Approximation Solution of Mixed Boundary-Contact Problems and their Applications

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Abstract: - In this article non-classical diffusion models of theory coupled-elasticity of static systems for isotropic inhomogeneous elastic materials with thermal and diffusion variables has been investigated. Approximate solutions for boundary-contact problems for theory coupled-elasticity in Cauchy hypothesis conditions has been constructed. The tools applied in this development are based on the boundary integral methods and Greens functions applications.

Key-Words: - Approximation solution; boundary-contact problems.

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1 Introduction

This paper is devoted to the development of a new method of approximate solutions of partial differential equations (PDEs). In particular, a non-classical diffusion models of theory coupled-elasticity of static systems for isotropic inhomogeneous elastic materials with thermal and diffusion for two dimensional (2-D) areas variables have been provided [1-4]. For consideration basic boundary-contact problems for theory coupled-elasticity (BCPTCE) in Cauchy hypothesis (CH) conditions in infinite and finite domains of isotropic inhomogeneous elastic media with inclusion of several elastic materials and mixed contact conditions have been investigated. Decomposition Methods to build a numerical solution of partial differential equations has been used [6].

Throughout of paper we introduce the following notations: E^2 is two-dimensional Euclidean space, $x = (x_j)$; $y = (y_j)$; $j = 1,2$-points of this space, $(SL_{(2)}(\mathrm{D})$ −Hilbert space, $D^{(0)}$ is infinite domain with inclusion another elastic material $D^{(r)}$ ($D^{(\mathrm{r})} \subset E^2, D^{(0)} = E^2 \backslash \overline{D^{(r)}}$, r = 1,2..) bounded by the close surface $S \in L_{(2)}(\alpha)$, $\alpha > 0$ with outward positive normal vector.

2 Problem Formulation

The generalized model static systems of partial differential equations of theory of coupled thermo-diffusion model for 2-D isotropic inhomogeneous elastic materials has the form [1]:

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$$(\mu_r + \alpha_r)\Delta u(x) + (\lambda_r + \mu_r - \alpha_r) graddivu + 2\alpha_r rotu_3 - \sum_{i=1}^{2} \gamma_{ir}\, gradu_{i+3} = f^{(1r)}(x) \quad (1)$$

$$(\nu_r + \beta_r)\Delta u_3(x) + 2\alpha_r rotu - 4\alpha_r u_3 = f_{3r}(x)$$

$$\aleph_{1r}\Delta u_4(x) - a_{1r}u_4 - a_{12r}u_5 - \gamma_{1r} divu = f_{4r}(x)$$

$$\aleph_{2r}\Delta u_5(x) - a_{2r}u_5 - a_{12r}u_4 - \gamma_{2r} divu = f_{5r}(x)$$

Where $u = (u_1, u_2)$ is the displacement vector, u_3 is a characteristic of rotation, u_4 is the temperature variation, u_5 is the chemical potential, $rotu_3 = \left(\frac{\partial u_3}{\partial x_2}, -\frac{\partial u_3}{\partial x_1}\right)'$, $rotu = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$, $\mu_r > 0, \alpha_r > 0$, $3\lambda_r + 2\mu_r > 0, \nu_r > 0, \beta_r > 0, \gamma_{rk} > 0, a_{rk} > 0, k = 1,2$, $a_{1r}a_{2r} - a_{12r}^2 > 0$ −Constants of elasticity of $D^{(\mathrm{r})}$ domains (r = 0,1). Δ is a two-dimensional Laplacian operator [3], $\Gamma^{(\mathrm{r})} = \left(\Gamma^{(1\mathrm{r})}, \Gamma_{3\mathrm{r}}, \Gamma_{4\mathrm{r}}, \Gamma_{5\mathrm{r}}\right) = (\Gamma_{1\mathrm{r}}, \Gamma_{2\mathrm{r}}, \Gamma_{3\mathrm{r}}, \Gamma_{4\mathrm{r}}, \Gamma_{5\mathrm{r}}) \subset C^{0,\alpha}\left(D^{(\mathrm{r})}\right)$, $\alpha > 0$ is a given vectors. Allow us to introduce the matrices of differential operators of static systems:

$$L^{(r)}(\partial x,\tau)=(L_{ij}^{(r)})_{5x5}$$

Where

$$L_{ij}^{(1r)}(\partial x)=\delta_{ij}(\mu_r+\alpha_r)\Delta+(\lambda_r+\mu_r-$$

$$-\alpha_r)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$L_{ij}^{(2r)}(\partial x)=L_{ij}^{(3r)}(\partial x)=-2\alpha_r\sum_{p=1}^{2}\xi_{ijp}\frac{\partial}{\partial x_p},$$

$$L_{i}^{(4r)}(\partial x)=\left((\nu_r+\beta_r)\Delta-4\alpha_r\right)$$

$$G(\partial x)=(\partial x_1,\partial x_2)$$

$$L_{ij}^{(r)}(\partial x)=\delta_{ij}\mu_r\Delta+(\lambda_r+\mu_r)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$i,j=1,2,3$$

Where δ_{ij} is Kronecker's symbol, ξ_{ijp} is of Levi-Chamita's symbol. Therefore (1) can be written in the form:

$$L^{(r)}(\partial x)U^{(r)}(x)=f^{(r)}(x),\qquad r=0,1\quad(2)$$

The generalized stress operators of coupled thermo-elasticity in $D^{(r)}$ domains have the form [3]:

$$R^{(r)}(\partial x,n(x))=\begin{vmatrix}\left|T^{(r)}(\partial x,n(x))\right|_{3x3} & \left|-N(x)\sum_{i=1}^{2}\gamma_{i\tau r}\right|_{3x2}\\ \left|\delta_{5q}\frac{\partial}{\partial n}\right|_{1x5} & \end{vmatrix}_{5x5}$$

Where

$$q=\overline{0,3};N(x)=(n,0,0),n=(n_1,n_2),$$

$T^{(r)}(\partial x,n(x))=\left|T_{jk}^{(r)}(\partial x,n(x))\right|_{3x3}$-matrices of stress operators on the plain [4]:

$$T_{jk}^{(r)}(\partial x,n(x))=\lambda_r n_j(x)\frac{\partial}{\partial x_k}+(\mu_r-\alpha_r)n_k(x)\frac{\partial}{\partial x_j}-$$

$$-(\mu_r+\alpha_r)\delta_{kj}\frac{\partial}{\partial n(x)},\ j,k=1,2\ ;$$

$$T_{jk}^{(r)}(\partial x,n(x))=-2\alpha_r\sum_{p=1}^{2}\xi_{jkp}\,n_p(x),j=1,2,k=3;$$

$$T_{jk}^{(r)}(\partial x,n(x))=(\nu_r+\beta_r)\delta_{kj}\frac{\partial}{\partial n(x)},j=3,k=1,3.$$

3 Problem Solution

In this work basic BCPTE in the case that couple-stresses components, displacement components, rotation, heat flux and temperature, concentration and chemical potential are represented on the surface of Holder class has been formulated.
The Cauchy-hypothesis condition for isotropic inhomogeneous elastic materials with a center of symmetry is implemented [3]:

$$\frac{\mu_0}{\lambda_0}=\frac{\mu_r}{\lambda_r},r=1,2,...$$

It is assumed that surfaces are sufficiently smooth.

Problem. It is required to find regular solutions $U^{(r)}(x)=(u_1^{(r)},u_2^{(r)},u_3^{(r)})$ of boundary contact problems:

$$\forall\,x\in D^r,\mathrm{r}=0,1:\ \ L^{(r)}(\partial x)U^{(r)}(\mathrm{x})=f^{(r)}(x),$$

$$\forall z\in\mathrm{S}\in\mathrm{L}_2(\alpha),\alpha>0:$$

$$\{U^{(1)}(z)\}^+=\{U^{(0)}(z)\}^-\qquad\textbf{(P)}$$

$$\{R^{(1)}(\partial z,n)\{U^{(1)}(z)\}^+=\{R^{(0)}(\partial z,n)\{U^0(z)\}^-$$

In the radiation conditions for an infinite domain.

In the first iteration the Neiman's [1-7] boundary value problem is considered:

$$\begin{cases}\forall \mathrm{x}\in D^0:\ L^{(0)}(\partial x)U_1^{(0)}(\mathrm{x})=f^{(0)}(x)\qquad(3)\\ \forall z\in\mathrm{S}\in\mathrm{L}_2(\alpha),\alpha>0:\ \ \{R^{(0)}(\partial z,n)U_1^{(0)}(z)\}^-=0\end{cases}$$

The solution of (3) is presented in the following form:

$$U_1^{(0)}(\mathrm{x})=\frac{1}{2}\int_{D_0}G^{(0)}(y-x)f^{(0)}(y)dy$$

Where $G^{(0)}(y-x)$is Green's tensor [2] defined for problem (3).

Further the Dirichlet's boundary value problems [1-5] is considered***:***

$$\begin{cases} \forall x \in D^1: \ L^{(1)}(\partial x)U_1^{(1)}(x) = f^{(1)}(x) \\ \forall z \in S \in L_2(\alpha), \alpha > 0: \quad \left\{U_1^{(1)}(z)\right\}^+ = \left\{U_1^{(0)}(z)\right\}^- \end{cases} (4)$$

The solution of (4) is presented in the following form:

$$U_1^{(1)}(x) = -\frac{1}{2}\int_{D_1} G^{(1)}(x,y) f^{(1)}(y)dy + \\ +\frac{1}{2}\int_S \left[R^{(1)}(\partial y,n)G^{(1)}(x,y)\right]^T \left\{U_1^{(0)}(y)\right\}^- d_y s$$

Where $G^{(1)}(y-x)$is Green's tensor defined for problem (4).

Let us introduce the following notations:

$$U_{k+1} = \begin{cases} U_{k+1}^{(0)}, & x \in D^0 \\ U_{k+1}^{(1)}, & x \in D^1 \end{cases}$$

$k = 1,2,..$

On the ***k***-iteration the approximation solution for the Neiman's boundary value problem will be constructed as follow:

$$\begin{cases} \forall x \in D^0: \ L^{(0)}(\partial x)U_{k+1}^{(0)}(x) = f^{(0)}(x) \\ \forall z \in S \in L_2(\alpha), \alpha > 0: \\ \{R^{(0)}(\partial z,n)U_{k+1}^{(0)}(z)\}^- = \{R^{(0)}(\partial z,n)U_k^{(1)}(z)\}^+ \end{cases}$$

Then corresponding solutions in Green's tensor will be presented in the form:

$$U_{k+1}^{(0)}(x) = -\frac{1}{2}\int_{D_1} G^{(0)}(x,y) f^{(0)}(y)dy + \\ +\frac{1}{2}\int_S G^{(0)}(x,y)\left\{R^{(0)}(\partial y,n)U_k^{(1)}(y)\right\}^+ d_y s$$

Accordingly previous iteration the solution for Dirichlet,s boundary value problems will be constructed as follow***:***

$$\begin{cases} \forall x \in D^1: \ L^{(1)}(\partial x)U_{k+1}^{(1)}(x) = f^{(1)}(x) \\ \forall z \in S: \quad \{U_{k+1}^{(1)}(z)\}^+ = \{U_k^{(0)}(z)\}^- \end{cases} (6)$$

The solution will be presented in Green's tensor:

$$U_{k+1}^{(1)}(x) = -\frac{1}{2}\int_{D_1} G^{(1)}(x,y) f^{(1)}(y)dy + \\ +\frac{1}{2}\int_S \left[R^{(1)}(\partial y,n)G^{(1)}(x,y)\right]^T \left\{U_{k+1}^{(0)}(y)\right\}^- d_y s$$

Let us introduce the following notation:

$$V_k(x) = U_k(x) - U_{k-1}(x), \quad k = 1,2....$$

Then with respect to $V_k(x)$ the following conditions is satisfied:

$$L(\partial x)V_k(x) = \begin{cases} f(x), k = 1 \\ 0, k > 1 \end{cases} \qquad x \overline{\in} S$$

$$\{R^{(0)}(\partial z,n)V_k\}^- = \begin{cases} \left\{R^{(0)}(\partial z,n)V_{k-1}(z)\right\}^+, k > 1 \\ 0, k = 1 \end{cases}$$

$$\{V_k(z)\}^+ = \begin{cases} \{V_{k-1}(z)\}^-, k > 1 \\ \left\{U_1^{(0)}(z)\right\}^-, k = 1 \end{cases}, z \in S,$$

Where

$$f(x) = \begin{cases} f^{(1)}(x), x \in D^r \\ f^{(0)}(x), x \in D^0 \end{cases}$$

Accordingly of CH condition the following equations is received:

$$L^{(r)}(\partial x) = \frac{\mu_r}{\mu_0} L^{(0)}(\partial x),$$

$$R^{(r)}(\partial x,n) = \frac{\mu_r}{\mu_0} R^{(0)}(\partial x,n) \ (6)$$

Take in account the (6) in expressions of solutions in D^0 and D^1 domains the following results has been obtained

$$V_k^{(0)}(x) = \frac{1}{2}\int_S G^{(0)}(x,y)\left\{R^{(1)}(\partial y,n)V_{k-1}(y)\right\}^+ d_y s$$

$$V_k^{(1)}(x) = \frac{1}{2}\int_S \left[R^{(1)}(\partial y,n)G^{(1)}(x,y)\right]^T \left\{V_{k-1}(y)\right\}^- d_y s = \\ = \left(\frac{\mu_1}{\mu_0}\right)^k \int_S K^k(x,y)\left\{U_1^{(0)}(y)\right\}^- d_y s = \left(\frac{\mu_1}{\mu_0}\right)^k V_k^{(0)}(x)$$

Where

$$K(x,y)=\int_s \left[R^{(1)}(\partial_\xi,n)G^{(1)}(\xi,x)\right]^T R^{(0)}(\partial y,n)G^{(0)}(\xi,y)d_\xi s \quad (7)$$

Take in account the symmetry properties of Greens tensors for inner and outer problems:

$$G(x,y)=G^T(y,x)$$

We can rewrite (7) in the form:

$$K(x,y)=\frac{\mu_1}{\mu_0}\int_s R^{(0)}(\partial y,n)G^{(0)}(x,\xi)R^{(0)}(\partial y,n)G^{(0)}(\xi,y)d_\xi s=$$

$$=\frac{\mu_1}{\mu_0}K_1(x,y)$$

Obviously $V_k(x)$ will be represented in the following form:

$$V_k(x)=\frac{\mu_1}{\mu_0}\int_S K_1(y,x)\{V_{k-1}(y)\}^- d_y s=\frac{\mu_1}{\mu_0}K_1\{V_{k-1}(y)\}^-,$$

$k=2,..$ (8)

Where

$$K_1\varphi=\int_s K_1(x,y)\varphi(y)d_y s \quad (9)$$

According to the following property of Green's tensor [3]:

$$\left|G^{(0)}(x-y)\right|\le\frac{c}{|x-y|};\quad \left|\frac{\partial}{\partial x_i}G^{(0)}(x-y)\right|\le\frac{c}{|x-y|^2},$$

$i=1,2,3$

Then

$$K_1(x,y)\in L_2(D)$$

4 Conclusion

Thus, an approximate method solution of basic boundary contact problems has been developed. The static system of theory coupled-elasticity of thermo-diffusion models for two-dimensional areas has been investigated. The Greens functions applications in the CH conditions to construct the numerical solutions for materials with sufficiently smooth surfaces has been provided.

The result is provided on the fundamental level and useful tools for implementation in other theories to solve mixed boundary-contact problems for statics and oscillation systems.

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Conflict of Interest

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