, B \in \mathbb{M}_n(\mathbb{R})$ and that A is a G- matrix, so that there exist $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$ such that $A^{-T} = D_1AD_2$. Assume that (A, B) form a Hyper G-matrix pair, so that there exist $D_3, D_4 \in \mathbb{D}_n(\mathbb{R})$ such that $A^{-T} = D_3BD_4$. Hence,

$$D_1AD_2 = D_3BD_4, \tag{3}$$

$$D_3^{-1}D_1AD_2D_4^{-1} = B \tag{4}$$

and A and B are diagonally equivalent.

Conversely, suppose A and B are diagonally equivalent, so that there exist D_5, D_6 such that $A = D_5BD_6$. Then

$$A^{-T} = D_1AD_2 = D_1D_5BD_6D_2 \tag{5}$$

so that (A, B) form a Hyper G-matrix pair. We thus have the following theorem.

Theorem 2.1. *Suppose $A, B \in \mathbb{M}_n(\mathbb{R})$ and A is a G-matrix. Then, (A, B) form a Hyper G-matrix pair if and only if A and B are diagonally equivalent.*

Lemma 2.2. *If $A \bowtie B$, then $A^{-T} \bowtie B^{-T}$.*

Proof. If $A \bowtie B$, then there exist some matrices $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$ such that

$$A^{-T} = D_1BD_2. \tag{6}$$

It is easily calculated that

$$D_1^{-1}B^{-T}D_2^{-1} = A = (A^{-T})^{-T} \tag{7}$$

so that

$$A^{-T} \bowtie B^{-T}. \tag{8}$$

□

Lemma 2.3. *([4]) A 2×2 matrix is a G-matrix if and only if it is nonsingular and has four or two nonzero entries.*

3 Main Results

We begin this section with a short general Algorithm: Start with $n \times n$ matrix B. Form the diagonal equivalence D_1BD_2 and call that product A^{-T}. Then find A.

Let us suppose that B is a STRUCTURED matrix. We first consider the case where B is a nonsingular upper triangular matrix. Note that for an upper (or lower) triangular matrix, nonsingular means that there are no zeros on the diagonal. We first consider

Algorithm 3.1. *Suppose that B is a nonsingular upper triangular matrix. Then D_1BD_2 is a nonsingular upper triangular matrix. Now, the inverse of a nonsingular upper triangular matrix is of the same type. The transpose of that inverse is a nonsingular lower triangular matrix.*

Using the Algorithm, we obtain a Hyper G-matrix pair (A, B), where B is a nonsingular upper triangular matrix, and A is a nonsingular lower triangular matrix.

Proposition 3.2. *For a nonsingular $B \in \mathbb{M}_n(\mathbb{R})$, if B is lower or upper triangular, then $A \bowtie B$, where A is upper or lower triangular respectively.*

Proof. The transpose operation makes it easy to see the proof. □

Example 3.3. The matrix $B = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & 4 & -1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ is a nonsingular upper triangular matrix. For $D_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \in \mathbb{D}_4(\mathbb{R})$, we have

$$D_1BD_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & 4 & -1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -8 & 6 & 0 \\ 0 & 30 & 12 & -15 \\ 0 & 0 & 2 & 30 \\ 0 & 0 & 0 & 140 \end{bmatrix}, \tag{9}$$

and hence it can be seen that with $D_1BD_2 = A^{-T}$,

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{2}{15} & \frac{1}{30} & 0 & 0 \\ -\frac{23}{10} & -\frac{1}{5} & \frac{1}{2} & 0 \\ \frac{71}{140} & \frac{13}{280} & -\frac{3}{28} & \frac{1}{140} \end{bmatrix}, \tag{10}$$

so that

$$A \bowtie B. \tag{11}$$

The matrix A is a nonsingular lower triangular matrix. Now, $B^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ -\frac{11}{10} & -\frac{2}{5} & \frac{1}{2} & 0 \\ \frac{37}{140} & \frac{13}{70} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$ is a nonsingular lower triangular matrix. And,

$$D_1^{-1}B^{-T}D_2^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ -\frac{11}{10} & -\frac{2}{5} & \frac{1}{2} & 0 \\ \frac{37}{140} & \frac{13}{70} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{2}{15} & \frac{1}{30} & 0 & 0 \\ -\frac{23}{10} & -\frac{1}{5} & \frac{1}{2} & 0 \\ \frac{71}{140} & \frac{13}{280} & -\frac{3}{28} & \frac{1}{140} \end{bmatrix} = A. \tag{12}$$

We have

$$A^{-T} \bowtie B^{-T}. \tag{13}$$

We next consider Companion matrices, which in general can have various forms. A fairly recent article which makes use of various companion matrices is the interesting and informative paper [12] by Van-der Meulen and Vanderwoerd.

Algorithm 3.4. *Suppose B is a nonsingular Companion matrix. Then D_1BD_2 is nonsingular and has the same "form".*

Proof. If B is a nonsingular Companion matrix, then without loss of generality

$$B = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{14}$$

and we use

$$D_1 = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_{21}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{d_{21}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_{21}} \end{bmatrix}$$

,

$$D_2 = \begin{bmatrix} d_{21} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{2n} \end{bmatrix} \in \mathbb{D}_n(\mathbb{R}). \tag{15}$$

We calculate

$$D_1BD_2 = \begin{bmatrix} -a_1d_{11}d_{21} & -a_2\frac{d_{11}}{d_{21}} & -a_3\frac{d_{11}}{d_{21}} & \cdots & -a_nd_{11}d_{2n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{16}$$

□

Corollary 3.5. *If B is a nonsingular Companion matrix and*

$$D_1 = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_{21}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{d_{21}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_{21}} \end{bmatrix},$$

$$D_2 = \begin{bmatrix} d_{21} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{2n} \end{bmatrix} \in \mathbb{D}_n(\mathbb{R}), \tag{17}$$

$$D_1BD_2 = \begin{bmatrix} -a_1d_{11}d_{21} & -a_2\frac{d_{11}}{d_{21}} & -a_3\frac{d_{11}}{d_{21}} & \cdots & -a_nd_{11}d_{2n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{18}$$

then

$$A \bowtie B, \text{ where } A^{-T} \text{ is defined by Equation (16).} \tag{19}$$

Proof. We calculate

$$D_1BD_2 = \begin{bmatrix} -a_1d_{11}d_{21} & -a_2\frac{d_{11}}{d_{21}} & -a_3\frac{d_{11}}{d_{21}} & \cdots & -a_nd_{11}d_{2n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = A^{-T}. \tag{20}$$

Then

$$A = \begin{bmatrix} 0 & \frac{1}{d_{11}d_{22}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{d_{12}d_{23}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_{1(n-1)}d_{2n}} & 0 \\ \frac{b_{n1}}{d_{1n}d_{21}} & \frac{b_{n2}}{d_{1n}d_{22}} & \frac{b_{n3}}{d_{1n}d_{23}} & \cdots & \frac{b_{n(n-1)}}{d_{1n}d_{2(n-1)}} & \frac{b_{nn}}{d_{1n}d_{2n}} \end{bmatrix}, \quad (21)$$

and

$$A \bowtie B. \quad (22)$$

□

The product of diagonal matrices and triangular matrices preserves triangularity.

Starting with any nonsingular matrix $A \in \mathbb{M}_n(\mathbb{R})$ and any diagonal matrices $D_1, D_2 \in \mathbb{D}_n(\mathbb{R})$, we can constructively obtain the equation $A^{-T} = D_1A_2D_2$, so that $(A, A_2) \in HG(D_1, D_2)$.

This is somewhat of an opposite approach to that employed in Algorithms 3.1 and 3.4.

Algorithm 3.6. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be any nonsingular matrix. We write $A^{-T} = D_1A_1$, where $A_1 \in \mathbb{M}_n(\mathbb{R})$ and $D_1 \in \mathbb{D}_n(\mathbb{R})$, and $A_1 = A_2D_2$, where $A_2 \in \mathbb{M}_n(\mathbb{R})$ and $D_2 \in \mathbb{D}_n(\mathbb{R})$, so that*

$$A^{-T} = D_1A_2D_2. \quad (23)$$

At the end of the algorithm, $A \bowtie A_2$ is obtained.

Example 3.7. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ such that $A^{-T} = D_1A_2D_2$. For a randomly chosen nonsingular diagonal matrix $D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, we can easily write the equation

$$A^{-T} = D_1A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{3} & -\frac{1}{15} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{25} & \frac{4}{25} \end{bmatrix}. \quad (24)$$

Similarly, we write the equation

$$A_2 = A_1D_2^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{3} & -\frac{1}{15} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{25} & \frac{4}{25} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{5} & \frac{1}{5} \\ \frac{1}{9} & -\frac{1}{30} & \frac{1}{5} \\ -\frac{1}{15} & \frac{3}{50} & -\frac{4}{25} \end{bmatrix}, \quad (26)$$

for nonsingular diagonal matrix $D_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

We obtain $A^{-T} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 1 & -\frac{1}{5} & -\frac{3}{5} \\ -1 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ such that $A^{-T} = D_1A_2D_2$.

We now transition to a different topic.

Let us give the definition of an escort element.

For each element a of a commutative and binary group, if there are b, c elements such that $ba = ac$, then a is called an escort element [9]. We borrow this definition in our paper in order to have matrix escort elements. For a nonsingular escort element A, we write $BA = AC$, where B, C are nonsingular.

We can obtain an example for which $BA = AC$, A is a G-matrix, and (A, B) is a Hyper G-matrix pair.

Example 3.8. Let

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 & 9 \\ \frac{1}{3} & 2 \end{bmatrix}, C = \begin{bmatrix} \frac{289}{21} & \frac{125}{21} \\ -\frac{353}{21} & -\frac{142}{21} \end{bmatrix}$$

so that $BA = AC$. By Lemma 2.3, A is a G-matrix. Also, for the diagonal matrices

$$S_1 = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & -\frac{1}{42} \end{bmatrix}, S_2 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{15}{2} \end{bmatrix},$$

we have

$$S_1AS_2 = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & -\frac{1}{42} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -\frac{15}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{5}{7} \end{bmatrix} = A^{-T}.$$

Next, for diagonal matrices

$$S_3 = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, S_4 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{15}{2} \end{bmatrix}, \quad (27)$$

we have

$$S_3BS_4 = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & -\frac{1}{42} \end{bmatrix} \begin{bmatrix} 5 & 9 \\ \frac{1}{3} & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -\frac{15}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{5}{7} \end{bmatrix} = A^{-T}. \quad (28)$$

Thus, for the escort matrix A, which is a G-matrix, the pair (A, B) is a Hyper G-matrix pair.

Example 3.9. We can verify that (A, B) is a Hyper G-matrix pair, by the use of Theorem 2.1. We need to show that A and B are diagonally equivalent. Here is a sketch of the way: let

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix},$$

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, E = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}. \quad (29)$$

Then setting $DAE = B$, we obtain one solution as

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{12} \end{bmatrix}, E = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}. \quad (30)$$

In the next result, we show that a certain Hyper G-matrix pair is necessarily obtained by an escort matrix which is a G-matrix.

Theorem 3.10. *Let A be a nonsingular escort element. If matrix A is a G-matrix, then there exist some nonsingular matrices B_3 and C_2 associated with A such that $B_3 \bowtie C_2$.*

Proof. For a nonsingular escort element A, we write

$$BA = AC, \quad \text{where } B, C \text{ are nonsingular.} \tag{31}$$

We calculate

$$B^{-T}A^{-T} = A^{-T}C^{-T}. \tag{32}$$

If matrix A is a G-matrix, then there exist diagonal matrices S_1, S_2 such that $A^{-T} = S_1AS_2$,

$$B^{-T} = B_1S_1^{-1}, C^{-T} = S_2^{-1}C_1, \text{ where } B_1, C_1 \text{ are nonsingular.} \tag{33}$$

By Equation 32 ,

$$B_1AS_2 = S_1AC_1, \tag{34}$$

$$B_1 = B_2A^{-1}, C_1 = A^{-1}C_2, \text{ where } B_2, C_2 \text{ are nonsingular matrices.} \tag{35}$$

By Equation 34, we write

$$B_2S_2 = S_1C_2 \tag{36}$$

$$B_2 = S_1C_2S_2^{-1} \tag{37}$$

We know that if

$$B_2 := B_3^{-T} \tag{38}$$

then

$$B_3 \bowtie C_2. \tag{39}$$

□

Remark 3.11. Our result about $BA = AC$ in Theorem 3.10 exhibits a special case of Sylvester's theorem on linear matrix equations. The reader can see Theorem 2.4.4.1 in [13] and also reference in [8, 14].

In the following results, we give some clarifications of the matrices in Theorem 3.10. In these results, we use the notation in Theorem 3.10 and its proof.

The purpose is to show how these already defined matrices are further related

Lemma 3.12. *If a nonsingular escort matrix A is a G-matrix, then*

$$B_2 = S_1AS_2C^{-T}S_2^{-1}, \tag{40}$$

$$C_2 = S_1^{-1}B^{-T}S_1AS_2. \tag{41}$$

Proof. By Equation 33 and Equation 35, $B_2 = S_1AS_2C^{-T}S_2^{-1}$ is clear. And by Eqution 33 and Equation 35, $C_2 = S_1^{-1}B^{-T}S_1AS_2$ is clear. □

Corollary 3.13. *If a nonsingular escort matrix A is a G-matrix, then*

$$B_2 = B^{-T}S_1A, C_2 = AS_2C^{-T}$$

Proof. By Equation 37 and Lemma 3.12, it is clear.

□

Corollary 3.14. *If a nonsingular escort matrix A is a G-matrix, then*

$$B_3 = BS_1^{-1}A^{-T}. \tag{42}$$

Proof. By Corollary 3.13 and Equation 38, it is easy to obtain.

□

Corollary 3.15. *If a nonsingular escort matrix A is a G-matrix, then*

$$C_2^{-T} = S_1BS^{-1}A^{-T}S_2^{-1}. \tag{43}$$

Proof. By Equation 37 and Lemma 3.12, we write

$$B_2^{-T} = D_1^{-T}B^{-1}B_2^{-T}D_2^{-T}C. \tag{44}$$

□

Example 3.16. The matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ is an escort element and a G-matrix. For the selected matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 11 & 1 \end{bmatrix}, \tag{45}$$

And with $C = A^{-1}BA = \begin{bmatrix} \frac{16}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$,

$$AC = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{16}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 11 & 1 \end{bmatrix}. \tag{46}$$

Next,

$$A^{-T} = S_1AS_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \tag{48}$$

$$B_2 = S_1AS_2C^{-T}S_2^{-1} = \begin{bmatrix} -1 & \frac{11}{2} \\ 1 & -\frac{5}{2} \end{bmatrix}, \tag{49}$$

$$C_2 = S_1^{-1}B^{-T}S_1AS_2 = \begin{bmatrix} -\frac{1}{12} & \frac{11}{12} \\ \frac{1}{16} & -\frac{5}{6} \end{bmatrix}. \tag{50}$$

We have

$$B_2 = S_1 C_2 S_2^{-1} = \begin{bmatrix} -1 & \frac{11}{2} \\ 1 & -\frac{5}{2} \end{bmatrix}, \tag{51}$$

$$B_3 = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{11}{6} & -\frac{1}{3} \end{bmatrix}, \tag{52}$$

$$B_3 \bowtie C_2. \tag{53}$$

Remark 3.17. If we know $B_2 = B_3^{-T}$, then we could replace B_3 by B_2^{-T} in Equation 53.

As in Theorem 3.10, we have S_1, S_2 such that $A^{-T} = S_1 A S_2$. The following corollary gives some S_1, S_2, when we specifically know A, B, C, B_2, and C_2

Corollary 3.18. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be a nonsingular G-matix. There exist some selected matrices $S_1, S_2 \in \mathbb{D}_n(\mathbb{R})$ such that $A^{-T} = S_1 A S_2$ and for a nonsingular matrix $B \in \mathbb{M}_n(\mathbb{R})$, with $BA = AC$, if B_2, C_2 are known, then*

$$S_1 = B^T B_2 A^{-1}, S_2 = A^{-1} C_2 C^T \tag{54}$$

Proof. Let $A \in \mathbb{M}_n(\mathbb{R})$ a nonsinguler matix. There exist some selected matrices $S_1, S_2 \in \mathbb{D}_n(\mathbb{R})$ such that $A^{-T} = S_1 A S_2$ and for a nonsingular matrix $B \in \mathbb{M}_n(\mathbb{R})$, $BA = AC$, if B_2, C_2 are known, then by Corollary 3.13

$$S_1 = B^T B_2 A^{-1}, S_2 = A^{-1} C_2 C^T \tag{55}$$

□

The following example illustrates Corollary 3.18.

Example 3.19. For the matrices in Example 3.16, we have

$$S_1 = B^T B_2 A^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{56}$$

$$S_2 = A^{-1} C_2 C^T = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}. \tag{57}$$

The following example shows that the equality of B_2, C_2 in Lemma 3.12 and S_1, S_2 in Corollary 3.18 hold even though A is not an escort matrix and the pair (B_3, C_2) is not a Hyper G-matrix pair.

Example 3.20. For the matrices $B = \begin{bmatrix} \frac{3}{80} & -\frac{1}{40} \\ -\frac{1}{96} & \frac{1}{16} \end{bmatrix}$, $C = \begin{bmatrix} \frac{17}{128} & -\frac{3}{256} \\ \frac{3}{128} & \frac{7}{256} \end{bmatrix}$, we have $A = \begin{bmatrix} 3 & -1 \\ 2 & -6 \end{bmatrix}$ and if given matrices $B_2 = \begin{bmatrix} 480 & -240 \\ 288 & -384 \end{bmatrix}$, $C_2 = \begin{bmatrix} 45 & -2 \\ 46 & 180 \end{bmatrix}$, then

$$B^T B_2 A^{-1} = \begin{bmatrix} \frac{3}{80} & -\frac{1}{96} \\ -\frac{1}{40} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 480 & -240 \\ 288 & -384 \end{bmatrix} \begin{bmatrix} \frac{-6}{-16} & \frac{1}{-16} \\ \frac{-2}{-16} & \frac{3}{-16} \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = S_1,$$

$$A^{-1} C_2 C^T = \begin{bmatrix} \frac{-6}{-16} & \frac{1}{-16} \\ \frac{-2}{-16} & \frac{3}{-16} \end{bmatrix} \begin{bmatrix} 45 & -2 \\ 46 & 180 \end{bmatrix} \begin{bmatrix} \frac{17}{128} & \frac{3}{128} \\ -\frac{3}{256} & \frac{7}{256} \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = S_2.$$

$$S_1 B S_2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{80} & -\frac{1}{40} \\ -\frac{1}{96} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \tag{58}$$

$$= \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{16} & -\frac{3}{16} \end{bmatrix} = A^{-T}. \tag{59}$$

$$S_1 A S_2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 30 & 5 \\ -12 & 18 \end{bmatrix} = B^{-T},$$

and we have the Hyper G-matrix pairs (A, B) and (B, A). This (59) is also provided for other selected matrices $S_1 = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}$, $S_2 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$.

Remark 3.21. $A \bowtie B$ can be obtained for different nonsingular diagonal matrices S_1, S_2. The relation between matrices S_1, S_2 satisfying the equation $A^{-T} = S_1 B S_2$ can be investigated.Consideration to other non escort matrices, besides the matrix A Example 3.20).

4 Conclusion and Future Work

This work opens the door for even more progress on Hyper G-matrix pairs of matrices. In particular, other structured matrices besides triangular and Companion matrices could be investigated. More connections might be made with G-matrices. Giving diagonal matrices in a different way may create new research areas for other application areas. In general, for a G-matrix A which is an escort matrix with $AC = BA$, is it always the case that (A, B) is a Hyper G-matrix pair? Finally, similar to G-classes of matrices, what kind of results can be established for Hyper G-classes of matrices?

References:

[1] Saunders, B. D., Schneider, H. Flows on Graphs applied to Diagonal Similarity and Diagonal Equivalence for Matrices, *Discrete Mathematics*, 24, 1978, 205–220.

[2] Fiedler, M., Hall, F. J. G-matrices, *Linear Algebra and its Applications*, 436, 2012, 731–741.

[3] Golshan, S., Armandnejad, A., Hall, F. J. Some results on the intersection of G-classes of matrices, *Filomat*, 38(15), 2024, 5275–5283.

[4] Fiedler, M., Markham, T. L. More on *G*-matrices, *Linear Algebra and its Applications*, 438, 2013, 231–241.

[5] Golshan, S., Armandnejad, A., Hall, F. J. Two $n \times n$ *G*-classes of matrices having finite intersection, *Special Matrices*, 11, 2023, 1–4.

[6] Hall, F. J., Li, Z., Parnass, C. T., Rozložník, M. Sign patterns of J-orthogonal matrices, *Special Matrices*, 5, 2017, 225–241.

[7] Cheon, Gi-S., Curtis, B., Kim, H. Semi-involutory matrices and signed self-inverse, *Linear Algebra and its Applications*, 622, 2021, 294–315.

[8] Keleş, H. On New Developments in Division of Matrices for Linear Matrix Equations, *Journal of Applied and Pure Mathematics*, 7(1–2), 2025, 11–17.

[9] Keleş, H. Poloids and matrices, *Aligarh Bulletin of Mathematics*, 41(1), 2022, 41–52.

[10] Keleş, H. On New Developments for Hyper *G*-Matrices, *Equations*, 5, 2025, 23–28.

[11] Keleş, H. On Hyper *G*-matrices, *Troia 1st International Conference on Applied Sciences*, 21–23 February 2025, pp. 46–51, Çanakkale, Türkiye.

[12] Vander Meulen, K. N., Vanderwoerd, T. On polynomial roots using intercyclic companion matrices, *Linear Algebra and its Applications*, 539, 2018, 94–116.

[13] Horn, R. A., Johnson, C. R. *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013.

[14] Bhatia, R. *Matrix Analysis*, 2nd ed., Cambridge University Press, England, 1991.

[15] Farnell, S., Pries, R. Families of Artin–Schreier curves with Cartier–Manin matrix of constant rank, *Linear Algebra and its Applications*, 439(7), 2013, 2158–2166.

[16] Grammont, L., Largillier, A. On ϵ-spectra and stability radii, *Journal of Computational and Applied Mathematics*, 147(2), 2002, 453–469.

[17] Cullum, J., Ruehli, A. Pseudospectra Analysis, Nonlinear Eigenvalue Problems, and Studying Systems with Delays, *BIT Numerical Mathematics*, 41(2), 2001, 265–281.

[18] Ahmad, S. K. S., Mehrmann, V. Backward Errors and Pseudospectra for Structured Nonlinear Eigenvalue Problems, *Operators and Matrices*, 10(3), 2016, 539–556.

[19] Tisseur, F., Higham, N. J. Structured Pseudospectra for Polynomial Eigenvalue Problems, with Applications, *SIAM Journal on Matrix Analysis and Applications*, 23(1), 2001, 187–208.

[20] Lancaster, P., Psarrakos, P. On the Pseudospectra of Matrix Polynomials, *SIAM Journal on Matrix Analysis and Applications*, 27(1), 2006, 115–129.

[21] Golub, G. H., Van Loan, C. F. *Matrix Computations*, 3rd ed., Johns Hopkins University Press, Baltimore, 1996.

[22] Meyer, C. D. *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.

[23] Lancaster, P., Tismenetsky, M. *The Theory of Matrices: With Applications*, 2nd ed., Academic Press, New York, 1985.

[24] Axler, S. *Linear Algebra Done Right*, 3rd ed., Springer, New York, 2015.

[25] Strang, G. *Introduction to Linear Algebra*, 6th ed., Wellesley–Cambridge Press, 2016.

[26] Wilkinson, J. H. *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.

[27] Stewart, G. W. Error and perturbation bounds for subspaces associated with certain eigenvalue problems, *SIAM Review*, 15(4), 1973, 727–764.

[28] Higham, N. J. *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, 2002.

[29] Bernstein, D. S. *Matrix Mathematics: Theory, Facts, and Formulas*, 2nd ed., Princeton University Press, 2009.

[30] Gantmacher, F. R. *The Theory of Matrices*, AMS, Providence, RI, 2000.

[31] Paige, C. C. The computation of eigenvalues and eigenvectors of very large sparse matrices, 1971.

[32] Davis, C., Kahan, W. The rotation of eigenvectors by a perturbation, III, *SIAM Journal on Numerical Analysis*, 7(1), 1970, 1–19.

[33] Lanczos, C. An iteration method for the solution of the eigenvalue problem of linear differential and integral operators, *Journal of Research of the National Bureau of Standards*, 45(4), 1950, 255–282.

[34] Arnoldi, W. E. The principle of minimized iterations in the solution of the matrix eigenvalue problem, *Quarterly of Applied Mathematics*, 9(1), 1951, 17–29.

[35] Gallopoulos, E., Philippe, B., Sameh, A. Computing the Matrix Pseudospectrum, in *Parallel Algorithms for Matrix Computations*, Springer, 2016, 439–465.

[36] Angelova, V., Petkov, P. Componentwise Perturbation Analysis of the Singular Value Decomposition of a Matrix, *Applied Sciences*, 14(4), 2024, 1417.

[37] Watkins, D. S. The matrix eigenvalue problem: GR and QR, *SIAM Review*, 49(4), 2007, 513–534.

[38] Motlaghian, S. M., Armandnejad, A., Hall, F. J. A note on some classes of G-matrices, *Operators and Matrices*, 16, 2022, 251–263.

[39] Murrja, A., Sosoli, I., Tabaku, I., Totojani, O., Keco, R. Perception of Risk in Farm Activities: A Comparison of Matrix Analysis with Results from Multifactorial Linear Regression, *WSEAS Transactions on Environment and Development*, 21, 2025, 109–126.

[40] Vaziri, E., Jamshidi, M., Motallebi, H. Spectral clustering by considering stationary distribution vector and transition matrix, *Wavelets and Linear Algebra*, 10(2), 2023, 29–38.

[41] Rezagholi, S., Hosseini, M. C-spectral norm inequalities between operator matrices and their entries, *Wavelets and Linear Algebra*, 10(2), 2023, 39–50.

[42] Soleymani, M., Salemi, A. An equivalent condition for linear preservers of multivariate group majorization on matrices, *Wavelets and Linear Algebra*, 10(2), 2023, 71–80.

[43] Nazari, A., Nezami, A., Bayat, M. Inverse eigenvalues problem of distance matrices via unit lower triangular matrices, *Wavelets and Linear Algebra*, 10(1), 2023, 23–36.

[44] Ilkhanizadeh Manesh, A. On L-rays of Toeplitz matrices, *Wavelets and Linear Algebra*, 10(1), 2023, 45–51.

[45] Talebi, G., Aminizadeh, M. A note on Sonnenschein summability matrices, *Wavelets and Linear Algebra*, 10(1), 2023, 65–67.

[46] Beheshti, R., Fathi, J., Zangiabadi, M. Some classes of interval tensors and their properties, *Wavelets and Linear Algebra*, 9(1), 2022, 49–65.

[47] Melnikov, B., Melnikova, E. Application of Paired Correlation Algorithms for the Distance Matrices Between DNA Chains, *Molecular Sciences and Applications*, 4, 2024, 46–56.

[48] Porto, D. Fast Processing without Processors: The Conductance Matrices, *International Journal of Applied Mathematics, Computational Science and Systems Engineering*, 6, 2024, 112–118.

[49] Karkour, Y., Tajani, C., Khattabi, I. Generating a Set of Consistent Pairwise Comparison Test Matrices in AHP using Particle Swarm Optimization, *WSEAS Transactions on Mathematics*, 23, 2024, 206–215.

[50] Aasma, A., Natarajan, P. N. Matrix Transforms into the Set of a-Absolutely Convergent Sequences with Speed and the Regularity of Matrices on the Subspaces of c, *WSEAS Transactions on Mathematics*, 23, 2024.

[51] Erişir, T., Yıldırım, E. On the Fundamental Spinor Matrices of Real Quaternions, *WSEAS Transactions on Mathematics*, 22, 2023.

[52] Kuloğlu, B., Eser, E., Özkan, E. The r-circulant Matrices Associated with k-Fermat and k-Mersenne Numbers, *WSEAS Transactions on Mathematics*, 22, 2023.

[53] Gao, Y., Jiang, Z., Gong, Y. On the Determinants and Inverses of Skew Circulant and Skew Left Circulant Matrices with Fibonacci and Lucas Numbers, *WSEAS Transactions on Mathematics*, 12, 2013.

[54] Garcia-Planas, M. I., Dominguez-Garcia, J. L. A General Approach for Computing Residues of Partial-Fraction Expansion of Transfer Matrices, *WSEAS Transactions on Mathematics*, 12, 2013.

[55] Jiang, Z., Shen, N., Li, J. The Spectral Decomposition of Some Tridiagonal Matrices, *WSEAS Transactions on Mathematics*, 12, 2013.

[56] Zhou, J., Jiang, Z. Spectral Norms of Circulant-Type Matrices Involving Some Well-Known Numbers, *WSEAS Transactions on Mathematics*, 12, 2013.

[57] Pesaranghader, A., Muthaiyah, S. Semantic Similarity Using First and Second Order Co-occurrence Matrices and Information Content Vectors, *WSEAS Transactions on Computers*, 12, 2013.

[58] Li, C.-X., Wu, S.-L. Some Comparison Results with New Effective Preconditioners for L-Matrices, *WSEAS Transactions on Mathematics*, 13, 2014.

[59] Kririm, S., Hmamed, A. Robust H_∞ Filtering for Uncertain Differential Linear Repetitive Processes via LMIs and Polynomial Matrices, *WSEAS Transactions on Systems and Control*, 10, 2015.

[60] Zatopek, J. Using the Transformation Matrices not Only to Derive the Motion Equations, *WSEAS Transactions on Applied and Theoretical Mechanics*, 11, 2016.

[61] Liu, X., Jie, Y. New Construction of Deterministic Compressed Sensing Matrices via Singular Linear Spaces over Finite Fields, *WSEAS Transactions on Mathematics*, 15, 2016.

[62] Kantalo, J. Determinants and Permanents of Hessenberg Matrices with Perrin's Bivariate Complex Polynomials and Its Application, *WSEAS Transactions on Mathematics*, 22, 2023.

[63] Shen, N., Jiang, Z., Li, J. On Explicit Determinants of the RFMLR and RLMFL Circulant Matrices Involving Certain Famous Numbers, *WSEAS Transactions on Mathematics*, 12, 2013.

[64] Jiang, Z., Li, J., Shen, N. On the Explicit Determinants and Singularities of r-circulant and Left r-circulant Matrices with Some Famous Numbers, *WSEAS Transactions on Mathematics*, 12, 2013.

[65] Jiang, Z. Efficient Algorithms for Finding the Minimal Polynomials and the Inverses of Level-k FLS $(r_1, \ldots, r_k)$-Circulant Matrices, *WSEAS Transactions on Mathematics*, 12, 2013.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.