Cosmological aspects of the theory of equations of the Vlasov–Einstein type and their consequences

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Abstract: - The authors propose a general scheme of derivation from the general relativistic Einstein-Hilbert action for a system of gravitationally interacting charged particles, Hamilton's dynamics equations and field equations. In accordance with the proposed methodology, new forms of equations of Vlasov type are obtained in the general relativistic case, nonrelativistic and weakly relativistic limits. Expressions are established for the resulting corrections in the equation Poisson, which can contribute to the action of dark matter and dark energy. An effective approach to synchronizing the proper times of different particles of a many-particle system is proposed based on invariance of the form of action. Authors derived (using hydrodynamic substitution) and solved the Euler-type equations leading to the cosmological Friedmann and Milne-McCrea models.

Key-Words: Lorentz action, Lagrangian, Einstein–Hilbert action, Vlasov–Einstein equation

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1 Introduction

Description of the dynamics and kinetics of composite astrophysical systems taking into account the effects of general relativity requires the development and deepening theory of many-body systems of interacting particles [1]–[5]. In the publications of modern authors, the main attention is paid to the study of the relativistic movement of objects representing a collection or dust-like "test particles" (which reduces the consideration of the entire system of particles to the study of the movement of one particle in external fields), or particles interacting gravitationally (in the absence of generalization to the case of charges on these particles) [6]–[10]. An accurate picture of the dynamics of the system must take into account that the movement of each particle occurs along geodesic lines, along which it is necessary to introduce individual (natural) parameters. The "proper time" of the particle is usually chosen as such a parameter, moving along a given trajectory [11]–[15]. However, such a parameterization leads only to a special case of the form of the general relativistic dynamics of a system of particles. In the works [16]–[19] the authors developed a methodology use of an additional continuous variable in general relativistic dynamics, and the subsequent transition to the 7-dimensional formalism based on the use of "observer time", weakening the need to introduce conditions on the energy manifold [6],[14] in canonical impulses. In this case, the expression for the total Hilbert–Einstein action and the Hamiltonian equations the movements take on a new form (using the formalism of particle distribution functions), as well as the expression for the energy-momentum tensor in Einstein's equations [20]–[26].

In this paper, the authors consider applications of Vlasov-type equations to cosmological problems based on theory of geodesics with electromagnetic fields for classical Lagrangians and carrying out time synchronization particles for multiparticle tasks. For this purpose, a Hamiltonian formulation of the equations of motion was obtained and a new form of the Liouville equation was derived. Further, the article considers the transition to the 3-dimensional formalism for the Vlasov–Einstein equations; a hypothesis about the origin of the cosmological lambda term, based on the structure of the total composite action of the particle system and gravitational and electromagnetic fields. Next, the authors give examples of the simplest hydrodynamic consequences of the Vlasov-Einstein equations, including the study of hydrodynamic consequences of the Vlasov–Poisson–Poisson equations for the analysis of the generalized Milne-McCrea model.

2 Liouville equation in extended phase space

General relativistic action for moving charged (with charge e) particle of mass m in the presence of gravitational and electromagnetic fields can be written in the following form:

$$S_1 = -mc\int_0^{\lambda_m} (g_{\mu\nu}(\boldsymbol{X})\frac{dX^\mu(\boldsymbol{q},\lambda)}{d\lambda} \times \frac{dX^\nu(\boldsymbol{q},\lambda)}{d\lambda})^{1/2} d\lambda - \frac{e}{c}\int A_\mu \frac{dX^\mu}{d\lambda}d\lambda,$$

where: $g_{\mu\nu}(\boldsymbol{X})$ is a fundamental tensor of 4-dimensional space-time ($\boldsymbol{X} = \{X^\mu\}_{\mu=\overline{0,3}}$), $A_\mu(\boldsymbol{X}) \equiv \{\varphi(\boldsymbol{X}); \boldsymbol{A}(\boldsymbol{X})\}$ are 4-electromagnetic field potential, q is a Lagrangian particle parameter; the variable $\lambda(> 0)$ is proportional to affine parameter: $ds = \sqrt{I}d\lambda$, $I \equiv g_{\mu\nu}(dX^\mu/d\lambda)(dX^\nu/d\lambda)$.

Let write down the Euler–Lagrange equations for the action S_1:

$$\frac{mc}{\sqrt{I}}\frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dX^\nu}{d\lambda}\right) + \frac{e}{c}\frac{dA_\mu}{d\lambda} = \frac{mc}{2\sqrt{I}}\frac{\partial g_{\nu\zeta}}{\partial X^\mu}\frac{dX^\nu}{d\lambda}\frac{dX^\zeta}{d\lambda} + \frac{e}{c}\frac{\partial A_\nu}{\partial X^\mu}\frac{dX^\nu}{d\lambda}. \quad (1)$$

This shows that in the absence of electromagnetic interaction between particles, the value $mc/\sqrt{I}$ is reduced, and the equations of motion can be written equivalently using both the parameter λ and the parameter proper interval s. However, taking into account the electromagnetic interaction leads to different equations when using different parameters. Although, how can be seen from equation (1), one can in principle go to the affine parameter s by expressing $d\lambda$ in terms of ds and I: $ds = \sqrt{I}d\lambda$. In multiparticle systems this possibility is absent. Let consider an action similar to S_1, but for a system of many particles with differing masses m_a and charges e_a ($a = \overline{1,N}$):

$$S_{1,\Sigma} = -\sum_a m_a c\int\sqrt{g_{\mu\nu}\frac{dX_a^\mu}{d\lambda}\frac{dX_a^\nu}{d\lambda}}d\lambda - \sum_a \frac{e_a}{c}\int A_\mu \frac{dX_a^\mu}{d\lambda}d\lambda.$$

To describe the dynamics of a many-particle system associated with the action of $S_{1,\Sigma}$, canonical ones can be entered in a standard way impulses:

$$(Q_a)_\mu = \frac{\partial L}{\partial V_a^\mu} = -\frac{m_a c}{\sqrt{I_a}}g_{\mu\nu}(\boldsymbol{X}_a)V_a^\nu - \frac{e_a}{c}A_\mu(\boldsymbol{X}_a),$$
$$V_a^\nu \equiv \frac{\partial X_a^\nu}{\partial\lambda}.$$

Hamilton's equations associated with canonical variables $(\boldsymbol{X}_a, \boldsymbol{Q}_a)$:

$$V_a^\nu = -\frac{\sqrt{I_a}}{m_a c}g^{\mu\nu}(\boldsymbol{X}_a)((Q_a)_\mu + \frac{e_a}{c}A_\mu).$$

$$\frac{d(Q_a)_\mu}{d\lambda} = \sum_a \frac{\sqrt{I_a}}{m_a c}((Q_a)_\zeta + \frac{e_a}{c}A_\zeta(\boldsymbol{X}_a))\frac{\partial g^{\zeta\nu}}{\partial X_a^\mu}((Q_a)_\nu + \frac{e_a}{c}A_\nu(\boldsymbol{X}_a)) + \frac{e_a\sqrt{I_a}}{m_a c^2}((Q_a)_\zeta + \frac{e_a}{c}A_\zeta(\boldsymbol{X}_a))g^{\zeta\xi}\frac{\partial A_\xi(\boldsymbol{X}_a)}{\partial X_a^\mu}.$$

Let introduce the partial distribution function $f_a(\boldsymbol{X}, \boldsymbol{Q}, \lambda)$ over the extended 9-dimensional phase space. The corresponding Liouville equation for f_a takes the following form:

$$\frac{\partial f_a(X,Q,\lambda)}{\partial\lambda} - \frac{\sqrt{I_a}}{m_a c}g^{\mu\nu}(X_a)((Q_a)_\mu + \frac{e}{c}A_\mu)\frac{\partial f_a}{\partial X^\nu} + + (\frac{\sqrt{I_a}}{m_a c}((Q_a)_\zeta + \frac{e_a}{c}A_\zeta(X_a))\frac{\partial g^{\zeta\nu}}{\partial X_a^\mu}((Q_a)_\nu + \frac{e_a}{c}A_\nu(X_a)) + \frac{e_a\sqrt{I_a}}{m_a c^2}((Q_a)_\zeta + \frac{e_a}{c}A_\zeta(X_a))g^{\zeta\xi}\frac{\partial A_\xi}{\partial X_a^\mu})\frac{\partial f_a}{\partial Q_\mu} = 0 \quad (2)$$

Let present λ–stationary form of the Liouville equation when $f_a = f_a(\boldsymbol{X}, \boldsymbol{P})$, i.e. e. does not depend on the parametric variable λ:

$$-g^{\mu\nu}(\boldsymbol{X})P_\mu\frac{\partial f_a(\boldsymbol{X},\boldsymbol{P})}{\partial X^\nu} + (-\frac{1}{2}\frac{\partial g^{\nu\zeta}}{\partial X^\mu}P_\nu P_\zeta + \frac{e_a}{c}F_{\mu\nu}(\boldsymbol{X})g^{\zeta\nu}P_\nu)\frac{\partial f_a}{\partial P_\mu} = 0$$

(since $X^0 = ct$, the last equation in the general case is t-nonstationary).

Example 1. Let consider a special case of equation (1), when the metric $g_{\mu\nu}$ and components of the vector potential A_μ do not depend on the time coordinate. Then the right side of equality (1) at index $\mu = 0$ is canceled, and perhaps analytically integrate the left-hand side (we omit the index a):

$$\frac{mc}{\sqrt{I}}(g_{0\nu}\frac{dX^\nu}{d\lambda}) + \frac{e}{c}A_0 = -Q_0.$$

The meaning of the resulting integral can be clarified by taking the post-Galilean metric $g_{\mu\nu} = \mathrm{diag}(1 + 2\Phi/c^2, -1, -1, -1)$ (*Landau metric*), where $\Phi(X^j)$ is the Newtonian gravitational potential. Then the last relation transforms to the form

$$\frac{mc}{\sqrt{I}}(1 + \frac{2\Phi}{c^2})\frac{dX^0}{d\lambda} + \frac{e}{c}A_0 = -Q_0,$$

and the remaining Euler-Lagrange equations of system (1) take the form:

$$\frac{mc}{\sqrt{I}}\frac{d}{d\lambda}\frac{dX^j}{d\lambda}+\frac{e}{c}\frac{dA_j}{d\lambda}=\frac{mc}{2c^2\sqrt{I}}\frac{\partial\Phi}{\partial X^j}(\frac{dX^0}{d\lambda})^2+\frac{e}{c}\frac{\partial A_\nu}{\partial X^j}\frac{dX^\nu}{d\lambda}, j=1,2,3. \quad (3)$$

Replacing the parameter λ from equation (3) with time t, we obtain the equations of motion of a charged particle in an electrostatic field and in the gravitational potential Φ:

$$\frac{d}{dt}(M\frac{dX^j}{dt})=-M\frac{\partial\Phi}{\partial X^j}+\frac{e}{c}F_{\mu j}\frac{dX^\mu}{dt},$$

where $M=-(Q_0/c-eA_0/c^2)/(1+2\Phi/c^2)$ — the effective mass of the particle in the superposition of gravitational and electromagnetic fields. Let give an explicit expression for Q_0 and, through it, for M:

$$Q_0=-\frac{mc(1+2\Phi/c^2)}{\sqrt{1-\boldsymbol{v}^2/c^2+2\Phi/c^2}}-\frac{e}{c}A_0,$$

$$M=\frac{m}{\sqrt{1-\boldsymbol{v}^2/c^2+2\Phi/c^2}}.$$

In addition to the Landau metric, when passing to the post-Newtonian approximation, one can also considerthe *Fock metric*:

$$g_{\mu\nu}=\mathrm{diag}(1+2\Phi/c^2,-(1-2\Phi/c^2),-(1-2\Phi/c^2),-(1-2\Phi/c^2)).$$

The equation of motion in this case takes the following form:

$$\frac{d}{dt}(M\frac{dX^j}{dt})=-M\frac{1+\boldsymbol{v}^2/c^2}{1-2\Phi/c^2}\frac{\partial\Phi}{\partial X^j}+\frac{e}{c}F_{\mu j}\frac{dX^\mu}{dt},$$

and the explicit expressions for Q_0 and M are as follows:

$$Q_0=-\frac{mc(1+2\Phi/c^2)}{\sqrt{1-\boldsymbol{v}^2/c^2+2\Phi/c^2+2\Phi\boldsymbol{v}^2/c^4}}-\frac{e}{c}A_0,$$

$$M=-\frac{(Q_0/c-eA_0/c^2)(1-2\Phi/c^2)}{1+2\Phi/c^2}=\frac{m(1-2\Phi/c^2)}{\sqrt{1-\boldsymbol{v}^2/c^2+2\Phi/c^2+2\Phi\boldsymbol{v}^2/c^4}}.$$

Example 2. Let consider the case when gravitational and electromagnetic fields depend only on the time variable t (which means the Universe is completely homogeneous). In this case, equations (1) can be integrated methods of Hamiltonian mechanics. It is interesting to examine some particular aspects of the situation. We have here three integrals of motion

$$\frac{mc}{\sqrt{I}}(g_{k\mu}\frac{dX^\mu}{d\lambda})+\frac{e}{c}A_k=-Q_k, k=1,2,3.$$

We use the energy integral

$$I=g_{\alpha\beta}(dX^\alpha/d\lambda)(dX^\beta/d\lambda)$$

instead of the equation for the zeroth component. Spatial components of non-canonical momentum depend only on time: $P_k=eA_k/c+Q_k$. Zeroth momentum component, also depending only on the variable t, is determined from the energy condition $g^{\mu\nu}P_\mu P_\nu=m^2c^2$. The equations of motion will then take the form:

$$\frac{dX^\mu}{d\lambda}=-\frac{\sqrt{I}}{mc}g^{\alpha\mu}(X^0)P_\alpha.$$

Eliminating the variable λ from here by dividing the expressions for the three integrals of motion by the equation for $k=0$, we obtain

$$\frac{dX^k}{dX^0}=\frac{g^{\mu k}(X^0)P_\mu(X^0)}{g^{\nu 0}(X^0)P_\nu(X^0)}=\frac{g^{\mu k}(X^0)(eA_\mu(X^0)/c+Q_\mu)}{g^{0\nu}(X^0)(eA_\nu(X^0)/c+Q_\nu)}.$$

Example 3. The generalized De Sitter Universe:

$$ds^2=c^2dt^2-\exp(2Ht)(dx^2+dy^2+dz^2)=c^2dt^2-\exp(2Ht)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2).$$

We get $g^{\alpha\beta}=\mathrm{d}iag(1,-e^{-2Ht},-e^{-2Ht},-e^{-2Ht})$, therefore we have a simplification of the last formula for Example 2:

$$\frac{dX^k}{dX^0}=-\frac{P_k\exp(-2Ht)}{P_0}=-\frac{Q_k\exp(-2Ht)}{Q_0}.$$

In this case, the canonical momenta Q_k ($k=1,2,3$) are (conserved) integrals, and Q_0 is determined from the energy condition of Example 2:

$Q_0^2-\exp(-2Ht)\boldsymbol{Q}^2=m^2c^2, \boldsymbol{Q}^2=Q_1^2+Q_2^2+Q_3^2$, that is, $Q_0^2=m^2c^2+\exp(-2Ht)\boldsymbol{Q}^2$,

$$\frac{dX^k}{dt}\equiv V^k=\frac{cQ_k\exp(-2Ht)}{\sqrt{m^2c^2+\boldsymbol{Q}^2\exp(-2Ht)}}.$$

The Euler–Lagrange equations is a consequence of considering the variational problem, and the above equations (for $k=1,2,3$) represent together with a set of initial data $X^k(X^0=0)=X_0^k$, $V^k(X^0=0)=V_0^k$ Cauchy problem, the solution of which completely determines the spatial evolution of the particle in the De Sitter metric. These equations can be easily integrated, so we get:

$$X^k(t)=-\frac{cQ_k\sqrt{m^2c^2+\boldsymbol{Q}^2\exp(-2Ht)}}{H\boldsymbol{Q}^2}+C_X^k,$$

where arbitrary constants C_X^k are determined from the initial conditions:

$$C_X^k=\frac{cQ_k}{H\boldsymbol{Q}^2}\sqrt{m^2c^2+\boldsymbol{Q}^2}+X_0^k,$$

in this case, the values of the Q_k integrals are related to the Cauchy data V_0^k: $V_0^k=Q_k/\sqrt{m^2c^2+\boldsymbol{Q}^2}$. Let's integrate the equations of motion over the time interval $[0,t]$ for $k=1,2,3$, we get:

$$X^k(t)=X^k(0)+\frac{cQ_k}{H\boldsymbol{Q}^2}(\sqrt{m^2c^2+\boldsymbol{Q}^2}-\sqrt{m^2c^2+\exp(-2Ht)Q^2}).$$

For a light-like geodesic you should put $m=0$, then the last formula will be significantly simplified:

$$X^k(t) = X^k(0) + \frac{cQ^k}{H\boldsymbol{Q}^2}(1 - \exp(-Ht_0/2)).$$

*Example 4.*Most the general form of the "force" function corresponding to the case when a spherical 3-volume containing matter, can be considered by an external observer as a point mass (from the principles of symmetry, located in the center of a given volume), is as follows: $F(r) = Ar^{-2} + Br$ ($B \equiv \Lambda/\sigma$, $\sigma = const$). From this, based on the use of the "weak field" approximation, it was concluded about the need to correct the shape of the coefficients of the point mass metric:

$$g_{00} = (1 - 2Ar^{-1} - Br^2/3)c^2,$$
$$g_{11} = (1 - 2Ar^{-1} - Br^2/3)^{-1}.$$

Accordingly, the transition to the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric allows using of these considerations to understand the structure and evaluate the impact on the cosmological dynamics of dark matter and energy. The use of the post-Newtonian approximation based on the previously considered Fock metric allows us to verify these conclusions. To do this, consider the action for gravity in the approximation of weak relativism with the Λ term has the following form (in the Lagrangian representation):

$$S^L = \sum_{a,\boldsymbol{q}} \int \frac{m_a}{2} \boldsymbol{x}_a^2(\boldsymbol{q}, t)\, dct \; - \sum_{a,\boldsymbol{q}} \int m_a \Phi(\boldsymbol{x}_a(\boldsymbol{q}, t)) dct \; + $$
$$+ \frac{2\mathcal{K}}{c^4} \int\int (\nabla\Phi)^2 d^3x dct \; + \mathcal{K} \int\int \Lambda d^3x dct - \frac{2\mathcal{K}\Lambda}{c^2} \int\int \Phi d^3x dct.$$

Varying over the particles, we obtain the equation of motion in the post-Newtonian approximation, corresponding to the above action:

$$m_a \ddot{\boldsymbol{x}}_a = -m_a \nabla\Phi(\boldsymbol{x}_a)$$

(it turns out to coincide in form with the equation of classical dynamics). Let us rewrite the action S in the Eulerian representation, introducing the classical distribution function (on 7-dimensional extended phase space):

$$S^E = \sum_a \frac{1}{2m_a} \int \boldsymbol{p}^2 f_a(\boldsymbol{x}, \boldsymbol{p}, t) d^3x d^3p dt \; - \sum_a \int \Phi(\boldsymbol{x}, t) f_a(\boldsymbol{x}, \boldsymbol{p}, t) d^3p d^3x dt \; +$$
$$+ \frac{2\mathcal{K}}{c^4} \int\int (\nabla\Phi)^2 d^3x dt + \mathcal{K} \int\int \Lambda d^3x dct \; - \frac{2\mathcal{K}\Lambda}{c^2} \int\int U d^3x dt.$$

The inverse transformation to the Lagrangian representation can be done by substituting $f_a(\boldsymbol{x}, \boldsymbol{p}, t) = \sum_q \delta(\boldsymbol{x} - \boldsymbol{x}_a(q, t)) \delta(\boldsymbol{p} - \boldsymbol{p}_a(\boldsymbol{q}, t))$.

Let us vary S^E with respect to Φ and obtain the Poisson equation with the Λ term:

$$\Delta\Phi = 4\pi\gamma \sum_a m_a \int f_a(\boldsymbol{x}, \boldsymbol{p}, t)\, d^3p - \frac{1}{2}c^2\Lambda.$$

What does the second term on the right side give? Presence of an "effective" external field: solution of the equation $\Delta\Phi = -\frac{1}{2}c^2\Lambda$ can be chosen in its simplest form as $\Phi = -\frac{1}{12}c^2\Lambda(x^2 + y^2 + z^2)$, which leads to "pushing" of particles. What does this give us in a Milne–McCrea type solution? From the Poisson equation we obtain

$$\Phi = 4\pi\gamma \sum_a m_a \int \frac{f_a(\boldsymbol{x}', \boldsymbol{p}, t)}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3p d^3x' - \frac{c^2\Lambda}{12}\boldsymbol{x}^2.$$

We took advantage of the fact that the solution to an inhomogeneous linear equation is the sum a particular solution and a general solution of a homogeneous equation, i.e. harmonic function. Our choice of a particular solution is clearly dictated by the requirement of isotropy (invariance regarding rotations) solutions of Friedmann and Milne-McCrea.

Let us present the corresponding "Vlasov–Poisson equation with Λ-term" (for particle type a):

$$\frac{\partial f_a}{\partial t} + (\frac{p}{m_a}, \frac{\partial f_a}{\partial bfx}) - (\nabla\Phi, \frac{\partial f_a}{\partial p}) = 0,$$
$$\Delta\Phi = 4\pi\gamma \sum_a m_a \int f_a(x, p, t) d^3p - \frac{1}{2}c^2\Lambda.$$

3 Derivation of the Vlasov–Maxwell–Einstein equation in $(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{t})$–representation

General relativistic action for a system of many particles with differing masses m_r and charges e_a ($a = \overline{1, N}$): in the presence of gravitational and electromagnetic fields can be written as follows:

$$S = S_p + S_{pf} + S_{ff} + S_{EH}, \qquad (4)$$

$$S_p = -\sum_a m_a c \int \sqrt{g_{\alpha\beta} \frac{dX_a^\alpha}{d\lambda} \frac{dX_a^\beta}{d\lambda}}\, d\lambda,$$
$$S_{pf} = -\sum_a \frac{e_a}{c} \int A_\alpha(\boldsymbol{X}_a) \frac{dX_a^\alpha}{d\lambda}\, d\lambda,$$
$$S_{ff} = -\frac{1}{16\pi c} \int F_{\alpha\beta} F^{\alpha\beta} |g|^{1/2}\, d^4X,$$
$$S_{EH} = K \int |g|^{1/2} (R + \Lambda)\, d^4X,$$
$$A_\mu(X) \equiv \{\varphi(\boldsymbol{X}); \boldsymbol{A}(\boldsymbol{X})\}, \qquad \boldsymbol{X} = \{X^\mu\}_{\mu=0,\dots,3},$$

where: $g_{\mu\nu}(\boldsymbol{X})$ is a fundamental tensor of 4-dimensional space-time, $A_\mu(\boldsymbol{X})$ is an electromagnetic field potential, Λ is a cosmological constant, $K = -c^3/(16\pi\gamma)$; variable $\lambda(>0)$ is proportional to proper time of the particle, i.e. affine parameter of the a-th particle: $ds_a = \sqrt{I_a}d\lambda$, $I_a \equiv (g_{\mu\nu}(dX^\mu/d\lambda)(dX^\nu/d\lambda))_a$ (I_a is a conserved integral of motion).

We obtain the equations of motion of charged massive particles in given fields by varying $S_p + S_{pf}$ (for an individual particle, index $a = a_0$ we do not write out):

$$-mc\frac{d^2}{d\lambda^2}(\frac{g_{\alpha\mu}(dX^\mu/d\lambda)}{\sqrt{I}}) - \frac{e}{c}\frac{dA_\alpha}{d\lambda} = -\frac{mc}{2\sqrt{I}}\frac{\partial g_{\mu\nu}}{\partial X^\alpha}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda} - \frac{e}{c}\frac{\partial A_\mu}{\partial X^\alpha}\frac{dX^\mu}{d\lambda}.$$

Considering that the quantity I is the integral of motion, we obtain the following equation:

$$\frac{d^2X^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\lambda}\frac{dX^\beta}{d\lambda} = \frac{e\sqrt{I}}{mc^2}F^\mu_\alpha\frac{dX^\alpha}{d\lambda}, \qquad (5)$$
$$\alpha,\ldots,\mu = 0,\ldots,3.$$

Let us pose the following problem: rewrite equation (5), eliminating the natural parameter λ, and passing instead to the coordinate $X^0 \equiv ct$ (“observer time”). To do this, let us present the dynamics equations in speed variables:

$$\frac{dX^\mu}{d\lambda} = V^\mu, \frac{dV^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}V^\alpha V^\beta + \frac{e\sqrt{I}}{mc^2}F^\mu_\alpha V^\alpha. \quad (6)$$

Let us note here the appearance of the integral $\sqrt{I}$ in the second term on the right side of the second equation — it will not exist when using natural parameter s instead of λ. However, when s is introduced into consideration, the second-order homogeneity in the velocities of the right hand disappears. part that is necessary for further transformation. Namely, the following very general statement about the reduction of order by two powers holds.

Lemma (on reducing the order of an ODE system). Let a system of $2N$ ordinary differential equations be given ($A = \overline{0, N-1}$):

$$\frac{dX^A}{d\lambda} = f^A(\boldsymbol{X},\boldsymbol{V}), \qquad \frac{dV^A}{d\lambda} = F^A(\boldsymbol{X},\boldsymbol{V}).$$

Let the functions $f^A(\boldsymbol{X},\boldsymbol{V})$ be of the first degree of homogeneity with respect to the variable $\boldsymbol{V}$, and the functions $F^A(\boldsymbol{X},\boldsymbol{V})$ be of the second degree:

$$f^A(\boldsymbol{X},k\boldsymbol{V}) = kf^A(\boldsymbol{X},\boldsymbol{V}),$$
$$F^A(\boldsymbol{X},k\boldsymbol{V}) = k^2f^A(\boldsymbol{X},\boldsymbol{V}).$$

Then the system of $2N-2$ equations

$$\frac{dX^A}{dX^0} = \frac{f^A(\boldsymbol{X},\boldsymbol{U})}{f^0(\boldsymbol{X},\boldsymbol{U})}, \qquad \frac{dU^A}{dX^0} = \frac{F^A(\boldsymbol{X},\boldsymbol{U})}{f^0(\boldsymbol{X},\boldsymbol{U})} - U^A\frac{f^A(\boldsymbol{X},\boldsymbol{U})}{f^0(\boldsymbol{X},\boldsymbol{U})}, U^A \equiv \frac{V^A}{V^0}, U^0 \equiv 1, A = \overline{0, N-1}.$$

is valid. *Proof* is carried out by direct substitution.

Using this lemma, we rewrite system (6) in the form

$$\frac{dX^i}{dt} = U^i, \frac{dU^i}{dt} = G^i(\boldsymbol{X},\boldsymbol{U}), i = 1,2,3, \qquad (7)$$
$$G^i(\boldsymbol{X},\boldsymbol{U}) = -(\Gamma^i_{\mu\nu} - \frac{U^i}{c}\Gamma^0_{\mu\, nu})U^\mu U^\nu + \frac{e\sqrt{I}}{mc^2}(F^i_\mu - \frac{U^i}{c}F^0_\mu)U^\mu, \qquad I \equiv g_{\mu\nu}\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}$$

(it should be noted that for $i = 0$ the equation becomes an identity).

Let us write down the Liouville equation for the 7-dimensional distribution function $f(\boldsymbol{x},\boldsymbol{u},t)$ corresponding to system (7) (hereinafter $\boldsymbol{X} = \{ct, \boldsymbol{x}\}$, $\boldsymbol{U} = \{1, \boldsymbol{u}\}$):

$$\frac{\partial f(x,u,t)}{\partial t} + u^i\frac{\partial f}{\partial x^i} + \frac{\partial(fG^i)}{\partial u^i} = 0. \qquad (8)$$

Thus, we obtained the first part (kinetic) of the Vlasov-Maxwell-Einstein system of equations. To obtain equations for the fields $g_{\mu\nu}$ and F^μ_ν and relate these field characteristics to distribution function $f(\boldsymbol{x},\boldsymbol{u},t)$, it is necessary to rewrite the total action, replacing the “arbitrary parameter” λ with time t, and including in S_p and S_{pf} partial single-particle distribution function $f_a(\boldsymbol{x},\boldsymbol{u},t)$: including in S_p and S_{pf} partial single-particle distribution function $f_a(\boldsymbol{x},\boldsymbol{u},t)$:

$$S = -\sum_a m_a c\int\sqrt{g_{\alpha\beta}U^\alpha U^\beta}f_a(\boldsymbol{X},\boldsymbol{U},t)d\boldsymbol{x}dtd\boldsymbol{u} - \sum_a\frac{e_a}{c}\int A_\alpha(\boldsymbol{x}_a)U^\alpha f_a(\boldsymbol{X},\boldsymbol{U},t)d^3xdtd^3u - -\frac{1}{16\pi c}\int F_{\alpha\beta}F^{\alpha\beta}|g|^{1/2}\,d^3xdct + K\int|g|^{1/2}(R+\Lambda)\,d^3xdct, \qquad K = \frac{-c^3}{16\pi\gamma}.$$

Varying the last expression for S with respect to the electromagnetic field potentials, we obtain Maxwell’s equations:

$$-\frac{c}{16\pi}\frac{\partial(\sqrt{-g}F^{\alpha\beta})}{\partial X_\beta} = \sum_a\int f_a(\boldsymbol{X},\boldsymbol{U},t)e_aU^\alpha d\boldsymbol{u}. \; (9)$$

If we vary S by the metric $g_{\mu\nu}$, we obtain Einstein’s equations for the gravitational field:

$$K\sqrt{-g}(R^{\mu\nu} - \frac{1}{2}(R+\Lambda)g^{\mu\nu}) = -\sum_a\frac{m_a}{2}\int\frac{U^\mu U^\nu}{\sqrt{g_{\zeta\eta}U^\zeta U^\eta}}f_a(\boldsymbol{X},\boldsymbol{U},t)\,d\boldsymbol{u} + + \frac{1}{16\pi c}\left(-2F^{\beta\nu}F^{\alpha\mu}g_{\alpha\beta} + \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}\right)\sqrt{-g}. \; (10)$$

The system of equations (8)–(10) is the complete Vlasov–Maxwell–Einstein system.

Note that the resulting form of Einstein’s equations leads to the conclusion that contribution to the cosmological Λ-term can be made by the first three terms of the action S. The obvious conclusion from this is that that the first three terms make the same contribution to the energy-momentum tensor and to the equations of motion as the "formal" Λ-term:

$$\Lambda_S(\boldsymbol{X},t) = -\sum_a \frac{m_a c}{K\sqrt{-g}} \int \sqrt{g_{\mu\alpha}U^\mu U^\alpha} \times f_a(\boldsymbol{X},\boldsymbol{U},t)d^3u - \sum_a \frac{e_a}{cK\sqrt{-g}} \int A_\alpha(\boldsymbol{x}_a)U^\alpha \times f_a(\boldsymbol{X},\boldsymbol{U},t)d^3u - \frac{1}{16K\pi c} F_{\alpha\beta}F^{\alpha\beta}.$$

The notation Λ_S emphasizes that this expression is an analogue of the Λ-term, conditioned by the form of the action S.

4 Hydrodynamic applications of the consequences of the Vlasov–Einstein equations

In this section we will consider applications of the formalism developed above for two relativistic problems.

Example 5. Consider a special case of the action S_2 for the relativistic Lorentz metric $g_{\alpha\beta} = diag(1,-1,-1,-1)$ (for one particle):

$$S_L = mc\int (\sqrt{c^2-\boldsymbol{u}^2} + U/c)dt - \frac{1}{8\pi G}\int (\nabla U)^2 d\boldsymbol{x}dt - \frac{c^2\Lambda}{8\pi G}\int U\, d\boldsymbol{x}dt, \qquad \boldsymbol{u} \equiv \frac{d\boldsymbol{x}}{dt}.$$

We vary S_L by coordinates and obtain the equations of relativistic dynamics with Hamiltonian function $\mathcal{H}_L = mc^2\sqrt{1+\boldsymbol{p}^2/(mc)^2} + mU$. Let's write the action in terms of particle distribution functions:

$$S_L = -c\int m(\sqrt{c^2-\boldsymbol{u}^2} + U/c) \times f(t,\boldsymbol{x},\boldsymbol{u},m)d\boldsymbol{u}dmd\boldsymbol{x}dt - \frac{1}{8\pi G}\int (\nabla U)^2 d\boldsymbol{x}dt - \frac{c^2\Lambda}{8\pi G}\int U\, d\boldsymbol{x}dt.$$

Varying it across the potential U, we obtain the equation for the gravitational field:

$$\Delta U = 4\pi G\int mf(t,\boldsymbol{x},\boldsymbol{u},m)d\boldsymbol{u}dm - \frac{1}{2}c^2\Lambda.$$

When passing to the equations of the hydrodynamic level, we obtain the Hamilton–Jacobi–Euler–Poisson system:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^j}(u^j(\nabla W)\rho) = 0,$$
$$\frac{\partial W}{\partial t} + c\sqrt{m^2c^2 + (\nabla W)^2} + mU = 0,$$
$$u^j(\boldsymbol{p}) = \frac{\partial \mathcal{H}_L}{\partial p_j} = \frac{cp_j}{\sqrt{m^2c^2+\boldsymbol{p}^2}}.$$

Let us rewrite this system of equations for the isotropic case, when $\rho = \rho(t,r,m)$, $U = U(t,r)$, $W = W(t,r)$, in the form:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^k}\left(\frac{c(\partial W/\partial r)x^k}{r\sqrt{m^2c^2+(\partial W/\partial r)^2}}\right) = 0,$$
$$\frac{\partial W}{\partial t} + c\sqrt{m^2c^2+(\partial W/\partial r)^2} + mU(r,t) = 0,$$
$$3\left(\frac{\partial W/\partial r}{r}\right) + r\frac{\partial}{\partial r}\left(\frac{\partial W/\partial r}{r}\right) = 4\pi G\int m\rho\, dm - \frac{1}{2}c^2\Lambda.$$

Cosmological solutions correspond to the case when the value ρ does not depend on the spatial variable: $\rho = \rho(m,t)$. In this case, the solution to the last equation (Poisson) will be the following function:

$$U(t,r) = -\frac{A(t)}{r} + \frac{B(t)}{6}r^2,$$
$$B(t) \equiv 4\pi G\int m\rho\, dm - \frac{1}{2}c^2\Lambda.$$

From the first equation (continuity equation) of the above system we have:

$$\frac{\partial\rho}{\partial t} + 3H\rho = 0,$$

where $H(m,t)$ is the Hubble parameter. We obtain the Hamilton-Jacobi equations for variable W:

$$3\varphi + r\frac{d\varphi}{dr} = 3H,$$
$$\varphi(r) = \frac{(\partial W/\partial r)c}{r\sqrt{(\partial W/\partial r)^2 + m^2c^2}}.$$

Solving the equation for φ, we obtain $\varphi = H + \psi(m,t)/r^3$, $\psi(m,t)$ — some function. So we we get a system of equations

$$\frac{\partial\rho}{\partial t} + 3H\rho = 0, \frac{c\partial W/\partial r}{\sqrt{m^2c^2+(\partial W/\partial r)^2}} = rH + \frac{\psi(t)}{r^2},$$
$$\frac{\partial W}{\partial t} + c\sqrt{m^2c^2+(\partial W/\partial r)^2} + mU(r,t) = 0.$$

Let us denote $\Omega(r,t) = c^{-1}(rH + \psi(t)/r^2)$, $\Theta(r,t) = \frac{\partial W(r,t)}{\partial r}$. Then the equation is valid:

$$\frac{\partial\Theta}{\partial t} + \frac{c\Theta\Theta_{r'}}{\sqrt{m^2c^2+\Theta^2}} + mU'(r) = 0,$$

or, after simplification:

$$(c\Omega_{t'} + c^2\Omega\Omega_{r'})^2 - (U_r)^2(1-\Omega^2)^3 = 0.$$

Substituting here the explicit expressions $\Omega, \Omega_t, \Omega_r$, we get:

$$(rH_{t'} + \frac{\psi_{t'}}{r^2} + (rH + \frac{\psi}{r^2})(H - \frac{2\psi}{r^3}))^2 - (\frac{rB}{3} + \frac{A}{r^2})^2(1-(rH+\frac{\psi}{r^2})^2)^3 = 0.$$

Analysis of this expression leads us to the following conclusions. When expanding the left side of the above equation in powers of r it is necessary that all coefficients of r^s be equal to zero, therefore (since $B^2H^6r^8 = 0$) either $H = 0$ or $B(t) = 0$ (the latter means $8\pi G \int m\rho(m,t)\, dm = c^2\Lambda$, that is, the stationarity of the Minkowski metric must be ensured corresponding dependence on the density of matter). If $B(t) = 0$, then from the necessity of equality of the coefficient $\psi^6 A^2$ to zero it follows that $A = 0$, $U(r) = 0$. Consequently, the cosmological solution for the Minkowski–Lorentz metric exists only for $U(r) = 0$, Moreover, from the last equation we obtain a relation for the Hubble parameter of the form $H_{t'} + H^2 = 0$, the solution of which has form $H(t) = (C_0 + t)^{-1}$.

Example 6. Consider the case of non-relativistic action and obtain the GamilJacobi equations for it. Action in This case can be represented in the form:

$$S_{nr} = \int (m\boldsymbol{u}^2/2 - e\phi - mU) \times$$
$$f(t,\boldsymbol{x},\boldsymbol{u},m,e) d\boldsymbol{x} d\boldsymbol{u} dm de dt +$$
$$+\frac{1}{8\pi}\int (\nabla\phi)^2 d\boldsymbol{x} dt -$$
$$-\frac{1}{8\pi G}\int (\nabla U)^2 dx dt - \frac{c^2\Lambda}{8\pi G}\int U\, dx dt.$$

Varying the electric potential ϕ and the gravitational potential U, we obtain the Poisson equations through distribution functions:

$$\Delta\phi = -4\pi \int ef(t,\boldsymbol{x},\boldsymbol{u},m,e)\, d\boldsymbol{u} dm de,$$
$$\Delta U = 4\pi G \int mf(t,\boldsymbol{x},\boldsymbol{u},m,e)\, du dm de - \frac{1}{2}c^2\Lambda.$$

Considering the Liouville equation for a single-particle function, making a hydrodynamic substitution into it $f(t,\boldsymbol{x},\boldsymbol{u},m,e) = \rho(t,\boldsymbol{x},m,e)\delta(u - \boldsymbol{v}(t,\boldsymbol{x},\boldsymbol{u},m,e))$ passing to the Hamilton-Jacobi formalism ($v_i = \partial W(t,\boldsymbol{x},e,m)/\partial x^i$), we obtain the system Euler–Poisson–Hamilton–Jacobi equations:

$$\frac{\partial\rho}{\partial t} + \rho\Delta W + \frac{\partial\rho}{\partial x^j}\frac{\partial W}{\partial x^j} = 0,$$
$$\frac{\partial W}{\partial t} + \frac{1}{2}\sum_i (\frac{\partial W}{\partial x^i})^2 U + \frac{e}{m}\phi = 0,$$
$$\Delta\phi = -4\pi \int e\rho\, dm de,$$
$$\Delta U = 4\pi G \int m\rho\, dm de - \frac{1}{2}c^2\Lambda.$$

Let rewrite the resulting system of equations for the isotropic case, i.e. e. in the case when $\rho = \rho(t,r,e,m)$, $W = W(t,r,e,m)$, $U = (r,t)$, $\phi = \phi(r,t)$:

$$\frac{\partial\rho}{\partial t} + \rho(\frac{3W_{r'}}{r} + r\frac{\partial}{\partial r}(\frac{W_{r'}}{r})) + \frac{\partial\rho}{\partial r}\frac{\partial W}{\partial r} = 0,$$
$$\frac{\partial W}{\partial t} + \frac{(W_{r'})^2}{2} + U + \frac{e\phi}{m} = 0,$$
$$\frac{3\phi_r}{r} + r\frac{\partial}{\partial r}(\frac{\phi_{r'}}{r}) = -4\pi \int e\rho\, dm de,$$
$$\frac{3U_r}{r} + r\frac{\partial}{\partial r}(\frac{U_{r'}}{r}) = 4\pi \int m\rho\, dm de - \frac{c^2\Lambda}{2}.$$

Let us now assume that the density does not depend on the spatial coordinate: $\rho = \rho(t,m.e)$ (homogeneity in space). Such solutions are usually called cosmological solutions, since on very large scales it is assumed that the density does not depend on the spatial coordinate at all. Then the last equation has a solution

$$U = -\frac{A_1(t)}{r} + \frac{A_2(t)}{6}r^2,$$
$$A_2(t) = 4\pi G \int m\rho\, dm de - \frac{c^2\Lambda}{2},$$

and the penultimate one:

$$\phi(r,t) = -\frac{A_3(t)}{r} + \frac{A_4}{6}r^2,$$
$$A_4(t) = -4\pi \int e\rho\, dm de.$$

The first and second equations of the system give the equations for the Hubble parameter:

$$\frac{\partial\rho}{\partial t} + 3H(e,m,t)\rho = 0,$$
$$3\kappa + r\frac{\partial\kappa}{\partial r} = 3H(t,m,e), \qquad \kappa \equiv \frac{W_r(t,r,m,e)}{r}.$$

Solving the equation for the variable κ, we obtain $\kappa = H + A_5(m,e,t)/r^3$. Substituting this quantity into the second equation of the system (Hamilton-Jacobi), we obtain:

$$\frac{\partial}{\partial t}(\frac{Hr^2}{2}\frac{A_5}{r}) + \frac{1}{2}(Hr + \frac{A_5}{r^2})^2 - \frac{A_1(t)}{r} + \frac{b(t)}{6}r^2 +$$
$$\frac{e}{m}(-\frac{A_3(t)}{r} + \frac{A_4(t)}{6}r^2) = 0.$$

From the second term we find $A_5(m,e,t) = 0$; collecting coefficients for r^{-1}, r^2, we obtain a system of evolutionary equations for density and for the Hubble parameter:

$$\frac{\partial\rho}{\partial t} + 3H\rho = 0,$$
$$\frac{\partial H}{\partial t} + H^2 + \frac{4\pi G}{3}\int m\rho\, dm de -$$
$$\frac{c^2\Lambda}{6} - \frac{4\pi Ge}{m}\int e\rho\, dm de = 0.$$

This provides another opportunity to explain the accelerated expansion Universe, along with the lambda term. It is clearly seen from the equations that the last two terms work in in the same correct direction, creating the missing repulsion. In case of charged particles, this equality of the last term on

the left side to zero means neutrality. Let us denote
$$\eta(t) = (4\pi G/3)\int m\rho\, dmde - (4\pi Ge/m)\int m\rho dmde.$$
Then, assuming that $H = H(t)$, we can go to the system of two ordinary differential equations:
$$\frac{d\eta}{dt} = -3H\eta, \qquad \frac{dH}{dt} = -H^2 - \eta + \zeta,$$
$$\zeta = -\frac{1}{6}c^2\Lambda.$$
Three cases can be distinguished: 1) $\Lambda = 0$, 2) $\Lambda > 0$, 3) $\Lambda < 0$. In all three cases, the modern physical region of the Universe must correspond to regions on the phase (η, H)-plane where $H > 0$ (redshift) and $dH/dt = \sigma - \eta - H^2 > 0$ (accelerated expansion).

In this case, for the case $\zeta = 0$, accelerated expansion $dH/dt > 0$ is possible for $H > 0$ and accelerated compression for $H < 0$, as well as slower expansion (for $H > 0$) with a transition to slower compression for $H = 0$.

Moreover, for the left ($H < 0$) and right ($H > 0$) half-planes of the (H, η)-phase plane of the velocity of the part directed in different directions; this can be interpreted as a “shrinking” of the system (this may be due to with such astronomical aspect as “violet shift”) and “expansion” of the system (respectively “red shift”).

5 Conclusion

In this paper, we consider the derivation of the Vlasov–Maxwell–Einstein equations system based on the Lagrangian formalism, while at the first stage we introduced the full action (Einstein–Hilbert–Pauli) of a system of massive charged particles, electromagnetic and gravitational fields. To do this, it was necessary to synchronize the proper times of various particles. This was done through the proper time of one particle and through an arbitrary parameter. We derived equations and obtained an expression for mass in stationary gravitational and electromagnetic fields. It is interesting to compare the resulting form of the Vlasov–Maxwell–Einstein equations with other versions and classify them. As a rule, various forms of the kinetics equations in a gravitational field are written only for the equations Vlasov–Einstein (without Maxwell) and with Christoffel symbols, which means not for impulses, but for velocities. They can also be derived according to this scheme.

The authors consider the use of AI to be very promising in obtaining from the action of a general form (including new types of interactions) new equations for fields and equations for the dynamics of particles in these fields. This is due to the fact that the terms corresponding to new fields (dilaton scalar, inflaton, etc.) in the approximation used enter the full Lagrangian additively, which leads to a standard technique for obtaining dynamic equations.

In the present literature, equations of Vlasov type are usually not derived from the basic principles, but are written out directly (apparently based on analogies with the classical case), which inevitably leads to inaccuracies and direct errors. When it comes to the Vlasov-Einstein equations, the derivation seems absolutely necessary for both sides of the Vlasov equation, that is, for the equation Liouville (particle transport) and equations for fields. When deriving the Liouville equation, we move from an arbitrary parameter (along particle trajectories) to the observer’s time, which leads to time synchronization in multiparticle systems.

In equations for fields, the form of the energy-momentum tensor is chosen quite arbitrarily in present publications, which is illegal. We obtained exact expressions for this tensor when passing to the distribution functions in the composite action for the system particles in gravitational and electromagnetic fields (which formally have the same effect as Einstein’s Λ–term). It seems promising to explore for this equations are all classical substitutions that are known in the Vlasov equation: energetic and hydrodynamic substitutions, as well as stationary Vlasov equation solutions. It appears current and interesting task to classify all decisions depending on time (spatially homogeneous solutions). This leads to cosmological solutions that are now being actively studied. The Hamilton–Jacobi equation methods are useful here. An extremely urgent task is to obtain for equations of Vlasov type the consequences of the assumption ``time averages coincide with Boltzmann extremals’’, which should lead to a complete self-consistent description of the evolution of structures in the Universe as a sequence of states of relative equilibrium.

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