On the Homotopy-First Integral method for non-conservative oscillators

ANDRÉS GARCÍA
GESE-Departamento de Ingeniería Eléctrica
Universidad Tecnológica Nacional-Facultad Regional Bahía Blanca
11 de abril 461, Bahía Blanca, Buenos Aires
ARGENTINA

Abstract: - This paper presents a ready-to-use formula for determining the number and approximate location of periodic orbits in second-order Lienard systems. As a result of the exact closed-form derived in [16], in which an ordinary differential equation (ODE) must be solved to determine the existence and location of periodic orbits for general non-conservative oscillators, a homotopy functional is defined for Lienard-type systems. This provides a closed-form and ready-to-use polynomial formula with roots as an approximation of the periodic orbit's amplitude.
In addition, some examples are analyzed, along with conclusions and future plans.

Key-Words: -Periodic orbits, Homotopy method, First-integral, Series expansion

Received: May 6, 2023. Revised: May 3, 2024. Accepted: June 15, 2024. Published: July 16, 2024.

1 Introduction

It is well known that dynamical systems written as a single or a collection of ordinary differential equations (ODEs) need to be evaluated to determine their parametric properties (see for instance [1], [2] and [3]).

Oscillations and periodic orbits are key behaviors of general ODEs ([4]). Determining periodic orbits can be complex and sometimes impossible using closed-form or accurate approximate formulas or reduced-order models ([5], [6]).

Even second-order ODEs in the plane can exhibit intricate periodic patterns that defy closed-form analysis (see [7]).
Simplified versions of these systems have been proposed as toy models for such complexity ([8]).
Two key studies in this field are discussed in [9]:

- Determine all possible periodic orbits with a given initial amplitude
- Time parameterization in time

The first case involves determining a single number, the period, while the second case is more complex and intricate. See references [4] and [10] for further details.

It is well-known that a second-order oscillator has a non-differential relation between its period and amplitude known as a first- integral (see for instance [11] and [12]). However, computing this first-integral can be challenging due to complex integrals.

In [13], a reduced dynamics approach was introduced to calculate a subset of periodic orbits by solving an equivalent first-order ODE. Applying this method to Mickens' oscillator (referenced in [14]-[15]) yields a closed-form formula for their amplitude-period relation ([16]).

Solving the reduced order ODE presented in [16] provides a complete solution to upper bound the number of limit cycles for Lienard systems ([6]).

However, estimating the location and amplitudes of periodic orbits remains challenging.

This paper presents a formula for identifying the number and location of periodic orbits in second - order Lienard systems. It introduces a homotopy and a reduced order ODE from a previous study, offering a polynomial formula with roots to approximate the amplitude of the periodic orbits.

This paper is organized as follows: Section 2 outlines the method with an appropriate homotopy, Section 3 provides application examples, Section 4 discusses precision and the potential Artificial Computational Intelligence extensions, and Section 5 offers some conclusions and future research directions.

2 The Homotopy-First Integral method

According to [13] and [16], given a second order ODE:

$$\ddot{x}(t) = f(x(t), \dot{x}(t))$$

Where $\dot{x}$ means time derivatives, a periodic orbit of amplitude A exists if and only if the following first order ODE possess solution for $x \in [0, A]$:

$$\frac{d\phi(x)}{dx} = \frac{f(x, \phi(x))}{\phi(x)}, \phi \in \mathbb{R}, \phi(A) = 0$$

If we restrict ourselves to the Lienard's case:

$$f(x, \dot{x}) = -x - \frac{dF(x)}{dx} \cdot \dot{x}$$

Then, it is possible to define an homotopy (see for instance [17]):

$$(1-p)\cdot\left(\phi(x)\cdot\frac{dF(x)}{dx}\right)+ \\ +p\cdot\left(\phi(x)\cdot\frac{dF(x)}{dx}+x+\phi(x)\cdot\frac{d\phi(x)}{dx}\right)=0, \quad p\in[0,1] \quad (1)$$

Assuming the classical series representation for $\phi(x)$ ([17]):

$$\phi(x) = \sum_{i=1}^{\infty} p^i \cdot \phi_i(x) \quad (2)$$

Replacing (2) into (1), the homotopy equation to solve leads:

$$\frac{dF(x)}{dx} \cdot \sum_{i=1}^{\infty} p^i \cdot \phi_i(x) + p \cdot \left(x + \sum_{i=1}^{\infty} p^i \cdot \phi_i(x) \cdot \sum_{j=1}^{\infty} p^j \cdot \frac{d\phi_j(x)}{dx}\right) = 0, \; p \in [0,1] \quad (3)$$

To maintain the equality with zero, every all coefficients in this series must be zero:

p^1

$$\frac{dF(x)}{dx} \cdot \phi_1 + x = 0 \Rightarrow \phi_1(x) = -\frac{x}{F(x)'}$$

Where $F(x)' = \frac{dF(x)}{dx}$.

p^2

$$\frac{dF(x)}{dx} \cdot \phi_2 = 0 \Rightarrow \phi_2(x) = 0$$

p^3

$$\frac{dF(x)}{dx} \cdot \phi_3(x) + \phi_1(x) \cdot \phi_1(x)' = 0 \Rightarrow$$

$$\phi_3(x) = -\frac{\phi_1(x) \cdot \phi_1(x)'}{F(x)'}$$

Where $\phi_1(x)' = \frac{d\phi_1(x)}{dx}$. The sequence can be continued, by adding more terms, but (2) can be also truncated at *i=3*:

$$\phi(x) \cong \phi_1(x) + \phi_3(x) = -\frac{x}{F(x)'} - \frac{\phi_1(x)\cdot\phi_1(x)'}{F(x)'} = -\frac{\left(x - \left(\frac{x}{F(x)'}\right)\cdot\phi_1(x)'\right)}{F(x)'}, \; \phi(A) = 0 \quad (4)$$

Equation (4) can be further developed:

$$\phi(x) \cong -\left[\frac{x - \frac{x}{F(x)'}\cdot\left(-\frac{F(x)' - x\cdot F(x)''}{F(x)'^2}\right)}{F(x)'}\right], \quad \phi(A) = 0 \quad (5)$$

Then, replacing *x=A* in (5):

$$\left[\frac{A-\frac{A}{F(A)'}\cdot\left(-\frac{F(A)'-A\cdot F(A)''}{F(A)'^2}\right)}{F(A)'}\right]=0\Leftrightarrow A-\frac{A}{F(A)'}\cdot\left(-\frac{F(A)'-A\cdot F(A)''}{F(A)'^2}\right)=0 \quad (6)$$

Where $F(A)' = F(x)'|_{x=A}$ and $F(A)'' = F(x)''|_{x=A}$. It is important to note that:

If $F(A)'=0 \Rightarrow$ (3) does not depend on $F(x)$

Finally, from (6) with $F(A)' \neq 0$:

$$F(A)'^3 = -F(A)' + A \cdot F(A)'' \quad (7)$$

3 Application examples

Using (7), we can predict the presence of limit cycles and estimate their amplitudes.

Example 1

Considering the well-known Van der Pol's oscillator ([18]):

$$f(x,\dot{x}) = -x - (3\cdot x^2 - 1)\cdot\dot{x}$$

Equation (7) leads:

$$F(A)'^3 = -F(A)' + A\cdot F(A)'' \Leftrightarrow (3\cdot A^2-1)^3 = -(3\cdot A^2 - 1) \quad + A\cdot 6\cdot A$$

That is $A = 0.96443$. In numerical simulations, this oscillator's the amplitude is approximately *A=1.16*.

Example 2

In [19], the following Lienard system was considered:

$$f(x,\dot{x}) = -x - (0.8\cdot x - 4\cdot x^2 + 0.32\cdot 5\cdot x^4)\cdot\dot{x}$$

In other words:

$$\begin{cases} F(x) = 0.8\cdot x - \frac{4}{3}\cdot x^3 + 0.32\cdot x^5 \\ F'(x) = 0.8 - 4\cdot x^2 + 0.32\cdot 5\cdot x^4 \\ F''(x) = -8\cdot x + 0.32\cdot 20\cdot x^3 \end{cases}$$

Then, equation (7) yields:

$$\left(F(A)'\right)^3 = -F(A)' + A\cdot F(A)'' \Leftrightarrow$$
$$\Leftrightarrow 4.096\cdot A^{12\prime}30.72\cdot A^{10} + 82.944\cdot A^{8\prime}94.72\cdot A^6 +$$
$$+36.672\cdot A^{4\prime}3.68\cdot A^2 + 1.312 = 0$$

Finding the real roots:

$$A_1 = 1.72721$$
$$A_2 = 0.803071$$

Based on the values provided in [19]: $A_1 = 1.8$, $A_2 = 1$, the accuracy obtained is around 80% in the worst case.

Example 3

For instance, in [19], the paper introduces the following parameterized system:

$$f(x,\dot{x}) = -x - (20\cdot x^3 - 6\cdot\mu\cdot x)\cdot\dot{x}$$

Two limit cycles are present for μ>2.5, while no limit cycles are observed for μ<2. Applying equation (7):

$$\begin{cases} F(x) = x - \mu\cdot x^3 + x^5 \\ F'(x) = 1 - 3\cdot\mu\cdot x^2 + 5\cdot x^4 \Rightarrow \text{Replacing } A^2 = y \Rightarrow \\ F''(x) = -6\cdot\mu\cdot x + 20\cdot x^3 \end{cases}$$

$$\Rightarrow \left(1 - 3\cdot\mu\cdot y + 5\cdot y^2\right)^3 = 15\cdot y^2 - 3\cdot\mu\cdot y - 1$$

Obtaining Fig. 1 and Fig. 2:

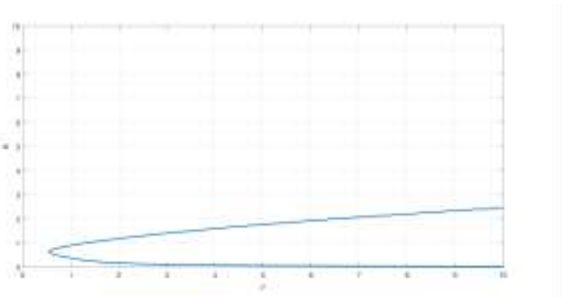

Figure 1: Parametric curve for Example 2

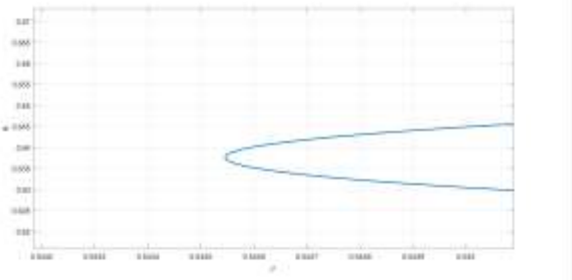

Figure 2: Zoom in the parametric curve

Clearly, for μ>0.55 two limit cycles exist.

Example 4

Building on the previous discussion in paper [19], we can now consider a final example:

$$\ddot{x} = f(x,\dot{x}) =$$
$$= -x - 7\cdot(x^6 - 11.1143\cdot x^4 + 29.6914\cdot x^2 - 13.1657)\cdot \dot{x}$$

Again, equation (7) yields:

$$\begin{cases} F(x) = x\cdot\left(x^2-1.6^2\right)\cdot\left(x^2-4\right)\cdot\left(x^2-9\right) \\ F'(x) = 7\cdot\left(x^6-11.1143\cdot x^4+29.6914\cdot x^2-13.1657\right) \Rightarrow \\ F''(x) = 42\cdot\left(x^5-7.40952\cdot x^3+9.89714\right) \end{cases}$$
$$\Rightarrow A_1 = 0.7701, \quad A_2 = 1.8380, \quad A_3 = 2.7133$$

Figures 3 and 4 in Matlab illustrate a potential limit cycle with an approximate amplitude of 3.1 in *x(t)*. In this way, the homotopy-first integral serves as both a numerical method for approximating periodic orbit amplitudes and an upper bound for the number of limit cycles.

Based on the conjecture from reference [18] of a single limit cycle without specifying its amplitude, the method in this paper approximates this amplitude value with A_3 above, suggesting a possible maximum of three limit cycles.

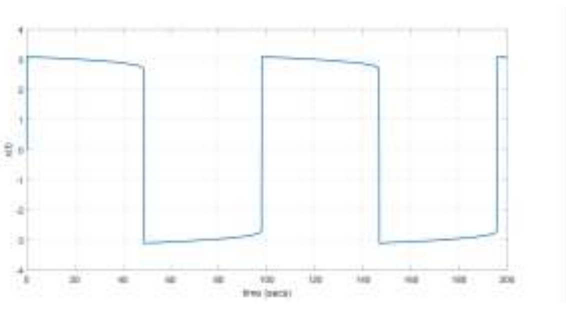

Figure 3: Limit cycle in *x(t)*

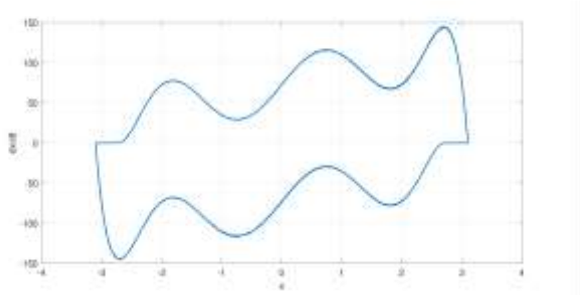

Figure 4: Limit cycle's phase portrait

Example 5

Practical applications utilizing oscillation circuits in power electronics, including DC-DC, DC-AC, and LED drivers, can be achieved through the use of relaxation oscillator circuits [21]:

$$f(x,\dot{x}) = -x - \lambda\cdot(x^2-1)\cdot\dot{x}$$

An harmonic oscillator is obtained when λ=0 in equation (7):

$$A = -1$$

The polynomial approximation is inaccurate for pure harmonic oscillators due to the third-order truncation in equation (7). However, for λ≠0, the condition for periodic orbits can be determined:

$$\lambda \geq \frac{1}{27}$$

The amplitude obtained wit the border value λ=1/27 using equation (7) is A=5.3805. This value is higher than the conjectured value in [20] which falls between 2 and 2.0235, indicating an error of more than double. The following section will provide insights into the accuracy issue.

4 Discussion

4.1 Approximation accuracy

To assess the accuracy of our approximation formula (7), we can analyze the classical Van der Pol equation in more depth:

$$f(x,\dot{x}) = -x - \mu\cdot(x^2-1)\cdot\dot{x}$$

According to [20] pp.7, the unique limit cycle has an amplitude between 2 and 2.0235 for all *μ>0*. When formula (7) is applied:

$$\mu^3 \cdot \left(A^2 - 1\right)^3 = -\mu \cdot \left(A^2 - 1\right) + 2 \cdot \mu \cdot A^2$$

In other words:

$$\mu^2 = \frac{A^2 + 1}{\left(A^2 - 1\right)^3} \qquad (8)$$

Plotting the curve μ-A as shown in Fig. 5:

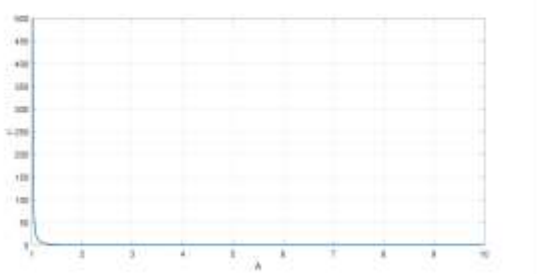

Figure 5: A-μ curve using (7)

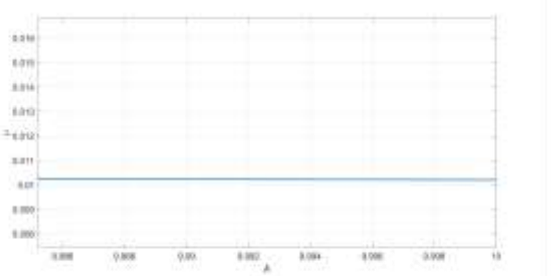

Figure 6: Figure 5 zoom

The conjectured interval $A \in [2,2.0235]$ is approximated by the curve in Fig. 5 with $\mu \in [0.416,0.43]$.

On the other hand, for $A<10$, an interval $\mu \in [0.01, \infty)$ is covered. Furthermore, by calculating the estimated amplitude using equation (8) and comparing it to the minimum amplitude suggested in [20], we observe that near-zero or zero error is achieved when μ values are close to unity (refer to Figures 7 and 8). It is also beneficial to compare the small values approximation in [20]: ($A \approx 2 + 7/96 \cdot \mu^2$) with equation (8), as illustrated in Figure 9.

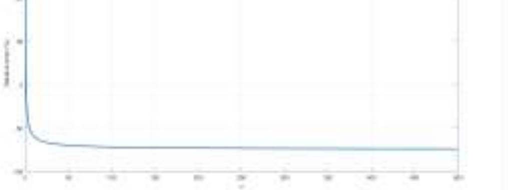

Figure 7: Relative error

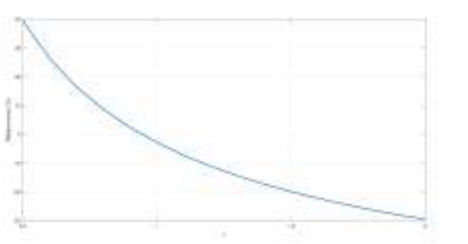

Figure 8: Figure 7 zoom

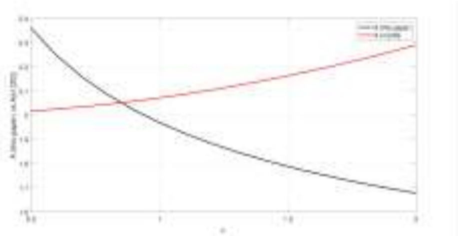

Figure 9: Comparison with reference [20] for small values of μ

4.2 Artificial/Computational Intelligence extensions

Possible extensions to a broader scope include utilizing Deep Neural Networks (DNN) for simulations based on collected data from the non-linear oscillator ([22] and [23]). This approach can help focus on central points for more accurate results.

5 Conclusion

This paper presents a method for determining periodic orbits of general nonlinear oscillators by using closed-form reduced order ODEs derived in a previous study. An homotopy approach is developed to combine with the resulting polynomial from applying the reduced order ODE to Lienard systems.

By defining a classical infinite sum as a solution for the homotopy and truncating it around its third term, formula (7) offers a quick and efficient way to determine both:

- Approximate number of periodic orbits
- Approximate amplitude of periodic orbits

The real roots of the polynomial indicate the locations and quantities of periodic orbits in a Lienard system.

In future work, a higher truncation order will be used in the homotopy defined in this paper. The extensions discussed here will be further investigated for general non-linear oscillators, beyond just Lienard oscillators, as the scope of the paper [16] is broad.

As suggested exploring the potential combination of artificial intelligence or computational intelligence is an important future research direction.

The focus will be on extending the formulas and homotopy from this study to develop a general method for approximating and validating oscillations with diverse behaviors, particularly in neuroscience and power electronics, where non-linear oscillators play a vital role.

Acknowledgement:
This work is supported by Universidad Tecnológica Nacional-Facultad Regional Bahía Blanca, Departamento de Ingeniería Eléctrica and GESE.

References:

[1] Carmen Chicone. Ordinary Differential Equations with Applications. Springer. 2006, Volume 34. ISBN: 978-0-387-30769-5
[2] Morris W. Hirsch, Stephen Smale, Robert L. Devaney. Differential Equations, Dynamical Systems, and an Introduction to Chaos. Academic Press; 3rd edition. 2012.
[3] Kazem Meidani, Amir Barati Farimani. Identification of parametric dynamical systems using integer programming. Expert Systems with Applications. 2023. Volume 219, ISSN 0957-4174. https://doi.org/10.1016/j.eswa.2023.119622.
[4] Cveticanin L. Strong Nonlinear Oscillators. Springer. 2018. ISBN : 978-3-319-58825-4
[5] Liu, C.-S.; Chang, C.-W.; Chen, Y.-W.; Chang, Y.-S. Periodic Orbits of Nonlinear Ordinary Differential Equations Computed by a Boundary Shape Function Method. *Symmetry* 2022,*14*,1313.https://doi.org/10.3390/sym14071313
[6] A. García. Bounding periodic orbits in second order systems. WSEAS TRANSACTIONS on SYSTEMS and CONTROL. Vol. 17, 2022.DOI: 10.37394/23203.2022.17.55.
[7] Daniel S. Graça and Ning Zhong. Computing the exact number of periodic orbits for planar flows. Trans. Amer. Math. Soc. 375 (2022), 5491-5538
[8] Steve Smale. Mathematical Problems for the Next Century. Mathematical Intelligencer. 20 (2): 7–15. 1998. CiteSeerX 10.1.1.35.4101. doi:10.1007/bf03025291. S2CID 1331144.
[9] Santana, M.V.B. Exact Solutions of Nonlinear Second-Order Autonomous Ordinary Differential Equations: Application to Mechanical Systems. Dynamics 2023, 3, 444-467. https://doi.org/10.3390/dynamics303002
[10] Kwari, L.J.; Sunday, J.; Ndam, J.N.; Shokri, A.; Wang, Y. On the Simulations of Second-Order Oscillatory Problems with Applications to Physical Systems. Axioms 2023, 12, 282.
[11] L. D. Landau and E.M. Lifshitz. Mechanics. Elsevier. Vol. 1. 1982.
[12] El-Dib YO. A review of the frequency-amplitude formula for nonlinear oscillators and its advancements. Journal of Low Frequency Noise, Vibration and Active Control. 2024. doi:10.1177/14613484241244992
[13] First Integrals vs Limit Cycles. arXiv:1909.07845 [math.DS]. Andrés García. 2019.
[14] Investigation of the properties of the period for the nonlinear oscillator $\ddot{x}+(1+\dot{x}^2)\cdot x=0$. R.E. Mickens, Journal of Sound and Vibration, 292, 1031-1035, 2006.
[15] Truly nonlinear oscillations: Harmonic balance, parameter expansions, iteration, and averaging methods. R. E. Mickens. World Scientific. 2010.
[16] J. H. He and A. García. The simplest amplitude-period formula for non-conservative oscillators. Reports in Mechanical Engineering, 2(1), 143–148. 2021. https://doi.org/10.31181/rme200102143h
[17] D-N Yu, H-J He and A. Garcıa. Homotopy perturbation method with an auxiliary parameter for nonlinear oscillators. Journal of Low Frequency Noise, Vibration and Active Control. Vol. 38 (3-4):1540-1554. 2019. doi:10.1177/1461348418811028
[18] B. van der Pol. A theory of the amplitude of free and forced triode vibrations, Radio Review, Vol. 1, pp. 701–710, 754–762. 1920.
[19] H. Giacomini and S. Neukirch. On the number of limit cycles of the Liénard equation. PHYSICAL REVIEW E, Vol.56, 1997.
[20] Kenzi Odani. On the limit cycle of the Liénard equation. Archivum Mathematicum, Vol. 36 1, pp. 25-31. 2000.
[21] Voglhuber-Brunnmaier, T. and Jakoby, B. Understanding Relaxation Oscillator Circuits Using Fast-Slow System Representations. 2023. IEEE Access.

[22] Rahman, J.U.; Danish, S.; Lu, D. Oscillator Simulation with Deep Neural Networks. *Mathematics.* Volume 12. 2024. https://doi.org/10.3390/math12070959.

[23] Khan, N.A.; Alshammari, F.S.; Romero, C.A.T.; Sulaiman, M. Study of Nonlinear Models of Oscillatory Systems by Applying an Intelligent Computational Technique. *Entropy* Volume 23. 2021. https://doi.org/10.3390/e23121685

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The author would like to acknowledge Departamento de Ingeniería Eléctrica at Universidad Tecnológica Nacional and GESE.

Conflict of Interest

The author have no conflicts of interest to declare that are relevant to the content of this article.