A Study of Applied Reduced Differential Transform Method Using Volterra Integral Equations in Solving Partial Differential Equations

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Abstract: - Nowadays, integration is one of the trending fields applied in calculus, especially in partial differential equations. Researchers are contributing to support useful utilities to solve partial differential equations in many kinds of methods. In this paper, we perform an application of Volterra Integral Equations in a reduced differential transform method (we call VIE-RDTM) to find the approximate solutions of partial differential equations. The aim is to find the approximate solutions approach to the exact solutions with more general forms. We also extend some new results for basic functions and compare the solutions using the reduced differential transform method and VIE-RDTM by depicting the approximate solutions in some partial differential equations. The results showed that the VIE-RDTM method gets the state-of-the-art general form of the solutions when the errors approach zero.

Key-Words: - Volterra Integral, Reduced Differential Transform Method, RDTM, VIE-RDTM.

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1 Introduction

Recently, linear and nonlinear differential equations have been applied in many kinds of life, especially in the real world of computer science, and economic problems (readers could see more in [1, 2, 3, 4, 5]). Whilst applying in economics looked at a newfangled approach to reveal the nature of the solutions or control the models in optimal problems, partial equations are developed with more and more various types of form, and more complete structures (shown in [6]). The next generation of partial equations in fractional partial equations contributed to the development of technology, and economics. Fractional calculus appeared as the consequence of modern technology development and has been expanded in different aspects of science (Shown in [7], and [8]). Fractional calculus has rapidly become a significant tool to control the models in economics such as controlling price problems, controlling options, or controlling inflation problems. Besides that, fractional calculus also inherited all the powerful properties from partial differential calculus, and its application in variation structure resembling majors (See in [9]). However, partial differential equations have played an important role and have not completed the stages in the future. The development from the partial differential equation will create the hard roots and basic preliminaries for construing the high floors of calculus. With the state-of-the-art contribution of fractional partial differential calculus, the scenario of various types of methods have created variants to apply easier to specific problems. The homotopy perturbation method (see in [10]) is one of the most popular methods applied to solve partial differential equations. This method is useful in finding approximate solutions to economic problems. Similar to the homotopy perturbation method, the Variational Iteration method is also applied in many kinds of partial differential equations and has got a good performance in approximate solution results (see in paper [11]). Besides that, the Laplace transform method is a classical method that contributed to the development of calculus (in paper [12]). Compared to other methods, the Laplace method is useful in solving related integral equations, and some equations about economics, and finance (see more in [6]). The Adomian Decomposition method (see in paper [13, 14]) is used in solving Volterra equations, by using an iterative way, the Adomian Decomposition method establishes the sequences based on initial values to find approximate solutions. In general, most of the methods have shown good ways to directly or indirectly illustrate the advantages and powerful tools to find the solutions. The Reduce Differential Transform method was introduced by Y. Kensin et al. (See the series [15, 16, 17, 18]) and has been extended with effective application immensely in some different branches of partial differential equations (shown in [19, 20, 21, 22, 23]). After that, the Reduce Differential Transform method was expanded into the fractional Reduce Differential Transform method to follow the fractional differential equations (see more in [24, 25, 26, 27, 28]). The Fractional differential derivative is the hottest trend variation to connect the real world and is applied in the majority of calculus. Many books and papers have focused on stochastic problems wherein integrals are a good presentation with respect to the components in stochastic optimization problems. With the hope to combine integration and derivative, the purpose is to find exact solutions based on initial conditions, we applied Volterra Integral Equations as the facility to reveal the exact solutions. This paper considers the differential partial equations written formed (1).

$$Af(x,t)+Bf(x,t)+ Mf(x,t)+Lf(x,t) = g(x,t) \quad (1)$$

with the initial condition f(x,0) = h(x), where A $=\frac{\partial^m}{\partial x^m}$, and B = $\frac{\partial^n}{\partial x^n}$ are partial differential derivative, M, L represent the linear or nonlinear terms having partial derivative derivatives, and g(x,t), h(x) are given functions. In some fractional differential partial equations, the constant values are hidden and the integration will reveal the exact solution under terminal conditions.

2 Methodology

2.1 Some Basic Definitions for Calculus

To see more details about the notations and definitions, readers could read in [29, 30, 31, 32]. Now we would remind some related basic reports in partial differential derivative theory. First, we consider some definitions

Definition 1 (Gamma function) (Shown in [29], P.8) For z ∈ C, and Re(z) > 0, the integral as follows is defined

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt \quad (2)$$

Proposition 1 Base on the equation 2, we have some specific properties related to the gamma function using integration by part and direct integration Γ(1) = 1:

$$\Gamma(z+1) = z\Gamma(z) \quad (3)$$

$$\Gamma(n+1) = n! \quad (4)$$

Definition 2 (Convert to Volterra Integral equations)

$$\underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n-fold\ \text{int}egration} f(w,t)dw_n\cdots dw_1 = \frac{1}{(n-1)!}\int_0^x (x-w)^{n-1} f(w,t)dw$$

To understand more clearly equation (5), we could start practicably from the equation (6) and (7):

$$\int_0^x\int_0^x f(w,t)dwdw=\int_0^x(x-w)f(w,t)dw \quad (6)$$

$$\int_0^x\int_0^x\int_0^x f(w,t)dwdwdw=\frac{1}{2!}\int_0^x(x-w)^2 f(w,t)dw \quad (7)$$

Now we can embark form the first step of equation (6): From the right-hand side, set $h(x)=\int_0^x(x-w)f(w,t)dt$, and we have differentiation $\frac{\partial}{\partial x}h(x)=\int_0^x f(w,t)dt$. After that, integrating by x and get the equation as the following:

$$h(x)=\int_0^x\int_0^x f(w,t)dtdw \quad (8)$$

and the equation (8) is equal the left-hand side of the equation (6). So, the equation (6) has been proved. Similar to the equation (7), and using the inductive method to get the transformation formula (5).

2.2 Reduced Differential Transformation Method

The reduced Differential Transformation Method is very useful in solving linear and nonlinear equations. This method has shown a reduced differential transform to find the approximate solution in the differential equation (See more in [31, 32, 34, 35, 36, 37]. Now we will summarize method as follows. Let's examine the two-variable function expressed as f(x,t), we can write as the following

$$f(x,t) = w(x)v(t) \tag{9}$$

Now, we can express equation (9) be formed

$$F_k(x,t) = \left(\sum_{i=0}^{\infty} w(i)x^i\right)\left(\sum_{j=0}^{\infty} v(j)t^j\right) = \sum_{k=0}^{\infty} H_k(i,j)t^k \tag{10}$$

where $H_k(i,j) = w(i)v(j)$ denotes the spectrum of F(x,t). Then the of F(x,t) is formed by

$$H_k(x) = \frac{1}{\Gamma(k+1)}\left[\frac{\partial^k}{\partial t^k} F_k(x,t)\right]_{t=t_0} \tag{11}$$

where k is the order of time derivative. Then the inverse transformation of H_k is defined by

$$F_k(x,t) = \sum_{k=0}^{\infty} H_k(x)(t-t_0)^k \tag{12}$$

Combine equation (11) and equation (12), we have

$$F_k(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}\left[\frac{\partial^k}{\partial t^k} F_k(x,t)\right]_{t=t_0} (t-t_0)^k \tag{13}$$

If we choose $t_0 = 0$, from equation (13) we have

$$F_k(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}\left[\frac{\partial^k}{\partial t^k} F_k(x,t)\right]_{t=t_0} t^k \tag{14}$$

By applying inductive method for equation (14), we have solution:

$$f(x,t) = \lim_{k\to\infty} F_k(x,t) \tag{15}$$

2.3 Using Volterra Integral Equations - Reduced Differential Transformation Method

With the reduced differential transform method, the approximate solution has been found when approaching exact solutions. In some cases, the solution could be expressed more generally when it could be different by a constant. Now we perform a new extension of this method, which we call a Volterra Integral Equations of reduced differential transform method (VIE-RDTM). Let's examine the two-variable function expressed as f(x,t), we can write as the following

$$f(x,t) = w(x)v(t) \tag{16}$$

we can integrate the equation (16) respect to x from 0 to x, and then integrate the equation (16) respect to t from 0 to t (or respect to x one more time):

$$F(x,t) = \int_0^x \int_0^t f(w,t)\,dt\,dw \left(or \int_0^x \int_0^x f(w,t)\,dt\,dw \right) \tag{17}$$

Now, we can express equation (17) by formed

$$F_k(x,t) = \int_0^x \sum_{i=0}^{\infty} u(i)x^i \sum_{j=0}^{\infty} v(j)t^j dw$$
$$= \int_0^x \sum_{k=0}^{\infty} H_k(i,j)t^k dw \left(or \int_0^x (x-w)H_k(t)dw \right) \tag{18}$$

where $H_k(i, j) = u(i)v(j)$ is the spectrum of F(x,t). Then the fractional reduced differential transform method of F(x,t) is formed by

$$H_k(x) = \frac{1}{\Gamma(k+1)}\left[\frac{\partial^k}{\partial t^k} F_k(x,t)\right]_{t=t_0} \tag{19}$$

Then the inverse transformation of H_k is defined by

$$F_k(x,t) = \sum_{k=0}^{\infty} H_k(x)(t-t_0)^k \tag{20}$$

Combine equation (19) and equation (20), we have

$$F_k(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}\left[\frac{\partial^k}{\partial t^k} F_k(x,t)\right]_{t=t_0} (t-t_0)^k \tag{21}$$

If we choose $t_0 = 0$, from equation (21) we have

$$F_k(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}\left[\frac{\partial^k}{\partial t^k} F_k(x,t)\right]_{t=t_0} t^k \tag{22}$$

By applying the inductive method for equation (22):

$$f(x,t) = \lim_{k\to\infty} F_k(x,t) \tag{23}$$

Finally, the solution is called from equation (22), and equation (23) is also the original solution of the equations.

2.4 Basic Functions Using Transformation Results

Now we will illustrate the fundamental function results of the extension of composed of Volterra Integral equations and reduced differential transform method (VIE-RDTM) (shown in [38, 39, 40, 41, 42]). Let T_f be the transformation of this method and g(x,t), h(x,t) are fundamentally analytic functions. By setting $G_k = T_f(g)$, $H_k = T_f(h)$ is the results of the transform after integrating by x, we have some results as shown in Table 1 (See more in [43],[44],[45]). We will prove some specific terms as the following properties:

Proposition 2

$$f(x,t)=\int_0^x g(w,t)\,dw \text{ then}$$
$$F_k(x)=\frac{1}{k!}\frac{\partial^k}{\partial t^k}\int_0^x g(w,t)\,dw=\int_0^x G_k(w)\,dw \tag{24}$$

Proposition 3

$$f(x,t)=\int_0^x\int_0^t g(w,t)\,dw \text{ then}$$
$$F_k(x)=\frac{1}{k!}\frac{\partial^{k-1}}{\partial t^{k-1}}\int_0^x g(w,t)\,dw=\frac{1}{k}\int_0^x G_{k-1}(w)\,dw \tag{25}$$

Proposition 4

$$f(x,t)=\int_0^x \frac{\partial}{\partial t} g(w,t)\,dw \text{ then}$$
$$F_k(x)=\frac{1}{k!}\frac{\partial^{k+1}}{\partial t^{k+1}}\int_0^x g(w,t)\,dw \tag{26}$$
$$=\int_0^x (k+1)G_{k+1}(w)\,dw$$

Proposition 5

$$f(x,t)=\int_0^x\int_0^x \frac{\partial}{\partial t} g(w,t)\,dw \text{ then}$$
$$F_k(x)=\frac{1}{k!}\frac{\partial^k}{\partial t^k}\int_0^x (x-w)\,g(w,t)\,dw \tag{27}$$
$$=\int_0^x (x-w)(k+1)G_{k+1}(w)\,dw$$

Proposition 6

$$f(x,t)=\int_0^x \frac{\partial^2}{\partial t^2} g(w,t)\,dw \text{ then}$$
$$F_k(x)=\frac{1}{k!}\frac{\partial^{k+2}}{\partial t^{k+2}}\int_0^x g(w,t)\,dw \tag{28}$$
$$=\int_0^x (k+1)(k+2)G_{k+2}(w)\,dw$$

Proposition 7

$$f(x,t)=\int_0^x\int_0^x \frac{\partial^2}{\partial t^2} g(w,t)\,dw \text{ then}$$
$$F_k(x)=\frac{1}{k!}\frac{\partial^{k+2}}{\partial t^{k+2}}\int_0^x g(w,t)\,dw \tag{29}$$
$$=\int_0^x (k+1)(k+2)G_{k+2}(w)\,dw$$

Proposition 8

$$f(x,t)=\int_0^x t^m g(w,t)\,dw \text{ then}$$
$$F_k(x)=\int_0^x \sum_{r=0}^{k}\delta(r-m)G_{k-r}(w)\,dw \tag{30}$$

3 Numerical Results

In this section, we illustrate the solutions by using Volterra Integral Equation-Reduced Differential Transform Method compared to the solutions using the Reduced Differential Transform Method ([46, 47, 48]) through the examples as the following:

Example 1 We consider the Black-Scholes equation (See in [6]) formed

$$-\frac{\partial f(x,t)}{\partial t}+x^2\frac{\partial^2 f(x,t)}{\partial x^2}=0 \tag{31}$$

satisfy the terminal condition $f(x,0) = x^2$.

Using RDTM method Applying RDTM method (see more detailed in [49]), from equation (31), we turn the equation (31) into new form

$$(k+1)H_{k+1}(x)=\frac{1}{k+1}x^2\frac{\partial^2 f(x,t)}{\partial x^2}H_k(x) \tag{32}$$

Using the condition, we have the following equations (33), (34):

$$H_0(x)=x^2;\ H_1(x)=2x^2 \tag{33}$$
$$H_2(x)=2x^2;\ H_1(x)=\frac{4}{3}x^2 \tag{34}$$

By inductive method, we have the specific solution as (35).

$$f(x,t)=x^2\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots\right) \tag{35}$$

This result approach to exact solution

$$u(x,t) = x^2 e^{2t}.$$

Using VIE-RDTM method: From equation (31), we integrate both sides of the equation using integration by part respect to x from 0 to x, and then integration respect to t from 0 to t to turn (31) into Volterra integration equation, we rewrite the equation (31) as equation (36):

$$\int_0^x f(w,t)\,dw = \int_0^x \left[x^2 \frac{\partial}{\partial x} f(x,t) - 2xf(x,t) \right] d\omega$$
$$+2\int_0^x \int_0^x d(w,\omega)\,dw\,d\omega \tag{36}$$

Apply transformation in Table 1 by using VIE-RDTM, from equation (36), we have the new form of equation (37):

$$\int_0^x H_k(w)\,dw = \frac{1}{k}\left[x^2 H_{k-1}(x) - 2xH_{k-1}(x) \right] + \frac{2}{k}\int_0^x H_{k-1}(w)\,dw \tag{37}$$

From (37), we will construe the inductive step by counting values H1, H2, H3, ...

$$H_0(x)=x^2;\ H_1(x)=2x^2 \tag{38}$$

$$H_2(x)=2x^2;\ H_1(x)=\frac{4}{3}x^2 \tag{39}$$

We got the exact solution using the VIE-RDTM method using the formula equation:

$$f(x,t) = x^2\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots\right) \tag{40}$$

Finally, the solution in equation (40) lead to another form of exact solution $f(x,t) = x^2e^{2t}$, and is illustrated in Figure 1 and Figure 2.

Example 2 In the second example, a partial differential equation for a European option is given (See in [6]) by:

$$\frac{\partial f(x,t)}{\partial t} + x^2\frac{\partial^2 f(x,t)}{\partial x^2} + x\frac{\partial f(x,t)}{\partial x} = 0 \tag{41}$$

subject to initial condition $u(x,0) = x^2$.

Using RDTM method Applying the transformation using RDTM, we have equation (42):

$$H_{k+1}(x) = \frac{1}{k+1}\left[x^2\frac{\partial^2}{\partial x^2}H_k(x) + x\frac{\partial}{\partial x}H_k(x) \right] \tag{42}$$

By using inductive step, we have equation (43), (44):

$$H_0(x) = x^2;\ H_1(x) = 4x^2 \tag{43}$$

$$H_2(x) = 8x^2;\ H_3(x) = 32x^2/2;\ H_4(x) = 64x^2/3;\cdots \tag{44}$$

Finally, using RDTM, we have the solution shown in (45):

$$f(x,t) = x^2\left(1 + 4t + \frac{(4t)^2}{2!} + \frac{(4t)^3}{3!} + \cdots\right). \tag{45}$$

The equation (45) will lead to the exact solution $f(x,t) = x^2e^{4t}$.

Using VIE-RDTM method From equation (41), integrating both sides respect to x then applied the formula we have equation 46:

$$\int_0^x (x-w)\frac{\partial}{\partial t} f(x,t)\,dw = x^2 f(x,t) - 3\int_0^x wf(w,t)\,dw + \int_0^x (x-w) f(w,t)\,dw \tag{46}$$

$$\int_0^x (x-w)H_k(w)\,dw = \frac{1}{k}\left[x^2H_k(x) - 3\int_0^x wH_k(w)\,dw + \int_0^x (x-w)H_k(w)\,dw \right] \tag{47}$$

We will construe the inductive values as equation (48), (49):

$$H_0 = x^3/3;\ H_1 = 4x^3/3 \tag{48}$$

$$H_2 = 8x^2;\ H_3 = 32x^2/2;\ H_4 = 64x^2/3;\ \cdots \tag{49}$$

Finally, we have the exact solution using VIE-RDTM as equation (50):

$$f(x,t) = x^2\left(1 + 4t + \frac{(4t)^2}{2!} + \frac{(4t)^3}{3!} + \cdots\right). \tag{50}$$

The equation (50) will lead to the exact solution $f(x,t) = x^2e^{4t}$ and will be graphed in Figure 3 and Figure 4.

Example 3 In this example, a fractional diffusion PDE is to demonstrate the relation between variable stable distribution (See in [6]) given by

$$t\frac{\partial f(x,t)}{\partial t} + x\frac{\partial f(x,t)}{\partial x} + t^2 f(x,t) - t^2 = 0 \tag{51}$$

subjected to the term condition $u(x,0) = 1$, and $\frac{\partial}{\partial t}u(x,0) = x$.

Using RDTM method Apply RDTM method, we have the form:

$$(k+1)H_{k+1} = x\frac{\partial}{\partial t}H_{k+1} - H_{k-1} + \delta(k-1) \tag{52}$$

By iterative method, we have the following terms in equation (53):

$$H_0(x) = 1;\ H_1(x) = x;\ H_2(x) = 0 \tag{53}$$

$$H_3(x) = x/2;\ H_4(x) = 0 \tag{54}$$

$$H_5(x) = x/8;\ H_6(x) = 0 \tag{55}$$

The solution of equation (51) has form:

$$f(x,t) = 1 + xt\left(1 + \frac{t^2}{2} + \frac{t^4}{8} + \cdots\right). \tag{56}$$

The equation (56) will be convergent to exact solution $f(x,t) = 1 + xte^{\frac{t^2}{2}}$.

Using VIE-RDTM method: From equation (51), integrating both sides using integration by part respect to x, we have

$$\int_0^x t\frac{\partial}{\partial t} f(w,t)dw = xf(x,t) - \int_0^x f(w,t)d\omega + \int_0^x t^2 f(w,t)dw - \int_0^x t^2 dw \quad (57)$$

Applying transformation using Table 1 and we have equation (58):

$$\int_0^x (k+1)H_{k+1}(w)dw = xH_k(x)\delta(k+1) - \int_0^x \sum_{r=0}^{k} \delta(r+1)H_{k-r}(w)dw + \int_0^x \sum_{r=0}^{k} \delta(r-1)H_{k-r}(w)dw - \int_0^x \delta(k-1)dw \quad (58)$$

Applying inductive for equation (58), we have equation (59), (60), (61):

$$H_0(x) = 1;\ H_1(x) = x. \quad (59)$$

$$H_2(x) = 0;\ H_3(x) = x/2;\ H_4(x) = 0. \quad (60)$$

$$H_5(x) = x/8;\ H_6(x) = 0. \quad (61)$$

Similarly, using VIE-RDTM we have the exact solution expressed as equation (62):

$$f(x,t) = 1 + xt\left(1 + \frac{t^2}{2} + \frac{t^4}{8!} + \cdots\right) \quad (62)$$

The result will lead to exact solution form

$f(x,t) = 1 + xte^{\frac{t^2}{2}}$ and has been depicted in Figure 5 and Figure 6.

Example 4 In this example, we illustrate a diffusion process defined by the Burgers equation as the following (see in [9]):

$$\frac{\partial f(x,t)}{\partial t} + \left(1 + f(x,t)\right)\frac{\partial f(x,t)}{\partial x} = 0 \quad (63)$$

satisfy the initial condition: $u(x,0) = x-1$.

Using RDTM method

Similarly, from equation (63), we have

$$H_{k+1}(x) = \frac{1}{k+1}\left[-x\frac{\partial}{\partial x}H_k - \sum_{r=0}^{k} H_{k-r}H_r\right] \quad (64)$$

By taking inductive for the terms, we have equation (65), (66):

$$H_1(x) = -x;\ H_2(x) = x; \quad (65)$$

$$H_3(x) = -x;\ H_4 = x;\ \cdots \quad (66)$$

Finally, we have the exact solution using the RDTM method shown as equation (67):

$$f(x,t) = -1 - x\left(1 - t + t^2 - t^3 + t^4 - \cdots\right) \quad (67)$$

The equation (67) will be convergent to the exact solution $f(x,t) = -1 + \frac{x}{1+t}$.

Using VIE-RDTM method From equation (63), we integrate both sides:

$$\int_0^x \frac{\partial}{\partial t} f(w,t)dw = -f(x,t) - \int_0^x f(w,t)\frac{\partial}{\partial x} f(w,t)dw \quad (68)$$

Applied transformation by using RDTM, we have equation (69):

$$\int_0^x \frac{\partial}{\partial t} f(w,t)dw = \frac{1}{k}\left[\begin{array}{l} -H_k(x) \\ -\int_0^x H(w)\frac{\partial}{\partial x} f(w)dw \end{array}\right] \quad (69)$$

Counting the terms by an inductive method, we have equation (70), (71):

$$H_0(x) = x-1;\ H_1(x) = -x;\ H_2(x) = x. \quad (70)$$

$$H_3(x) = -x;\ H_4(x) = x;\ \cdots \quad (71)$$

Finally, we have the exact solution using the VIERDTM method shown as equation (72):

$$f(x,t) = -1 - x\left(1 - t + t^2 - t^3 + t^4 - \cdots\right) \quad (72)$$

The equation (72) will lead to exact solution $f(x,t) = \frac{x}{1+t} - 1$ and delineated in Figure 7 and Figure 8.

Example 5 In this example, we illustrate a linear differential equation defined by the following (see in [50]):

$$4\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\partial^2 f(x,t)}{\partial x^2} - \frac{\partial^2 f(x,t)}{\partial x \partial t} \quad (73)$$

satisfy the initial condition:

$u(x,0) = x^2$; $u_t(x,0) = e^x$.

Using VIE-RDTM method From equation (73), we integrate both sides:

$$4\int_0^x \int_0^x \frac{\partial^2 f(w,t)}{\partial t^2}dw = f(x,t) - \int_0^x \frac{\partial f(,t)}{\partial t}dw \quad (74)$$

Applied transformation by using RDTM, we have equation (69):

$$4\int_0^x \int_0^x (k+1)(k+2)H_k(w)dw = H_k(t) - \int_0^x (k+1)H_k(w)dw \quad (75)$$

Counting the terms by an inductive method, we have equation (70), (71):

$H_0(x) = x^2$; $H_1(x) = e^x$; $H_2(x) = 1/4 - 3e^x/8$. (76)
$H_3(x) = 13e^x/96$; $H_4(x) = -17e^x/512$; $\cdots$ (77)
Finally, we have the exact solution using the RDTM method shown as equation (72):

$$f(x,t) = e^x + e^x t + \frac{4}{5}e^x\left(1+\frac{t}{4}+\frac{t^2}{32}+\frac{t^3}{384}+\frac{t^4}{6144}+...\right) + \frac{4}{5}e^x\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}+...\right) \quad (78)$$

The equation (78) will lead to exact solution $f(x,t) = x^2 + \frac{t^2}{4} + \frac{4}{5}\left(e^{x+\frac{t}{4}} - e^{x-t}\right)$. compared to [50] and performed in Figure 9 and Figure 10.

4 Conclusion

Partial differential calculus is usefully applied in many fields of applied mathematics and technology science. It is also applied in a huge number of branches of finance, physics, viscoelastic mechanics, or economics [51, 52]). In this paper, we suggested the combination of integration and differential method, and this method showed the most convenient shortcut to find the exact solution. The method is omitting the derivative terms and replacing them with reduced terms and integration. The method has performed successfully to get the analytic solution approaching the exact solution.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Nguyen Minh Tuan carried out all the content of the paper consisting of conceptualization, data curation, investigation, formal analysis, methodology, software, visualization, writing-original draft, and writing review and editing.

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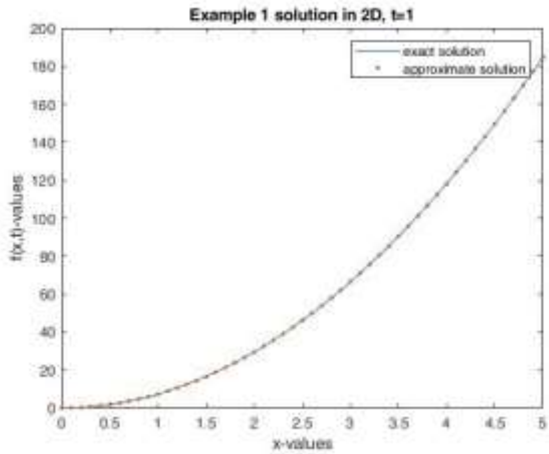
The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.



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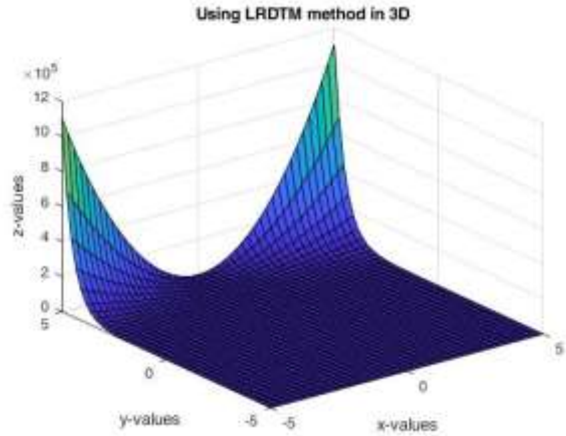

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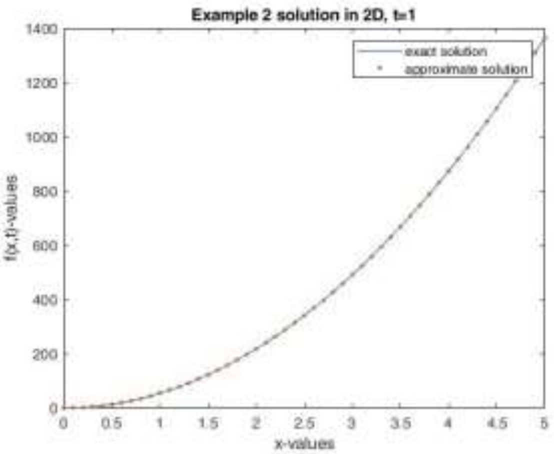

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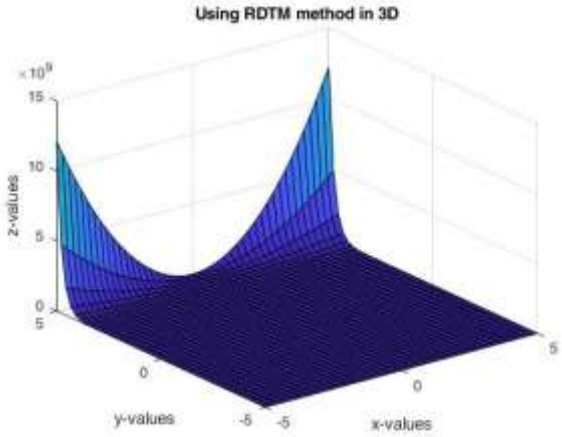

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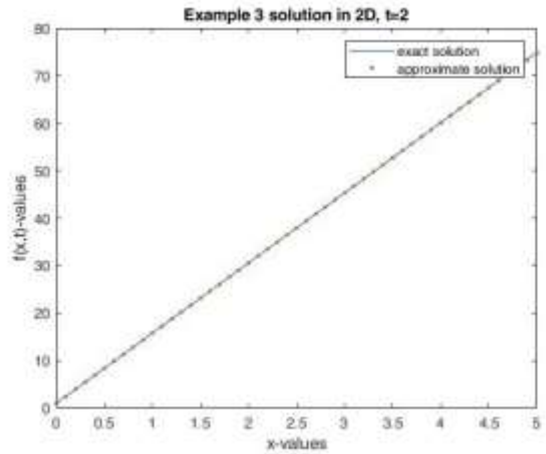

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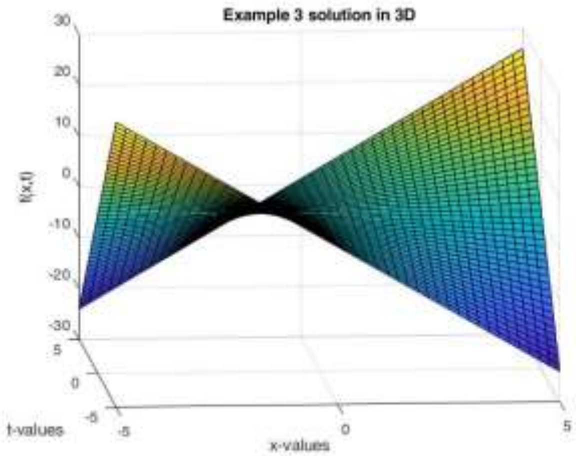

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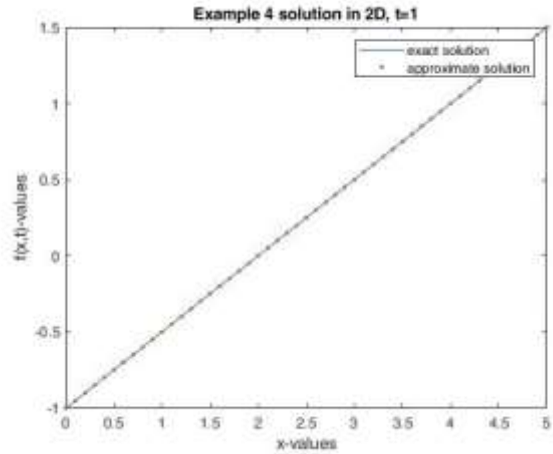
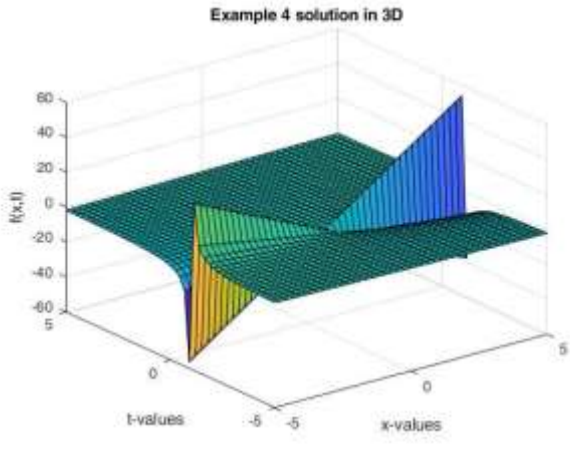


Figure 7: The fourth example in 2D



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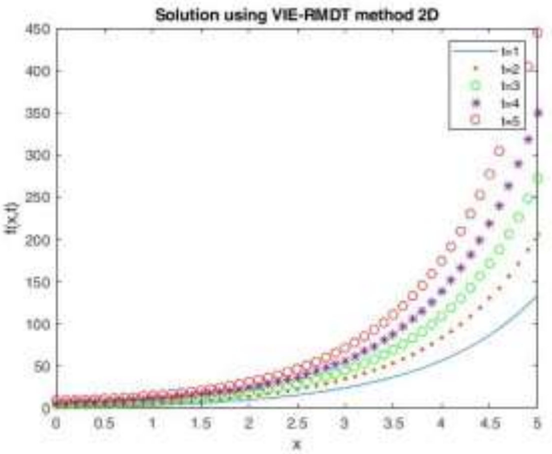

Figure 9: The fifth example in 2D

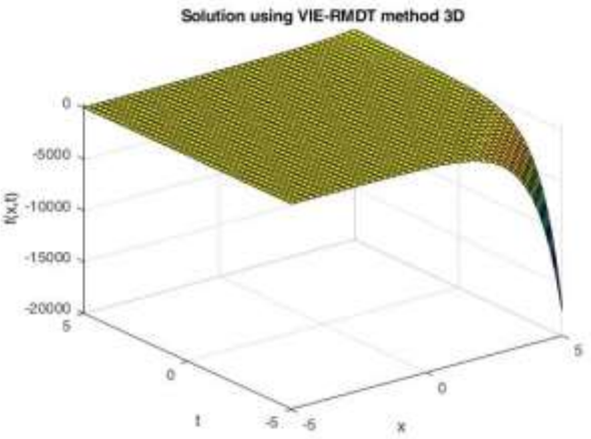

Figure 10: The fifth example in 3D

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Nguyen Minh Tuan carried out all the content of the paper consisting of conceptualization, data curation, investigation, formal analysis, methodology, software, visualization, writing-original draft, and writing review and editing.

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