The number of fillings a $2 \times 2 \times n$ prism with $1 \times 1 \times 2$ prisms

RADOVAN POTŮČEK
Department of Mathematics and Physics
University of Defence
Kounicova 65, 662 10 Brno
CZECH REPUBLIC

Abstract: This paper is inspired by very interesting YouTube video of Burkard Polster, professor of mathematics at Monash University in Melbourne, Australia, which, among other things, concerned the number of ways to fill a part of the plane with dominoes, i.e. 1×2 rectangles. First we deal with the numbers of fillings the $2 \times 2 \times n$ prism with elementary $1 \times 1 \times 2$ prisms for $n = 1, 2, 3, 4, 5$. Special symbolism and figures showing the filling of the prism are used as well as the concept of matching from graph theory and the corresponding graph diagrams. Then we generalize these specific considerations and derive a general recurrence formula for any $n \geq 3$, which expresses the number of fillings of the $2 \times 2 \times n$ prism with $1 \times 1 \times 2$ elementary prisms, which in a way can be considered as spatial domino cubes, if we do not consider their marking with pairs of numbers from 0 to 6.

Key-Words: prism filling, elementary prism, heuristic search, domino, recurrence formula, enumeration

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1 Introduction

This article, inspired by YouTube video [8], was written, among other reasons, because no article concerning the number of fillings of a $2 \times 2 \times n$ prism with $1 \times 1 \times 2$ prisms was available on the website even to the average reader, only mentions [2] within the social question-and-answer website and the article [3].

Domino tilling is still a very hot topic in combinatorics and recreational mathematics – see e.g. books [4] and [5] or e.g. papers [6], [7], [8], [9].

When filling a $2 \times 2 \times n$ prism (box, brick), which we will further briefly denote as the $P(n)$ prism, with $1 \times 1 \times 2$ elementary prisms, which we will briefly call *e-prisms*, they can be oriented in three directions – vertically, horizontally to the right and horizontally up. These three types of e-prisms we will briefly call *v-prism*, *r-prism* and *u-prism*.

For example, the filling of the $P(3)$ prism in the following figure:

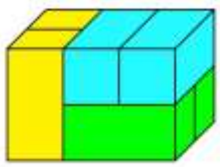


Figure 1: One $P(3)$ prism filling with six e-prisms.

we can represent as the 2×3 table containing appropriate of seven possible elements labelled by the symbols □, ≡, ≡, ⊢, ⊣, ⊢ and ⊣.

These symbols represent v-prism, two u-prisms placed above each other in both horizontal layers, two r-prisms placed above each other in both horizontal layers, left part of a r-prism placed in the upper layer above a u-prism, right part of a r-prism placed in the upper layer above a u-prism, lower part of a u-prism placed in the upper layer above a r-prism and upper part of a u-prism placed in the upper layer above a r-prism, respectively.

So the filling shown in Figure 1 can be represented as the following 2×3 table:

□	⊤	⊤
□	⊥	⊥

Here and further on in the pictures, we will distinguish the individual types of e-prisms by colour: v-prisms in yellow, r-prisms in green and u-prisms in light blue. Mostly, however, we will represent a filling of the $P(n)$ prism using a $2 \times n$ table.

It is obvious that the number of v-prisms in the $P(n)$ prism must be an even number. Otherwise, it would not be possible to fill the two remaining horizontal layers formed by an odd number of unit cubes with horizontal e-prisms.

If we rotated the $P(n)$ prism by 90 degrees around its axis passing perpendicular to the 2×2 base, then the previously u-prisms would become v-prisms, so the number of u-prisms must also be an even number. Moreover, since the volume of the $P(n)$ prism is always an even number, the number of r-prisms must also be an even number.

Therefore, for example, for the $P(3)$ prism there are seven possible (v, u, r) configurations describing the numbers of pairs of prisms: $(3, 0, 0)$, $(2, 1, 0)$, $(2, 0, 1)$, $(1, 1, 1)$, $(0, 2, 1)$, $(0, 1, 2)$, $(0, 3, 0)$. It is clear that the remaining $(0, 0, 3)$ configuration does not exist, because six r-prisms do not fill the $P(3)$ prism. Obviously, the $(0, 0, n)$ configuration exists only for the $P(n)$ prism, where n is even number.

Let k denote the number of pairs of horizontal e-prisms used. Then the number of remaining pairs of v-prisms is $n - k$. In the following text, we will use the notation $F_k(n)$ for the number of fillings of the $P(n)$ prism when using k horizontal pairs of e-prisms and the notation $F(n)$ for the number of all filings of the $P(n)$ prism.

When determining the number of fillings of the $P(n)$ prism, we will first determine all possible numbers of pairs of horizontal e-prisms and subsequently for each such partial filling we will determine the number of all possible fillings $F(n)$ of the $P(n)$ prism.

2 The number of fillings of the $P(1)$ prism

It is clear that for the $P(1)$ prism there are only two (v, u, r) configurations: $(1, 0, 0)$ and $(0, 1, 0)$, so

$$F(1) = F_0(1) + F_1(1) = 1 + 1 = 2, \qquad (1)$$

as is shown in the following figure:

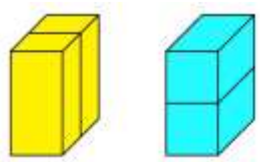


Figure 2: The only two fillings of the $P(1)$ prism.

These two fillings of the $P(1)$ prism can of course also be represented by these two 2×1 tables:

□
□

⫴
⫴

If each of the four unit cube will represent a vertex of the undirected graph $G_1 = (V_1, E_1)$, where the number of vertices $|V_1| = 4$ and the number of edges $|E_1| = 4$, then we can represent both fillings using two matchings, shown in the following figure:

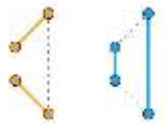


Figure 3: Two matchings in the graph G_1.

3 The number of fillings of the $P(2)$ prism

For the $P(2)$ prism, there are six (v, u, r) configurations: $(2, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 2, 0)$, $(0, 1, 1)$ and $(0, 0, 2)$. Because the $P(2)$ prism contains $2^3 = 8$ unit cubes, then it can be filled with at most four horizontal e-prisms. Therefore, the number k of pairs of horizontal e-prisms can take the values $0, 1, 2$. We will now analyse these three partial cases.

▸ For $k = 0$, corresponding to the $(2, 0, 0)$ configuration, there is only one filling of the $P(2)$ prism, namely with four v-prisms, represented by 2×2 table

□	□
□	□

so we have $F_0(2) = 1$.

▸ For $k = 1$, corresponding to the $(1, 1, 0)$ and $(1, 0, 1)$ configurations, there are four fillings of the $P(2)$ prism consisting of two fillings with one pair of u-prisms and two fillings with one pair of r-prisms. These four fillings are represented by the tables

□	⫴
□	⫴

⫴	□
⫴	□

□	□
≡	≡

≡	≡
□	□

so we have $F_1(2 = 4$.

▸ For $k = 2$, corresponding to the $(0, 2, 0)$, $(0, 1, 1)$ and $(0, 0, 2)$ configurations, there are four fillings of the $P(2)$ prism: one with two pairs of u-prisms, two with one pair of u-prisms and one pair of r-prisms and one filling with two pairs of r-prisms as represented by the following tables:

‖‖	‖‖
‖‖	‖‖

⊢	⊣
⊢	⊣

⊤	⊤
⊥	⊥

≡	≡
≡	≡

so we have $F_2(2) = 4$.

We have thus derived that the total number of fillings of the $P(2)$ prism is

$$F(2) = F_0(2) + F_1(2) + F_2(2) = \\ = 1 + 4 + 4 = 9. \quad (2)$$

These nine fillings of the $P(2)$ prism are illustrated in Figure 4.

If each of the eight unit cube will represent a vertex of the undirected graph $G_2 = (V_2, E_2)$, where $|V_2| = 8$ and $|E_2| = 12$, then we can represent these nine fillings using the following nine matchings forming the vertex cover of the graph G_2 in Figure 5.

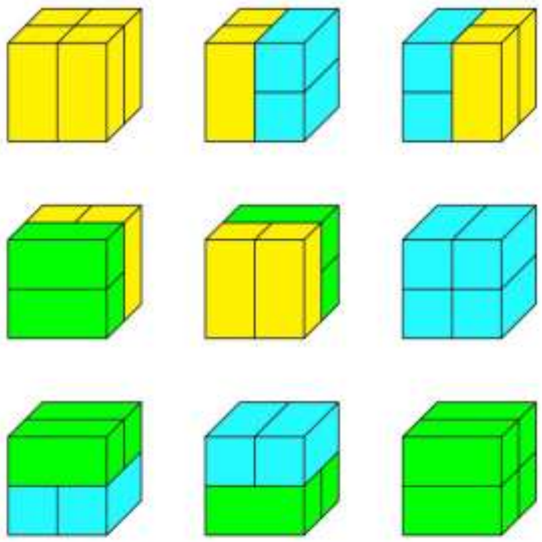


Figure 4: Nine fillings of the $P(2)$ prism.

When displaying graphs with matchings, the unit cubes in the top row and top layer will correspond to the highest-placed vertex group, and the unit cubes in the bottom row in the upper layer will correspond to the lowest-placed vertex group. The vertices corresponding to the unit cubes of the lower layer will be placed in the middle.

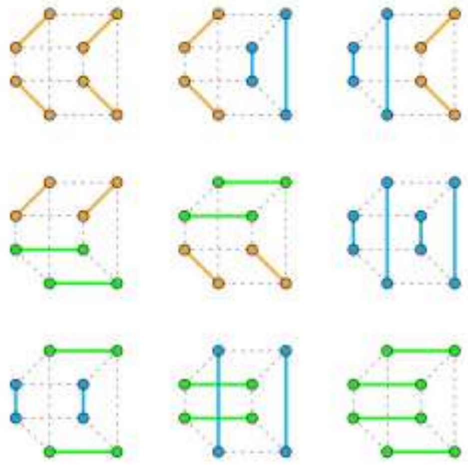


Figure 5: Nine matchings in the graph G_2.

The slanted edges apparently correspond to v-prisms, the vertical edges correspond to u-prisms and the horizontal edges correspond to r-prisms.

4 Case $n = 3$

For the $P(3)$ prism, there are nine (v, u, r) configurations: $(3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1), (1, 0, 2), (0, 3, 0), (0, 2, 1)$ and $(0, 1, 2)$ (but not $(0, 0, 3)$).

Because the $P(3)$ prism contains $2 \cdot 2 \cdot 3 = 12$ unit cubes, then it can be filled at most with 6 horizontal e-prisms. Therefore, the number k of pairs of horizontal e-prisms can take the values $0, 1, 2, 3$.

▸ For $k = 0$, corresponding to the $(3, 0, 0)$ configuration, there is only one filling of the $P(3)$ prism represented by 2×3 table

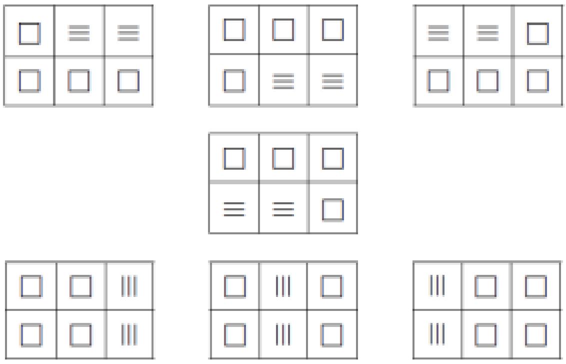
□	□	□
□	□	□

so we have $F_0(3) = 1$, in accordance with the remark at the end of the first paragraph.

▸ For $k = 1$, corresponding to the $(2, 1, 0)$ and $(2, 0, 1)$ configurations, there are four fillings of the $P(3)$ prism with one pair of r-prisms placed one above the other.

There are also three fillings with one pair of u-prisms placed one above the other, so we have $F_1(3) = 4 + 3 = 7$.

The following 2×3 tables represent these seven fillings:



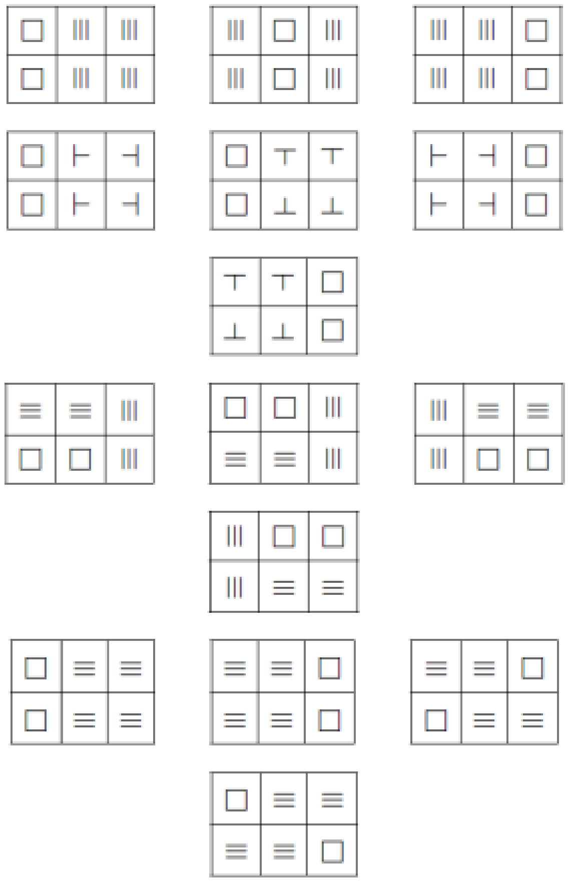
▸ The case $k = 2$ corresponds to three (v, u, r) configurations: $(1,2,0)$, $(1,1,1)$ and $(1,0,2)$.

For the configuration $(1,2,0)$ there are three fillings of the $P(3)$ prism with two pairs of u-prisms placed one above the other.

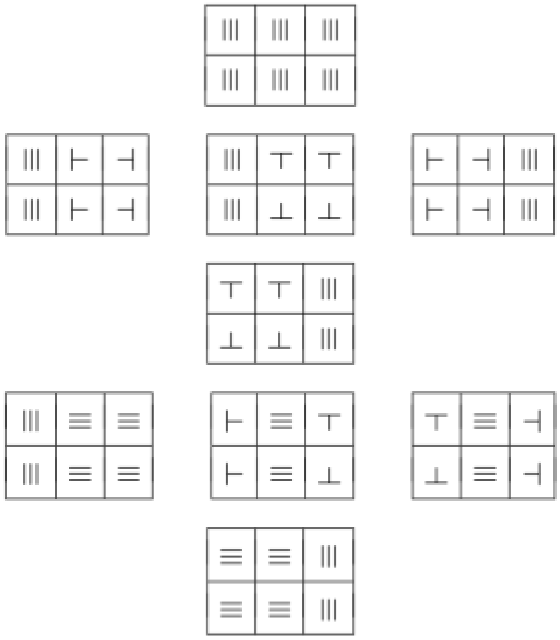
For the configuration $(1,1,1)$ there are four fillings of the $P(3)$ prism forming a $2 \times 2 \times 2$ cube containing one pair of u-prisms and one pair of r-prisms and four another L-shaped fillings consisting of one pair of u-prisms and one pair of r-prisms.

For the configuration $(1,0,2)$ there are four fillings of the $P(3)$ prism with two pairs of r-prisms forming a $2 \times 2 \times 2$ cube and also with two pairs of r-prisms shifted by one unit cube from each other.

So we have $F_2(3) = 3 + 4 + 4 + 4 = 15$ fillings, as represented by the following tables:



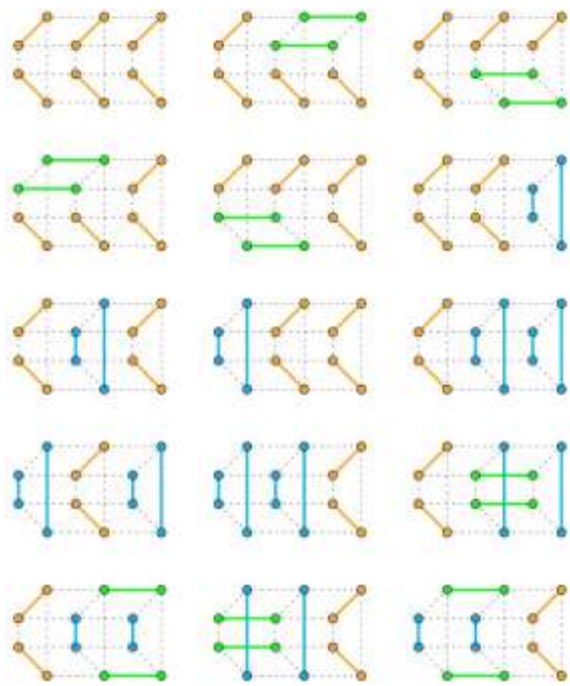
▸ For $k = 3$, corresponding to the $(0,3,0)$, $(0,2,1)$ and $(0,1,2)$ configurations, there are a total of $F_3(3) = 1 + 4 + 4 = 9$ fillings of the $P(3)$ prism with three pairs of horizontal e-prisms as represented by the following tables:



We have thus derived that the total number of fillings of the $P(3)$ prism of the e-prisms is

$$F(3) = F_0(3) + F_1(3) + F_2(3) + F_3(3) = \\ = 1 + 7 + 15 + 9 = 32. \quad (3)$$

In Figure 6 divided into two parts, 32 corresponding pairings are shown.



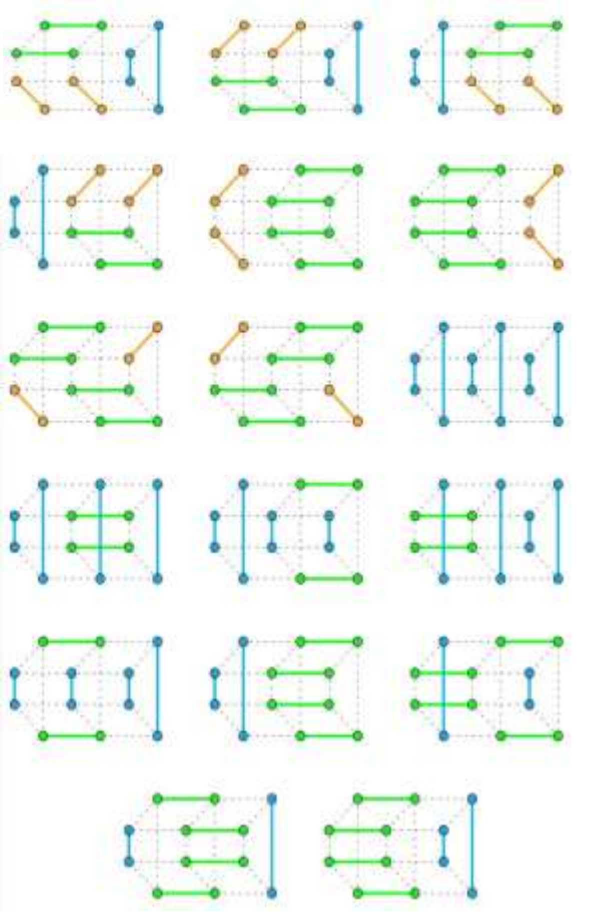


Figure 6: Thirty-two matchings in the graph G_3.

Note that the thirty-two graphs above with color-coded matchings represent graphs $G_3 = (V_3, E_3)$, where $|V_3| = 12$ and $|E_3| = 20$.

5 The number of L-prism fillings

Before deriving the main result in the following part of the article, we will define an L-prism and then derive the numbers of filling the L-prism of the e-prisms.

The hexagonal prism that for $n \geq 2$ results from a $2 \times 2 \times n$ prism after removing one of its two v-prisms or one of its two u-prisms located on the far right, we will call *L-prism* of length n and denoted by $L(n)$.

So there are exactly four L-prisms $L(n)$ that differ from each other by a 90 degree rotation about an axis passing perpendicular to the 2×2 base. Four such L-prisms $L(3)$ without their specific e-prism fillings are shown in the following figure:

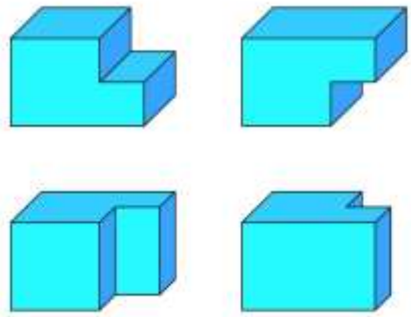


Figure 7: Four L-prisms $L(3)$.

The first of the L-prisms shown above, which is in the shape of two steps of a staircase, we will call the L-prism in the basic position and we will work with it in the next explanation. We will use the notation $H(n)$ for the number of all fillings of the $L(n)$ prism in the basic position.

Apparently, we have $H(1) = 1$, with the $L(1)$ prism filled by one u-prism. Now, we will determine the numbers of fillings $H(2), H(3), H(4)$ and then $H(n)$ for any $n \geq 2$.

▸ For $n = 2$ there are 3 fillings of the $L(2)$ prism, as shown in the following figure:

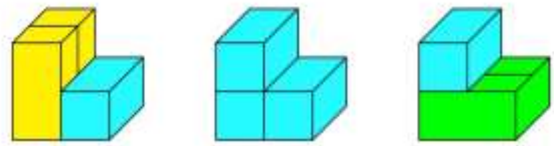


Figure 8: Three fillings of the $L(2)$ prism.

It is clear that the L-prism in the basic position is always filled with an even number of v-prisms and r-prisms and an odd number of u-prisms. Note that in Figure 8 the right overhang is formed in two cases by one u-prism and in one case by the right halves of two r-prisms.

Let us denote such two types of L-prism $L(n)$ by the symbols $L_u(n)$ and $L_r(n)$ and the corresponding filling numbers by the symbols $H_u(n)$ and $H_r(n)$.

The number of fillings $H_u(2) = 2$ apparently corresponds to the number $F(1) = 2$ of fillings the $P(1)$ prism, and the number of fillings $H_r(2) = 1$ corresponds to the number $H(1) = 1$ of fillings the $L(1)$ prism.

So we get

$$H(2) = H_u(2) + H_r(2) = F(1) + H(1) = \\ = F(1) + 1 = 2 + 1 = 3. \quad (4)$$

▸ For $n = 3$ there are the following 12 fillings of the $L(3)$ prism:

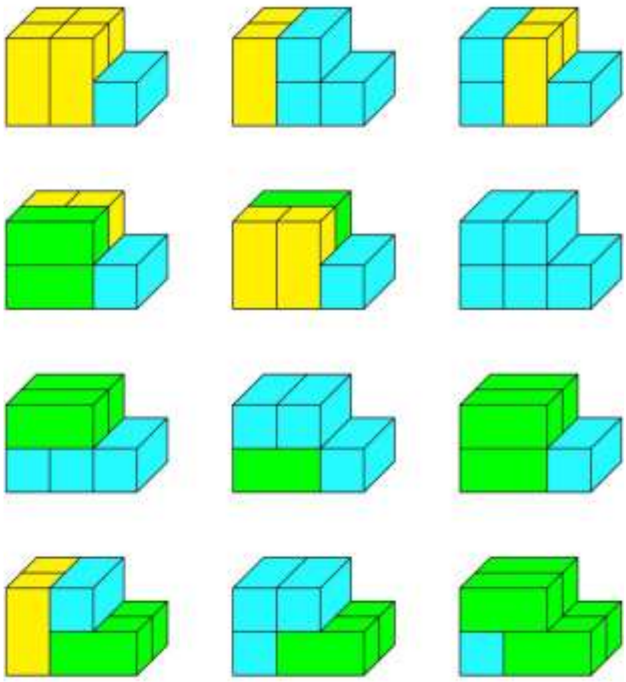


Figure 9: Twelve fillings of the $L(3)$ prism.

The number of fillings $H_u(3) = 9$ apparently corresponds to the number $F(2) = 9$ of fillings the $P(2)$ prism, and the number of fillings $H_r(3) = 3$ corresponds to the number $H(2) = 3$ of fillings the $L(2)$ prism. So we have

$$H(3) = H_u(3) + H_r(3) = F(2) + H(2) = \\ = F(2) + F(1) + 1 = 9 + 2 + 1 = 12. \quad (5)$$

▸ For $n = 4$, clearly there are $H_u(4) = F(3) = 32$ fillings of the $L(4)$ prism of type $L_u(4)$ and $H_r(4) = H(3) = 12$ fillings of the $L(4)$ prism of type $L_r(4)$. These two types of the $L(4)$ prism fillings are shown in the following figure:

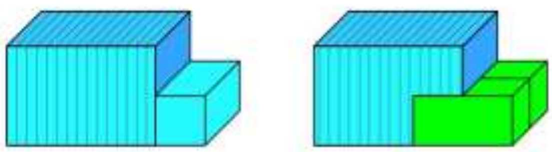


Figure 10: Two types of the $L(4)$ prism fillings.

So, in total, we get

$$H(4) = H_u(4) + H_r(4) = F(3) + H(3) = \\ = F(3) + F(2) + F(1) + 1 = 32 + 12 = 44 \quad (6)$$

fillings of the $L(4)$ prism.

6 Case $n = 4$

Assume that we have a prism $P(4)$ filled in some way. For the $P(4)$ prism, there can be three possibilities for the number of placements of the right halves of the r-prisms in the last 4th layer on the right. This layer:

i) may not contain the right half of any r-prism,

ii) may contain the right halves of 2 adjacent r-prisms,

iii) can be formed by the right halves of 4 r-prisms.

We will now analyse these three individual cases:

i) If the $P(4)$ prism contains no right half of the r-prism in the 4th layer on the right, then the first three layers form a $P(3)$ prism and the 4th layer itself on the right forms a $P(1)$ prism as shown in the following figure:

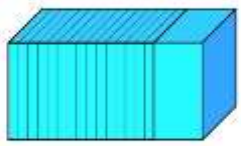


Figure 11: $P(4)$ prism without r-prisms in the 4th layer on the right.

In this case, there are $F(3) = 32$ fillings of the $P(3)$ prism and $F(1) = 2$ fillings of the $P(1)$ prism, so, in total, there are $F_i(4) = 32 \cdot 2 = 64$ fillings of the $P(4)$ prism in the considered configuration.

ii) If the $P(4)$ prism contains the right halves of 2 adjacent r-prisms in the 4th layer on the right, then it is formed by an L-prism $L(3)$ and by the complementary $L(2)$ prism, formed by two r-prisms as shown in Figure 12:

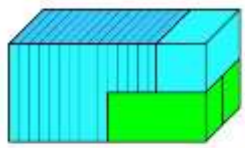


Figure 12: $P(4)$ prism with 2 right halves of adjacent r-prisms in the 4th layer on the right.

In this case, there are $H(3) = 12$ fillings of the $L(3)$ prism, which is in one of 4 possible positions as shown in Figure 7, and only $H_r(2) = 1$ filling of the specific $L(2)$ prism. So, in total, for this considered configuration, we have $F_{ii}(4) = 4 \cdot 12 \cdot 1 = 48$ fillings of the $P(4)$ prism.

iii) If the $P(4)$ prism contains the right halves of 4 r-prisms in the 4th layer on the right, then the first two layers form a $P(2)$ prism and the 3rd and 4th layers are formed by 4 r-prisms as shown in the following figure:

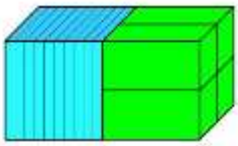


Figure 13: $P(4)$ prism with 4 right halves of r-prisms in the 4th layer on the right.

In this case, there are $F(2) = 9$ fillings of the $P(2)$ prism and only 1 filling of the $P(4)$ prism residue by 4 r-prisms. So we get $F_{iii}(4) = 9 \cdot 1 = 9$ additional partial fillings of the $P(4)$ prism.

In total we thus obtained

$$F(4) = F_i(4) + F_{ii}(4) + F_{iii}(4) = \\ = 64 + 48 + 9 = 121 \quad (7)$$

fillings of the $P(4)$ prism.

Due to the above considerations, we can write

$$F(4) = F(3)F(1) + 4H(3) + F(2). \quad (8)$$

If we consider the representation of r-prisms in the penultimate layer on the right side of the $P(4)$prism, we also receive the result $F(4) = 121$, as we will now justify and derive.

i) If the prism $P(4)$ does not contain any right half of the r-prism in the penultimate layer, i.e. in the 3rd layer on the right, then the first two layers on the left form a $P(2)$ prism and the remaining two layers on the right form two $P(1)$ prisms as shown in the following figure:

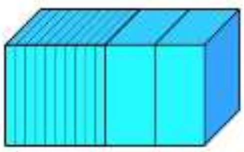


Figure 14: $P(4)$ prism without r-prisms in the 3th layer on the right.

In this case, there are $F(2) = 9$ fillings of the $P(2)$ prism and $F(1) = 2$ fillings in each $P(1)$ prism so there are $F_i'(4) = 9 \cdot 2 \cdot 2 = 36$ fillings of the $P(4)$ prism in the considered configuration.

ii) If the $P(4)$ prism contains two halves of 2 adjacent r-prisms in the 3rd layer on the right, then these 2 r-prisms can be located either in the 2nd and 3rd layers, or in the 3rd and 4th layers, as is shown in Figure 15:

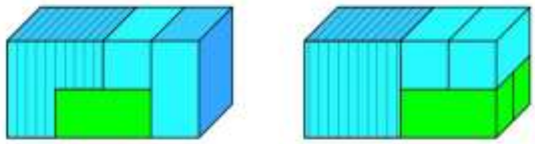


Figure 15: Two options for placing r-prisms in the 2nd and 3rd layers, or in the 3rd and 4th layers of the $P(4)$ prism.

In the first case shown in the figure above on the left, given the four possible positions of two adjacent r-prisms, there are $4 \cdot H(2) \cdot F(1) = 4 \cdot 3 \cdot 2 = 24$ ways of filling the $P(4)$ prism.

In the second case shown in the figure above on the right, given four possible positions of two adjacent r-prisms, there are $4 \cdot F(2) = 4 \cdot 9 = 36$ ways of filling the $P(4)$ prism.

In total we have $F_{ii}'(4) = 24 + 36 = 60$ fillings of the $P(4)$ prism in the considered configuration.

iii) If the $P(4)$ prism contains halves of 4 r-prisms in the 3rd layer on the right, then these 4 r-prisms can be located either in the 2nd and 3rd layers, or in the 3rd and 4th layers, or they can overlap with their halves as is shown in Figure 16:

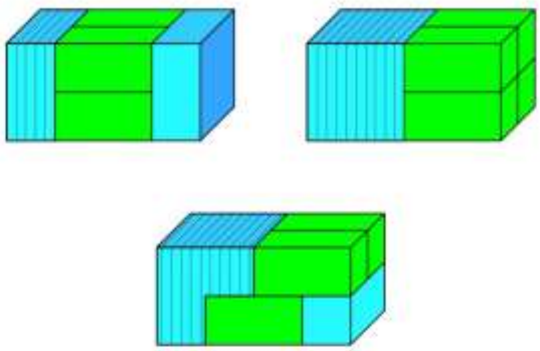


Figure 16: Three options for placing r-prisms in the 2nd and 3rd layers, or in the 3rd and 4th layers of the $P(4)$ prism, or the overlapping case.

In the first case shown in the figure above on the left there are $F(1) \cdot F(1) = 2 \cdot 2 = 4$ ways of filling the $P(4)$ prism.

In the second case shown in the figure above on the right, given the four possible positions of two adjacent r-prisms, there are $F(2) = 9$ ways of filling the $P(4)$ prism.

In the third case shown in the middle bottom figure, given the four possible positions of two adjacent r-prisms, there are $4 \cdot H(2) = 4 \cdot 3 = 12$ ways of filling the $P(4)$ prism.

In total we have $F'_{iii}(4) = 4 + 9 + 12 = 25$ fillings of the $P(4)$ prism in the considered configuration.

In total we thus obtained

$$F(4) = F'_i(4) + F'_{ii}(4) + F'_{iii}(4) = \\ = 36 + 60 + 25 = 121 \quad (9)$$

fillings of the $P(4)$ prism.

6 Case $n = 5$

Assume that we have a prism $P(5)$ filled in some way. For the $P(5)$ prism, as for the $P(4)$ prism, there are three possibilities for the number of placements of the right halves of the r-prisms in the last 5th layer on the right.

i) If the $P(5)$ prism contains no right half of the r-prism in the 5th layer on the right, then the first four layers form a $P(4)$ prism and the 5th layer itself on the right forms a $P(1)$ prism:

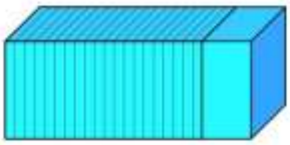


Figure 17: $P(5)$ prism without r-prisms in the 5th layer on the right.

In this case, there are $F(4) = 121$ fillings of the $P(4)$ prism and $F(1) = 2$ fillings of the $P(1)$ prism, so, in total, there are $F_i(5) = 121 \cdot 2 = 242$ fillings of the $P(5)$ prism in the considered configuration.

ii) If the $P(5)$ prism contains the right halves of 2 adjacent r-prisms in the 5th layer on the right, then it is formed by an L-prism $L(4)$ and by the complementary $L(2)$ prism, formed by two r-prisms as shown in Figure 18:

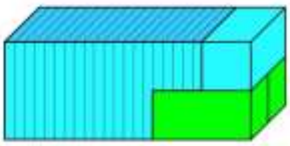


Figure 18: $P(5)$ prism with 2 right halves of adjacent r-prisms in the 5th layer on the right.

In this case, there are $H(4) = 44$ fillings of the $L(4)$ prism, which is in one of 4 possible positions, and only $H_r(2) = 1$ filling of the specific $L(2)$ prism. So, in total, for this considered configuration we have $F_{ii}(5) = 4 \cdot 44 \cdot 1 = 176$ fillings of the $P(5)$ prism.

iii) If the $P(5)$ prism contains the right halves of 4 r-prisms in the 5th layer on the right, then the first three layers form a $P(3)$ prism and the 4th and 5th layers are formed by 4 r-prisms as shown in Figure 19:

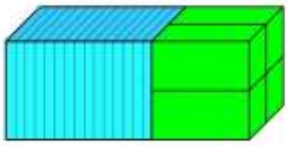


Figure 19: $P(5)$ prism with 4 right halves of r-prisms in the 5th layer on the right.

In this case, there are $F(3) = 32$ fillings of the $P(3)$ prism and only 1 filling of the $P(5)$ prism residue by 4 r-prisms. So we have $F_{iii}(5) = 32 \cdot 1 = 32$ additional partial fillings of the $P(5)$ prism.

In total we thus obtained

$$F(5) = F_i(5) + F_{ii}(5) + F_{iii}(5) = \\ = 242 + 176 + 32 = 450 \quad (10)$$

fillings of the $P(5)$ prism.

Due to the above considerations, we can write

$$F(5) = F(4)F(1) + 4H(4) + F(3). \quad (11)$$

Since $F(1) = 2$ according to equation (1), we can rewrite equations (8) and (11) in the form

$$F(4) = 2F(3) + 4H(3) + F(2) \quad (12)$$

and

$$F(5) = 2F(4) + 4H(4) + F(3). \quad (13)$$

After subtracting equation (12) from equation (13), we get the equation

$$F(5) - F(4) = 2[F(4) - F(3)] + \\ + 4[H(4) - H(3)] + F(3) - F(2). \quad (14)$$

Since according to equation (6) we have

$$H(4) - H(3) = F(3), \quad (15)$$

we can write $F(5)$ in the form

$$F(5) = 3F(4) + 3F(3) - F(2) \\ + 4[H(4) - H(3)] + F(3) - F(2).$$

i.e. in the recurrent form

$$F(5) = 3F(4) + 3F(3) - F(2)$$
$$F(4) = 121, \quad F(3) = 32, \quad F(2) = 9. \tag{16}$$

After substituting the values into the right side of the equation, we get that the number of fillings $P(5)$ prism is

$$F(5) = 3 \cdot 121 + 3 \cdot 32 - 9 = 450. \tag{17}$$

which is consistent with result (10).

8 General case for $n \geq 3$

Equations (4), (5) and (6) contain a series of these equalities:

$$H(2) - H(1) = F(1),$$
$$H(3) - H(2) = F(2),$$
$$H(4) - H(3) = F(3).$$

We now generalize these equalities to the equality

$$H(n+1) - H(n) = F(n). \tag{18}$$

Let us consider the following Figure 20 illustrating the $P(n+1)$ prism, which contains the $L(n+1)$ and $L(n)$ L-prisms, and derive the difference between the filling numbers $H(n+1)$ and $H(n)$ of these L-prisms.

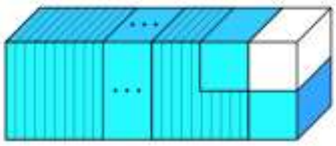


Figure 19: $P(n+1)$ prism containing $L(n+1)$ and $L(n)$ prisms.

Assume that we have a $P(n+1)$ prism filled in some way. For the reasons stated in the three previous paragraphs, when determining the number of fillings of the L-prism $L(n+1)$, we will again consider three cases for the number of placements of the right halves of the r-prisms in the last $(n+1)$th layer on the right.

i) If the $P(n+1)$ prism contains no right half of the r-prism in the $(n+1)$th layer on the right, then the first n layers form a $P(n)$ prism and the $(n+1)$th layer itself on the right forms a $P(1)$ prism as shown in the following figure:

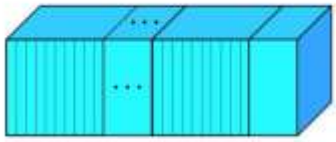


Figure 20: $P(n+1)$ prism without r-prisms in the $(n+1)$th layer on the right.

In this case, there are $F(n)$ fillings of the $P(n)$ prism and $F(1) = 2$ fillings of the $P(1)$ prism, so in total there are $F_i(n+1) = 2 \cdot F(n)$ fillings of the $P(n+1)$ prism in the considered configuration.

ii) If the $P(n+1)$ prism contains the right halves of 2 adjacent r-prisms in the $(n+1)$th layer on the right, then it is formed by an L-prism $L(n)$ and by the complementary $L(2)$ prism, formed by two r-prisms as shown in Figure 21:

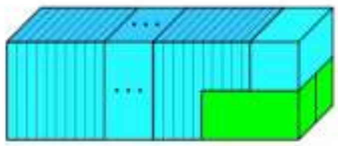


Figure 21: $P(n+1)$ prism with two right halves of adjacent r-prisms in the $(n+1)$th layer on the right.

In this case, there are $H(n)$ fillings of the $L(n)$ prism, which is in one of 4 possible positions, and only $H_r(2) = 1$ filling of the specific $L(2)$ prism. So in total for this considered configuration we have $F_{ii}(n+1) = 4 \cdot H(n) \cdot 1 = 4H(n)$ fillings of the $P(n+1)$ prism.

iii) If the $P(n+1)$ prism contains the right halves of 4 r-prisms in the $(n+1)$th layer on the right, then the first $n-1$ layers form a $P(n-1)$ prism and the nth and $(n+1)$th layers are formed by 4 r-prisms as shown in Figure 22:

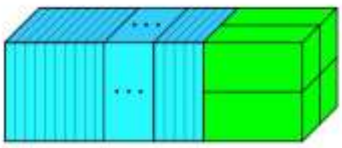


Figure 22: $P(n+1)$ prism with 4 right halves of r-prisms in the $(n+1)$th layer on the right.

In this case, there are $F(n-1)$ fillings of the $P(n-1)$ prism and only 1 filling of the $P(n+1)$ prism residue by 4 r-prisms. Therefore, we have $F_{iii}(n+1) = F(n-1) \cdot 1 = F(n-1)$ additional partial fillings of the $P(n+1)$ prism.

Since we have

$$F(n+1) = F_i(n+1) + F_{ii}(n+1) + F_{iii}(n+1),$$

we get the equation

$$F(n+1) = 2F(n) + 4H(n) + F(n-1) \quad (19)$$

expressing the number of fillings of the $P(n+1)$ prism, whence

$$4H(n) = F(n+1) - 2F(n) - F(n-1). \quad (20)$$

Analogously, we would derive the equation

$$4H(n-1) = F(n) - 2F(n-1) - F(n-2). \quad (21)$$

After subtracting equation (21) from equation (20), we get the equation

$$4[H(n) - H(n-1)] = F(n+1) - \\ - 3F(n) + F(n-1) + F(n-2). \quad (22)$$

According to equation (18) we can equation (22) for $n \geq 3$ write in the form

$$4F(n-1) = F(n+1) - 3F(n) + \\ + F(n-1) - F(n-2).$$

If we rewrite this equation in recurrent form, we obtain the main result of this paper:

Theorem 1. The number $F(n)$ of fillings a $2 \times 2 \times n$ prism with $1 \times 1 \times 2$ prisms is for all positive integer $n \geq 3$ given by the recurrent equation

$$F(n+1) = 3F(n) + 3F(n-1) - F(n-2), \\ F(1) = 2, \;\; F(2) = 9, \;\; F(3) = 32. \quad (23)$$

Remark 1. The first 40 numbers of fillings a $2 \times 2 \times n$ prism with $1 \times 1 \times 2$ prisms were calculated by using the following `for` statement written by the programming language in the computer algebra system Maple 2022. The results are stated into Table 1.

```
a:=2: b:=9: c:=32:
for n from 4 to 40 do
    d:=3*c+3*b-a: a:=b: b:= c: c:=d:
    print("F(",n,")=",d);
end do;
```

Table 1: The values of $F(n)$ for $n = 1, 2, \ldots, 40$.

n	$F(n)$	n	$F(n)$
1	2	21	637815097922
2	9	22	2380358351281
3	32	23	8883618307200
4	121	24	33154114877521
5	450	25	123732841202882
6	1681	26	461777249934009
7	6272	27	1723376158533152
8	23409	28	6431727384198601
9	87362	29	24003533378261250
10	326041	30	89582406128846401
11	1216800	31	334326091137124352
12	4541161	32	1247721958419651009
13	16947842	33	4656561742541479682
14	63250209	34	17378525011746267721
15	236052992	35	64857538304443591200
16	880961761	36	242051628206028097081
17	3287794050	37	903348974519668797122
18	12270214441	38	3371344269872647091409
19	45793063712	39	12582028104970919568512
20	170902040409	40	46956768150011031182641

4 Conclusion

In this paper the numbers $F(n)$ of fillings a $2 \times 2 \times n$ prism with $1 \times 1 \times 2$ prisms were determined. The recurrence formula

$$F(n+1) = 3F(n) + 3F(n-1) - F(n-2), \\ F(1) = 2, \;\; F(2) = 9, \;\; F(3) = 32.$$

for $n \geq 3$ was derived.

The first 40 numbers $F(n)$ were calculated by using the programming language in the computer algebra system Maple 2022. The obtained numerical results correspond to the results [10] given at The On-Line Encyclopedia of Integer Sequences.

Area of Further Development

The result in this paper can be generalized to the number of fillings a $k \times k \times n$ prism with $1 \times 1 \times k$ prisms for arbitrary integer $k \geq 3$.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.