Abelian groups derived from hypergroups

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Abstract: We introduce a new strongly regular relation α on a given group G and show that α is a congruence relation on G, with respect to module the commutator subgroup of G. Then we show that the composition of this relation with the fundamental relation β^* is equal to the fundamental and γ are is equal to the relation α. and we conclude that if ρ is an arbitrary strongly regular relation on the hypergroup H, then the effect of α on ρ, results in a strongly regular relation such that its quotients is an abelian group.

Key–Words: Hypergroup, fundamental relation, fundamental group, Strongly regular relation.

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1 Introduction

The hyperstructure theory, born in 1934 with Marty's paper at the $Viii$ Congress of Scandinavian Mathematicians , was subsequently developed around the $40's$ with the contribution of various authors especially in France and in the United States [3]. Marty showed that the characteristics of hypergroups can be used in solving some problems of groups, algebraic functions, and rational functions. Surveys of the theory can be found in [8]. A special equivalence relation which is called fundamental relations play important roles in the theory of algebraic hyperstructures. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. The fundamental relation β^* on hypergroups was defined by Koskas[7] and studied by many of authors(for more details see [2, 3] [4, 5, 6], [9] and Vogiouklis[10]).

2 Preliminaries

In this section, we provide the basic definitions of hypergroup and hyperring theory. For a complete introduction, we refer the readers to [3].
Let H be a set, elements of which will be denoted $a, b, ...$, and subsets of which will be denoted $A, B, ...$. Let $P^*(H)$ be the family of nonempty subsets of H and $\circ$ a *hyperoperation* or *join operation* in H, that is, $\circ$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, its image under $\circ$ in $P^*(H)$, is denoted by $a \circ b$ or ab. The join operation is extended to subsets of H in a natural way, so that $A \circ B$ or AB is given by $AB = \bigcap\{ab|a \in A, b \in B\}$. The notation aA and Aa is used for $\{a\}A$ and $A\{a\}$, respectively. Generally, the singleton $\{a\}$ is defined by its member a. A non-empty set H together with a hyperoperation $\cdot$ is called a hypergroupoid or a hyperstructures, and it is denoted by the pair $(H, \cdot)$. A hypergroupoid $(H, \cdot)$ is called a semihypergroup if for all x, y, z of H, the associativity is hold: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, which means that $\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v$. An element e of H is called an identity (resp. scalar identity) of $(H, \cdot)$ if for all $a \in H$, one has $a \in (e \cdot a) \cap (a \cdot e)$, $(\{a\} = (e \cdot a) \cap (a \cdot e))$.

Definition 2.1 *A semihypergroup $(H, \cdot)$ is a hypergroup if $x \cdot H = H \cdot x = H$, for all $x \in H$ (Reproduction axiom).*

A hypergroup $(H, \cdot)$ is commutative if $a.b = b \circ a$ for all $a, b \in H$.

Definition 2.2 *Let $(H, \circ)$ be a semihypergroup and ρ be an equivalence relation on H. Then ρ is said to be:*

(i) regular on the right (resp. on the left) if for all x of H, from $a\rho b$, it follows that:

$$(a \circ x)\overline{\rho}(b \circ x)(resp.(x \circ a)\overline{\rho}(x \circ b));$$

(ii) strongly regular on the right (resp. on the left) if for all x of H, from $a\rho b$, it follows that $(a \circ x)\overline{\overline{\rho}}(b \circ x)(resp.\ (x \circ a)\overline{\overline{\rho}}(x \circ b))$;

(iii) ρ is called (resp. strongly regular)regular if it is (resp. strongly regular) regular both on the right and on the left.

Theorem 2.3 *Let $(H, \circ)$ be a semihypergroup(resp. hypergroup) and ρ be an equivalence relation on H.*

The ρ is strongly regular if and only if $(H/\rho, \otimes)$ is a semigroup(resp. group), with respect to operation:

$$\overline{x} \otimes \overline{y} = \{\overline{z} | z \in x \circ y\}.$$

Definition 2.4 *Let $(H, \circ)$ be a semihypergroup and n be a nonzero natural number. We say that*

$$x\beta_n y \Leftrightarrow \exists (a_1, a_2, \ldots, a_n) \in H^n, \{x, y\} \subseteq \prod_{i=1}^{n} a_i.$$

Let $\beta = \bigcup_{n\geq 1} \beta_n$. Clearly, β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Theorem 2.5 *[5] [?] β^* is the smallest strongly regular relation on H.*

The smallest equivalence relation β^* in above is called the *fundamental relation* on (resp. semi)hypergroup $(H, \circ)$, and the derived(resp. semi)group H/β^* is called the *fundamental(resp. semigroup) group of H. R. Ameri in [?] shown that this relation is fanctorial, that is, the relation β^* induced a functor from category of (resp. semi)hypergroups to category of (resp. semi)groups.*

Theorem 2.6 *[6] If $(H, \circ)$ is a hypergroup, then $\beta^* = \beta$.*

Example 2.7

Let $(H, \circ)$ be a very thin hypergroup, such that for all, only a pair of the element of H the hypercomposition is a singleton set, that is there exists a unique pair $(a, b) \in H^2, |a \circ b| > 1$, and $|x \circ y| = 1$ for all $x, y \in H, (x, y) \neq (a, b)$. Then $\beta^*(x) = \{x\}$, for all $x \notin a \circ b$, and $\beta^*(y) = \beta^*(a \circ b)$, for all $y \in a \circ b$.

Remark 2.8 *Freni in [4] introduced a new relation γ on a hypergroup H as follows:*

$\gamma = \bigcup_{n\geq 1} \gamma_n$, where $\gamma_1 = \{(x, x); x \in H\}$, and for positive integer $n > 1$, γ_n is defined by

*$x\gamma_n y \Longleftrightarrow \exists a_i \in H,\ \exists \sigma \in S_n,\ 1 \leq i \leq n;$
$x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)}$.*

Evidently, for every $n \in \mathbb{N}$, the relations γ_n have symmetric and reflexive properties, and hence the relation $\gamma = \bigcup_{n\geq 1} \gamma_n$ has reflexive and symmetric properties. Assume γ^* be the transitive closure of γ. Also, the class of H/γ^* is considered $\gamma^*(z) = \{w \mid z\gamma^* w\}$, for $z, w \in H$. It was proved that the relation γ is transitive, and also γ^* has the smallest strongly regular equivalence property so that H/γ^* is an abelian group.

Theorem 2.9 *[4] The relation γ^* is the smallest strongly regular relation on a (resp. hypergroup)semihypergroup such that the quotient H/γ^* is commutative (resp. group) semigroup.*

Theorem 2.10 *[4] If H be a hypergroup, then $\gamma = \gamma^*$.*

3 A new Relation α

In this section we introduce a new relation α on a (resp. semi)hypergroup H, and reformulate the relation γ^* based on the relation α.

Definition 3.1 *Consider the relations α and δ on a group G as follows:*
$g_1\alpha g_2 \Longleftrightarrow \exists m \in \mathbb{N}, \exists (y_1, y_2, ..., y_m) \in G^m,$
$\exists \sigma \in \mathbb{S}_m : g_1 \in \prod_{i-1}^{m} y_i,$ *and* $g_2 \in \prod_{i=1}^{m} y_{\sigma(i)}$.
and

$$g_1\delta g_2 \Longleftrightarrow g_1 g_2^{-1} \in G',$$

where G' is derived group (or commutator subgroup) of G.

As usual, we usually use, the congruence relation instead, (strongly)regular relation on groups or semigroups.

Lemma 3.2 *The relation α and δ are congruence relations on G.*

Proof. *By definition it is clear that δ is a congruence relation on G. Also, α is a symmetric and reflexive relation. Let $g_1\alpha g_2$ and $g_2\alpha g_3$, then by definition α, there will be $m, n \in \mathbb{N}$, $\sigma \in S_n$, $\tau \in \mathbb{S}_m$, $(x_1, x_2, ..., x_m) \in G^m$ and $(y_1, y_2, ..., y_n) \in G^n$, such that $g_1 = \prod_{i=1}^{m} x_i$, $g_2 = \prod_{i=1}^{m} x_{\sigma(i)} = \prod_{j=1}^{n} y_j$, $g_3 = \prod_{j=1}^{n} y_{\tau(j)}$. So, $g_1 g_2 = x_1 x_2 ... x_m y_1 y_2 ... y_n \alpha y_{j_1} y_{j_2} ... y_{j_n} x_{i_1} x_{i_2} ... x_{i_m} = g_3 g g_2$.*

Now we can write
$x_1 x_2 ... x_m y_1 y_2 ... y_n (y_n^{-1} y_{n-1}^{-1} ... y_1^{-1}) \alpha y_{j_1} y_{j_2} ... y_{j_n} x_{i_1} x_{i_2} ... x_{i_m} (y_n^{-1} y_{n-1}^{-1} ... y_1^{-1})$. Therefore, $g_1\alpha g_2$. Suppose $g_1\alpha g_2$ and $g_3 \in G$, so there are $n \in \mathbb{N}$,$\sigma \in S_n$ and $(x_1, x_2, ..., x_n) \in G^n$, such that $g_1 = x_1 x_2 ... x_n \alpha x_{i_1} x_{i_2} ... x_{i_m} = g_2$. So, one has $g_3 g_1 = g_3 x_1 x_2 ... x_n \alpha g_3 x_{i_1} x_{i_2} ... x_{i_n} g_3 g_2$.

Lemma 3.3 *On a group $\alpha = \delta$.*

Proof. *We must show that $\delta \subseteq \alpha$ and $\alpha \subseteq \delta$. Because, α is a congruence relation, then $(G\alpha, \bullet)$ is a group, where $[a_1]_\alpha \bullet [a_2]_\alpha = [a_1 a_2]_\alpha$ and $e_{G/\alpha} = [e_G]_\alpha$, and $[a]_\alpha^{-1} = [a^{-1}]_\alpha$. Since, $a_1 a_2 \alpha a_2 a_1$, then*

$[a_1]_\alpha \bullet [a_2]_\alpha = [a_1a_2]_\alpha = [a_2a_1]_\alpha = [a_2]_\alpha \bullet [a_1]_\alpha$, *and hence* $(G/\alpha, \bullet)$ *is abelian group. Thus* $\delta \subseteq \alpha$, *because* δ *is the smallest strongly regular relation on* G *such that* G/α *is an abelian group. Conversely, suppose that* $a\alpha b$ *we must show that* $ab^{-1} \in G^{'}$. *Since,* $a\alpha b$ *there will be* $m \in \mathbb{N}$, $\sigma \in \mathbb{S}_m$ *and* $(x_1, x_2, a, x_m) \in G^m$, *that is* $a = \prod_{i=1}^m x_i$ *and* $b = \prod_{i=1}^m x_{\sigma(i)}$. *We know that* $x_ix_j = [x_i, x_j]x_jx_i$, *where* $[x_i, x_j] = x_ix_jx_i^{-1}x_j^{-1}$. *So, there exists natural number* k *and elements* a_j, b_j, $(1 \leq j \leq k)$ *such that*

$$x_{i_1}x_{i_2}...x_{i_m} = [a_1, b_1][a_2, b_2]...[a_k, b_k]x_1x_2...x_m,$$

where

$$g = [a_1, b_1][a_2, b_2]...[a_k, b_k] \in G^{'}.$$

Therefore, $ab^{-1} = x_1x_2...x_m(x_{i_1}x_{i_2}...x_{i_m})^{-1} = x_1x_2...x_m(gx_1x_2...x_m)^{-1} = g^{-1} \in G^{'}$.

Remark 3.4 *Let* ρ *be an strongly regular relation on* H. *Then it is easy to see that for each* $a, b \in H$; $a(\alpha * \rho)b \Longleftrightarrow [a]_\rho\alpha[b]_\rho$.

Lemma 3.5 *Let* ρ *is strongly regular relation on* H. *Then* $\alpha * \rho$ *is also an strongly regular relation on* H.

Proof. *It is clear that* $\alpha * \rho$ *is an equivalence relation on* H. *Let* $h_1, h_2, h \in H$, *and* $h_1(\alpha * \rho)h_2$. *Since* $h_1(\alpha * \rho)h_2$, *then* $[h_1]_\rho\alpha[h_2]_\rho$ *and* $[h]_\rho\alpha[h]_\rho$. *Given that* α *is a strongly regular relation. Then* $[h_1]_\rho \bullet [h_2]_\rho\alpha[h_2]_\rho \bullet [h_1]_\rho$, *and since* ρ *is strongly regular, it concluded that*

$$[h_1]_\rho \bullet [h_\rho] = [h_1oh]_\rho = [z_1]_\rho.$$

and

$$[h_2]_\rho \bullet [h_\rho] = [h_2oh]_\rho = [z_2]_\rho,$$

for all $z_1 \in h_1oh$ *and for each* $z_2 \in h_2oh$. *Therefore,*

$$[z_1]_\rho = [h_1oh]_\rho\alpha[h_2oh]_\rho = [z_2]_\rho,$$

and for each $z_1 \in h_1oh, z_2 \in h_2oh$; $z_1(\alpha * \rho)z_2$.

Theorem 3.6 $\alpha * \beta = \gamma$.

Proof. *By Lemma 3.5,* $(H/(\alpha * \beta), \star)$ *is a group. Let* $h_1, h_2 \in H$, *by definition of* α, *one has* $[h_1]_\beta \bullet [h_2]_\beta\alpha[h_2]_\beta \bullet [h_1]_\beta$. *Since* β *is strongly regular , then* $[z_1]_\beta = [h_1oh]_\beta\alpha[h_2oh]_\beta = [z_2]_\beta$, *for each* $z_1 \in h_1oh_2, z_2 \in h_2oh_1$. *This means that* $[h_1oh_2]_{\alpha*\beta} = [h_2oh_1]_{\alpha*\beta}$, *and since* $\alpha * \beta$ *is strongly regular, we have* $[h_1]_{\alpha*\beta}\star[h_2]_{\alpha*\beta} = [h_2]_{\alpha*\beta}\star[h_1]_{\alpha*\beta}$. *Since* $(H, (\alpha * \beta), \star)$ *is an abelian group and* γ *is the smallest relation such that* H/γ *is an abelian group, it conclude that* $\gamma \subseteq \alpha * \beta$. *Suppose* $h_1, h_2 \in H$ *and* $h_1(\alpha * \beta)h_2$. *So, by definition, there are* $m \in \mathbb{N}$ *and* $([x_1]_\beta, [x_2]_\beta, ..., [x_m]_\beta) \in (H/\beta)^m$ *and* $\sigma \in S_m$, *such that* $[h_1]_\beta = \prod_{i=1}^m [x_i]_\beta$ *and* $[h_2]_\beta = \prod_{i=1}^m [x_{\sigma(i)}]_\beta$. *Since* β *is strongly regular relation, then*

$$h_1] \in [h_1]_\beta = [x_1]_\beta \bullet [x_2]_\beta \bullet ... \bullet [x_m]_\beta = [x_1 \circ x_2 \circ ... \circ x_m]_\beta,$$

and

$$h_2 \in [h_2]_\beta = [x_{i_1}]_\beta \bullet [x_{i_2}]_\beta \bullet ... \bullet [x_{i_m}]_\beta = [x_{i_1} \circ x_{i_2} \circ ... \circ x_{i_m}]_\beta.$$

Let $x \in x_1 \circ x_2 \circ ... \circ x_m$ *and* $y \in x_{i_1} \circ x_{i_2} \circ ... \circ x_{i_m}$. *Then we have* $x\gamma y$. *Also,* $x \in [h_1]_\beta$ *and* $y \in [h_2]_\beta$ *implies* $h_1\beta x$ *and* $y\beta h_2$. *But,* $\beta \subseteq \gamma$. *Therefor,* $h_1\gamma x$ *and* $y\gamma h_2$, *this shows that* $h_1\gamma x\gamma y\gamma h_2$, *and hence* $h_1\gamma h_2$, *as desired.*

Theorem 3.7 *Let* ρ *be an strongly regular relation on a hypergroup* $(H, \circ)$. *Then* $H/(\alpha * \rho)$ *is an abelian group and* $H/(\alpha * \rho) = (H/\rho)^{'}$.

Proof. *By Theorem 2.7 we have* $\beta \subseteq \rho$, *and hence* $\alpha*\beta \subseteq \alpha*\rho$. *Also, by Theorem 3.6 we have* $\gamma \subseteq \alpha*\rho$.

Corollary 3.8 *Let* $\varphi : H_1 \rightarrow H_2$ *be a homomorphism of hypergroups. Let* ρ_1 *be a strongly regular relation on* H_1 *and* ρ_2 *be a strongly regular relation on* H_2. *Then* $\bar{\varphi} : H_1/\rho_1 \rightarrow H_2/\rho_2$, *where* $\bar{\varphi}([x]_{\rho_1}) = [\varphi(x)]_{\rho_2}$ *is a homomorphism of groups.*

Proof. *It is obvious.*

Corollary 3.9 *Let* $\mathcal{H}$, $\mathcal{G}$ *and* $\mathcal{A}$ *be the categories of hypergroups, groups and abelian groups, respectively. Let* ρ *be a strongly regular relation on* $\mathcal{H}$. *Then the mappings* $\mathcal{F}_\rho : \mathcal{H} \rightarrow \mathcal{G}$, *and* $\mathcal{F}_\alpha : \mathcal{G} \rightarrow \mathcal{A}$, *defined by* $\mathcal{F}_\rho(H) = H/\rho$ *and* $\mathcal{F}_\alpha(G) = G/\alpha$ *are functors. Moreover,* $\mathcal{F}_\alpha \cdot \mathcal{F}_\rho = \mathcal{F}_{\alpha*\rho} : \mathcal{H} \rightarrow \mathcal{A}$, *and* $\mathcal{F}_{\alpha*\rho}(H) = H/(\alpha * \rho)$.

4 Conclusion

A new characterization for the fundamental relation γ^* *on a hypergroup, such that its quotient space be abelian are given. In Precisely, it is shown that the* γ^* *can be obtained as combination the fundamental relation* β^* *and the commutator subgroup of the fundamental group derived from* β^*.

References:

[1] R. Ameri, On the categories of hypergroups and hypermodules, J. Discrete Math. Sci. Cryptogr., 6 (2003) 121–132. European J. Combin., 34 (2013)379–390.

[2] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, Advances in Mathematices, Vol. 5, Kluwer Academic Publishers, 2003.

[3] P. Corsini, Prolegomena of Hypergroup Theory, 2nd ed., Aviani Editore, Tricesimo, 1993.

[4] D. Freni, A new characterization of the derived hypergroup via strongly regular equivalences, Communication in algebra, 30(8) (2002) 3977–3989.

[5] D. Freni, On a Strongly Regular Relation in Hypergroupoids, Pure Math. Appl, Ser. A, 3-4 (1992) 191–198.

[6] D. Freni, Une note sur le coeur d'un hypergroupe et sur la cloture transitive β^* de β, (in French), [A note on the core of a hypergroup and the transitive closure β^* of β], Rivista. di Mat. Pura Appl., 8 (1991) 153–156.

[7] M. Koskas, Groupoides, Demi-hypergroupes et hypergroupes, J. Math. Pures Appl, 49(1970) 155–192.

[8] F. Marty, Sur une generalization de la notion de groupe, in: 8^{th} Congress Math., Scandinaves, Stockholm, Sweden, 1934, 45–49.

[9] S. Spartalis, T. Vougiouclis, The fundamental relations on H_v-rings, Rivista. di Math. Pura Appl., 13 (1994) 7–20.

[10] T. Vougiouklis, Hyperstructures and Their Representations, Hadronic Press Monographs in Mathematics, Hadronic Press, Florida, 1994.

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Reza Ameri proposed, carried out the research, and wrote the article. M. H. and A. S., edited the article. They also commented on it. They also selected the appropriate journal and submitted the article. All authors have agreed to the manuscript.
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