A New Biased Two-Parameter Estimator in Linear Regression Model

[1]OLASUNKANMI JAMES OLADAPO, [1]JANET IYABO IDOWU, [1]*ABIOLA TIMOTHY OWOLABI, [2]KAYODE AYINDE
[1,2]Department of Statistics
[1]Ladoke Akintola University of Technology
[2]Federal University of Technology
[1]Ogbomoso, Oyo State
[2]Akure, Ondo State
[1,2]NIGERIA
***Corresponding Author:** E-mail: atowolabi@lautech.edu.ng

Abstract: The most frequently used estimation technique in the linear regression model is the ordinary least squares (OLS) estimator. The presence of multicollinearity makes the technique inefficient and gives misleading results. This study proposed a new biased two-parameter estimator to deal with the multicollinearity problem. Theory and simulation results show that this estimator outperforms existing estimators considered under some conditions, according to the mean squares error (MSE) criterion. Finally, the real-life dataset illustrates the paper's findings, which agree with the theoretical and simulation results.

Keywords: Multicollinearity, Mean Square Error, Monte Carlo Simulation, Ordinary Least Squares, Biased Estimator.

Received: December 15, 2022. Revised: August 11, 2023. Accepted: September 7, 2023. Published: October 3, 2023.

1. Introduction

In the multiple linear regression model, it is assumed that there is no perfect correlation among the explanatory variables. Violation of this assumption is the term multicollinearity. OLS estimator in the multicollinearity is the best linear unbiased estimator (BLUE) [1], but it is inefficient and cannot give the desired results when explanatory variables are correlated. It provides a misleading conclusion.

Several researchers have proposed alternative estimators to the OLS to overcome these effects of multicollinearity. The authors include Hoerl and Kennard [2], Liu [3], Stein [4], Swindel [5], Ahmad and Aslam [6], Lukman *et al*. [7], Kibria and Lukman [8] , Efron *et al.* [9], Draper and Smith [10], Mansson *et al*., [11], Dempster *et* al. [12], Akdeniz and Roozbeh [13], Muniz *et al*., [14], Arashi and Valizadeh [15], Ayinde *et al*., [16], Yang and Chang [17], Owolabi *et al*., [18, 19], Oladapo *et al*., [20], and Idowu *et al.* [21]

This paper aims to introduce a new class of two-parameter estimator to estimate regression parameters when there is a problem of multicollinearity and compare the performance of the new estimator with the existing estimators.

The article is organized as follows. In section 2, the existing and new proposed estimators are introduced. Section 3 provides theoretical comparisons among the estimators and the choice of biasing parameters. A simulation study is conducted in Section 4 to evaluate the performance of the proposed estimator. A numerical example is given in Section 5 to illustrate the findings in the paper, and Section 6 is some concluding remarks.

2. Some Existing Biased Estimators and the Proposed Estimator

2.1 Existing estimators

Consider a multiple linear regression model of the form

$$y = X\beta + u \tag{1}$$

where y is an n × 1 vector of observations, β is a p × 1 vector of unknown regression parameters, X is an $n \times p$ observed matrix of the regression, and u is an $n \times 1$ vector of random errors, which is distributed as multivariate normal with mean 0 and covariance matrix $\sigma^2 I_n$, I_n being an identity matrix of order n. The OLS estimator of β in (1) is obtained as:

$$\hat{\beta} = (X'X)^{-1} X'y \tag{2}$$

The canonical form of (1) can be written as:

$$y = Z\alpha + u \quad (3)$$

Where Z=XQ, $\alpha = Q'\beta$ and Q is the orthogonal matrix such that $Z'Z = Q'X'XQ = \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)$. The OLS estimator of α is given as follows:

$$\hat{\alpha} = \Lambda^{-1}Z'y \quad (4)$$

And the mean squared error matrix (MSEM) of $\hat{\alpha}$ is given by

$$MSEM(\hat{\alpha}) = \sigma^2\Lambda^{-1} \quad (5)$$

The ordinary ridge regression (ORR) estimator of α by Hoerl and Kennard [2] is defined as:

$$\hat{\alpha}_k = LX'y \quad (6)$$

where $L = (\Lambda + kI)^{-1}$ and k is a biasing parameter such that k > 0.

$$MSEM(\hat{\alpha}(k)) = \sigma^2 L\Lambda L' + (L\Lambda - I)\alpha\alpha'(L\Lambda - I)' \quad (7)$$

The Liu estimator by Liu [3] is defined as:

$$\hat{\alpha}_d = F\hat{\alpha} \quad (8)$$

where $F = (\Lambda + I)^{-1}(\Lambda + dI)$ and d is a biasing parameter of the Liu Estimator.

$$MSEM(\hat{\alpha}_d) = \sigma^2 F\Lambda^{-1}F' + (F - I)\alpha\alpha'(F - I)' \quad (9)$$

The Modified One-Parameter Liu (ML) estimator by Lukman *et* al. [22] is defined as:

$$\hat{\alpha}_{ML} = T\hat{\alpha} \quad (10)$$

where $T = (\Lambda + I)^{-1}(\Lambda - dI)$ and d is a biasing parameter of the Liu Estimator.

$$MSEM(\hat{\alpha}_{ML}) = \sigma^2 T\Lambda^{-1}T' + (T - I)\alpha\alpha'(T - I)' \quad (11)$$

The Kibria-Lukman [8] (KL) estimator is defined as:

$$\hat{\alpha}_{KL} = G\hat{\alpha} \quad (12)$$

where $G = (\Lambda + kI)^{-1}(\Lambda - kI)$ and

$$MSEM(\hat{\alpha}_{KL}) = \sigma^2 G\Lambda^{-1}G' + (G - I)\alpha\alpha'(G - I)' \quad (13)$$

2.2 The proposed estimator

Following the same concept proposed by Liu [23] and Lukman and Kibria [22], the new proposed biased two-parameter estimator is named Modified Liu Kibria-Lukman estimator (MLKL) and is denoted as $\hat{\alpha}_{MLKL}$ is defined as follows:

$$\hat{\alpha}_{MLKL} = TG\hat{\alpha} \quad (14)$$

where k > 0 and 0 < d < 1.

Properties of the new estimator

The bias, covariance, and mean squared error matrix (MSEM) of the proposed estimator are given as follows:

$$B(\hat{\alpha}_{MLKL}) = (TG - I)\alpha \quad (15)$$

$$D(\hat{\alpha}_{MLKL}) = \sigma^2 TG\Lambda^{-1}T'G' \quad (16)$$

$$MSEM(\hat{\alpha}_{MLKL}) = \sigma^2 TG\Lambda^{-1}T'G' + (TG - I)\alpha\alpha'(TG - I)' \quad (17)$$

The following lemmas are useful to prove the statistical property of $\hat{\alpha}_{MLKL}$.

Lemma 1: Let S be an n x n positive definite matrix, that is S > 0, and α be some vector, then $S - \alpha\alpha' \geq 0$ if and only if $\alpha' S^{-1}\alpha \leq 1$. Farebrother [24]

Lemma 2: Let $\hat{\alpha}_i = B_i y$ i = 1, 2 be two linear estimators of α. Suppose that $D = \text{cov}(\hat{\alpha}_1) - \text{cov}(\hat{\alpha}_2) > 0$, where $\text{cov}(\hat{\alpha}_i)$ i = 1, 2 denotes the covariance matrix of $\hat{\alpha}_i$ and $b_i = Bias(\hat{\alpha}_i) = (B_i X - I)\alpha$, i =1, 2. Consequently,

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MSEM(\hat{\alpha}_1) - MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1 b_1' - b_2 b_2' > 0 \quad (18)$$

If and only if $b_2'\left[\sigma^2 D + b_1 b_1'\right]^{-1} b_2 < 1$, where $\text{MSEM}(\hat{\alpha}_i) = \text{cov}(\hat{\alpha}_i) + b_i b_i'$ Trenkler and Toutenburg [25]

3. Comparison and Choice of Biasing Parameter

3.1 Comparison among the estimators

In this section, the MSEM of the proposed estimator, $\hat{\alpha}_{MLKL}$, is compared with others theoretically.

Comparison between $\hat{\alpha}$ and $\hat{\alpha}_{MLKL}$.

The difference between $MSEM(\hat{\alpha}) - MSEM(\hat{\alpha}_{MLKL})$ is given as follows:

$$MESM(\hat{\alpha}) - MSEM(\hat{\alpha}_{MLKL}) = \sigma^2\Lambda^{-1} - \left(\sigma^2 TG\Lambda^{-1}T'G' + (TG - I)\alpha\alpha'(TG - I)'\right) \quad (19)$$

Let $k > 0$ and 0 < d < 1. Thus, the following theorem holds.

Theorem 3.1: The estimator $\hat{\alpha}_{MLKL}$ is superior to the estimator $\hat{\alpha}$ using the MSEM criterion, that is, MSEM ($\hat{\alpha}$) − MSEM ($\hat{\alpha}_{MLKL}$) > 0 if and only if

$$\alpha'(TG - I)'\left\lfloor \sigma^2\left(\Lambda^{-1} - TG\Lambda^{-1}T'G'\right)\right\rfloor \alpha(TG - I) < 1 \quad (20)$$

Proof

$$D[\hat{\alpha}]-D(\hat{\alpha}_{MLKL})=\sigma^2\left(\Lambda^{-1}-TG\Lambda^{-1}T'G'\right)$$

$$=\sigma^2 diag\left\{\frac{1}{\lambda_i}-\frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \quad (21)$$

$\Lambda^{-1}-TG\Lambda^{-1}T'G'$ will be a pdf if and only if $(\lambda_i+1)^2(\lambda_i+k)^2-(\lambda_i-d)^2(\lambda_i-k)^2>0.$

Comparison between $\hat{\alpha}_k$ and $\hat{\alpha}_{MLKL}$.

The difference between $MSEM(\hat{\alpha}_k)-MSEM(\hat{\alpha}_{MLKL})$ is given as follows:

$$MESM(\hat{\alpha}_k)-MSEM(\hat{\alpha}_{MLKL}) =\left(\sigma^2 L\Lambda L'+(L\Lambda-I)\alpha\alpha'(L\Lambda-I)'\right) -\left(\sigma^2 TG\Lambda^{-1}T'G'+(TG-I)\alpha\alpha'(TG-I)'\right) \quad (22)$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.2: The estimator $\hat{\alpha}_{MLKL}$ is superior to the estimator $\hat{\alpha}_k$ using the MSEM criterion, that is, MSEM ($\hat{\alpha}_k$) − MSEM ($\hat{\alpha}_{MLKL}$) > 0 if and only if

$$\alpha'(TG-I)'\left[\sigma^2\left(L\Lambda L'-TG\Lambda^{-1}T'G'\right)+(L\Lambda-I)\alpha\alpha'(L\Lambda-I)'\right]\alpha(TG-I)<1 \quad (23)$$

Proof

$$D(\hat{\alpha}_k)-D(\hat{\alpha}_{MLKL})=\sigma^2\left(L\Lambda L'-TG\Lambda^{-1}T'G'\right)$$

$$=\sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \quad (24)$$

$L\Lambda L-TG\Lambda^{-1}T'G'$ will be a pdf if and only if $\lambda_i^2(\lambda_i+1)^2-(\lambda_i-d)^2(\lambda_i-k)^2>0.$

Comparison between $\hat{\alpha}_d$ and $\hat{\alpha}_{MLKL}$.

The difference between $MSEM(\hat{\alpha}_d)-MSEM(\hat{\alpha}_{MLKL})$ is given as follows:

$$MESM(\hat{\alpha}_d)-MSEM(\hat{\alpha}_{MLKL}) =\left(\sigma^2 F\Lambda^{-1}F'+(F-I)\alpha\alpha'(F-I)'\right) -\left(\sigma^2 TG\Lambda^{-1}T'G'+(TG-I)\alpha\alpha'(TG-I)'\right) \quad (25)$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.3: The estimator $\hat{\alpha}_{MLKL}$ is superior to the estimator $\hat{\alpha}_d$ using the MSEM criterion, that is, MSEM ($\hat{\alpha}_d$) − MSEM ($\hat{\alpha}_{MLKL}$) > 0 if and only if

$$\alpha'(TG-I)'\left[\sigma^2\left(F\Lambda^{-1}F'-TG\Lambda^{-1}T'G'\right)+(F-I)\alpha\alpha'(F-I)'\right]\alpha(TG-I)<1 \quad (26)$$

Proof

$$D[\hat{\alpha}_d]-D(\hat{\alpha}_{MLKL})=\sigma^2\left(F\Lambda^{-1}F'-TG\Lambda^{-1}T'G'\right)$$

$$=\sigma^2 diag\left\{\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \quad (27)$$

$F\Lambda^{-1}F'-TG\Lambda^{-1}T'G'$ will be a pdf if and only if $(\lambda_i+d)^2(\lambda_i+k)^2-(\lambda_i-d)^2(\lambda_i-k)^2>0.$

Comparison between $\hat{\alpha}_{ML}$ and $\hat{\alpha}_{MLKL}$.

The difference between $MSEM(\hat{\alpha}_{ML})-MSEM(\hat{\alpha}_{MLKL})$ is given as follows:

$$MESM(\hat{\alpha}_{ML})-MSEM(\hat{\alpha}_{MLKL}) =\left(\sigma^2 T\Lambda^{-1}T'+(T-I)\alpha\alpha'(T-I)'\right) -\left(\sigma^2 TG\Lambda^{-1}T'G'+(TG-I)\alpha\alpha'(TG-I)'\right) \quad (28)$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.4: The estimator $\hat{\alpha}_{MLKL}$ is superior to the estimator $\hat{\alpha}_{ML}$ using the MSEM criterion, that is, MSEM ($\hat{\alpha}_{ML}$) − MSEM ($\hat{\alpha}_{MLKL}$) > 0 if and only if

$$\alpha'(TG-I)'\left[\sigma^2\left(T\Lambda^{-1}T'-TG\Lambda^{-1}T'G'\right)+(T-I)\alpha\alpha'(T-I)'\right]\alpha(TG-I)<1 \quad (29)$$

Proof

$$D[\hat{\alpha}_{ML}]-D(\hat{\alpha}_{MLKL})=\sigma^2\left(T\Lambda^{-1}T'-TG\Lambda^{-1}T'G'\right)$$

$$=\sigma^2 diag\left\{\frac{(\lambda_i-d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \quad (30)$$

$T\Lambda^{-1}T'-TG\Lambda^{-1}T'G'$ will be a pdf if and only if $(\lambda_i-d)^2(\lambda_i+k)^2-(\lambda_i-d)^2(\lambda_i-k)^2>0.$

Comparison between $\hat{\alpha}_{KL}$ and $\hat{\alpha}_{MLKL}$.

The difference between $MSEM(\hat{\alpha}_{KL}) - MSEM(\hat{\alpha}_{MLKL})$ is given as follows:

$$MESM(\hat{\alpha}_{KL}) - MSEM(\hat{\alpha}_{MLKL}) = \left(\sigma^2 G\Lambda^{-1}G' + (G-I)\alpha\alpha'(G-I)'\right) - \left(\sigma^2 TG\Lambda^{-1}T'G' + (TG-I)\alpha\alpha'(TG-I)'\right) \quad (31)$$

Let *k* > *0* and 0 < *d* < 1. Thus, the following theorem holds.

Theorem 3.5: The estimator $\hat{\alpha}_{MLKL}$ is superior to the estimator $\hat{\alpha}_{KL}$ using the MSEM criterion, that is, MSEM ($\hat{\alpha}_{KL}$) − MSEM ($\hat{\alpha}_{MLKL}$) > 0 if and only if

$$\alpha'(TG-I)' \left[\sigma^2\left(G\Lambda^{-1}G' - TG\Lambda^{-1}T'G'\right) + (G-I)\alpha\alpha'(G-I)'\right]\alpha(TG-I) < 1 \quad (32)$$

Proof

$$D[\hat{\alpha}_{KL}] - D(\hat{\alpha}_{MLKL}) = \sigma^2\left(G\Lambda^{-1}G' - TG\Lambda^{-1}T'G'\right)$$

$$= \sigma^2 diag\left\{\frac{(\lambda_i-k)^2}{\lambda_i(\lambda_i+k)^2} - \frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \quad (33)$$

$G\Lambda^{-1}G' - TG\Lambda^{-1}T'G'$ will be a pdf if and only if $(\lambda_i-k)^2(\lambda_i+1)^2 - (\lambda_i-d)^2(\lambda_i-k)^2 > 0.$

3.2 Choice of Biasing Parameters

Different authors such as Hoerl *et al*. [26], Wencheko [27], Kibria and Banik [28], Khalaf and Shukur [29], and Owolabi *et al*. [30], among others, have proposed different estimators of *k* and *d*. An optimal value of k is the value that minimizes the MSE of the proposed estimator, that is $\hat{\alpha}_{MLKL}$, considering *d* to be fixed.

$$MSEM(\hat{\alpha}_{MLKL}) = \sigma^2 TG\Lambda^{-1}T'G' + (TG-I)\alpha\alpha'(TG-I)'$$

$$g(k,d) = MSEM(\hat{\alpha}_{MLKL}) = trace[MSEM(\hat{\alpha}_{MLKL})]$$

$$g(k,d) = \sigma^2\sum_{i=1}^{p}\frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2} + \sum_{i=1}^{p}\frac{(2\lambda_i k + \lambda_i d - kd + \lambda_i + k)^2}{(\lambda_i+1)^2(\lambda_i+k)^2}\alpha_i^2 \quad (34)$$

Let $\frac{\partial g(k,d)}{\partial k} = 0$;

$$-2\sigma^2\sum_i^p\frac{(\lambda_i-d)^2(\lambda_i-k)}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2} - 2\sigma^2\sum_i^p\frac{(\lambda_i-d)^2(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^3} + 2\sum_i^p\frac{[2\lambda_i k + \lambda_i d - kd + \lambda_i + k]\alpha_i^2(2\lambda_i - d + 1)}{(\lambda_i+1)^2(\lambda_i+k)^2} - 2\sum_i^p\frac{[2\lambda_i k + \lambda_i d - kd + \lambda_i + k]^2\alpha_i^2}{(\lambda_i+1)^2(\lambda_i+k)^3} \quad (35)$$

$$k = \frac{\sigma^2\lambda_i(\lambda_i-d) + \alpha_i^2\lambda_i^2(d+1)}{\sigma^2(\lambda_i-d) + \alpha_i^2\lambda_i(2\lambda_i-d+1)} \quad (36)$$

Also, differentiating $g(k,d)$ with respect to d gives

$$\frac{\partial g(k,d)}{\partial d} = -2\sigma^2\sum_i^p\frac{(\lambda_i-d)(\lambda_i-k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2} + 2\sum_i^p\frac{[2\lambda_i k + \lambda_i d - kd + \lambda_i + k]\alpha_i^2(\lambda_i-k)}{(\lambda_i+1)^2(\lambda_i+k)^2} \quad (37)$$

Let $\frac{\partial g(k,d)}{\partial d} = 0$;

$$d = \frac{\sigma^2\lambda_i(\lambda_i-k) + \alpha_i^2\lambda_i(2\lambda_i k + \lambda_i + k)}{\sigma^2(\lambda_i-k) + \alpha_i^2\lambda_i(\lambda_i-k)} \quad (38)$$

For practical purposes, σ^2 and α_i^2 are replaced with $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, in equations (36) and (38), respectively. Consequently, (36) becomes

$$\hat{k} = \frac{\hat{\sigma}^2\lambda_i(\lambda_i-d) + \hat{\alpha}_i^2\lambda_i^2(d+1)}{\hat{\sigma}^2(\lambda_i-d) + \hat{\alpha}_i^2\lambda_i(2\lambda_i-d+1)} \quad (39)$$

And (38) becomes

$$\hat{d} = \frac{\hat{\sigma}^2\lambda_i(\lambda_i-k) + \hat{\alpha}_i^2\lambda_i(2\lambda_i k + \lambda_i + k)}{\hat{\sigma}^2(\lambda_i-k) + \hat{\alpha}_i^2\lambda_i(\lambda_i-k)} \quad (40)$$

4. Simulation Study

In this section, the Monte Carlo simulation scheme was conducted to study the performance of the proposed estimator to validate the theoretical comparison. We followed the similar simulation scheme used by many authors in their studies, such as Wichern and Churchill [31], Clark and Troskie [32], Dempster *et* al. [12], McDonald and

Galarneau [33], Gibbons [34], Ahmad and Aslam [6] and Newhouse and Oman [35].
The following expression generates the regressors

$$x_{ij} = \left(1-\rho^2\right)^{1/2} z_{ij} + \rho z_{i,p+1}, \qquad i = 1,2,...,n, \quad j = 1,2,...,p. \tag{41}$$

where z_{ij} is independent standard normal pseudorandom numbers, ρ is the correlation between two explanatory variables, and p is the number of explanatory variables. This study considers the values of ρ to be 0.8, 0.9, 0.95, and 0.99. Also, explanatory variables (p) will be taken to be three (3) and seven (7), and sample sizes 50 and 100 for the simulation study. The response variable is defined as:

$$Y_i = \beta_1 X_1 + \beta_2 X_2 + ... + \beta_p X_p + e_i \tag{42}$$

where e_i is normally distributed with mean 0 and variance σ^2. We choose β such that $\beta'\beta$ =1. The chosen values of σ are 3, 5, and 10, and the replication for the study is 1000 times. The mean square error is then obtained as:

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\alpha}_{ij} - \alpha_i\right)'\left(\hat{\alpha}_{ij} - \alpha_i\right) \tag{43}$$

4.1 Simulation Result

From Table 1-4, the simulation results presented show that the proposed estimator outperforms all other estimators used in this study. The proposed estimator, $\hat{\alpha}_{MLKL}$, in this study performs best at the two different sample sizes considered (n = 50 and 100), four sigma levels (σ = 1, 3, 5, and 10), four different levels of multicollinearity levels (ρ = 0.8. 0.9, 0.95 and 0.99) and the two different number of parameters (p=3 and 7). It provides a smaller MSE compared with other estimators in the study when the number of parameters is three and seven. The OLS estimator is the least performed estimator, which is expected due to the facts established in the literature. The following observations were also deduced from the result:

i. An increase in the level of correlation results in an increase in the MSE for all the estimators.
ii. The MSE increases for each estimator as the level of error variances increases.
iii. An increase in the sample size, *n*, decreases the MSE for all the estimators.

5. Numerical Example

In this section, Portland cement data was used to demonstrate the performance of the proposed estimator. The Portland cement data was originally adopted by Woods *et al*. [36] and was later adopted by Li and Yang [37] and then by Ayinde *et al*. [38]. The data set is widely known as the Portland cement dataset. The regression model for these data is defined as:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i \tag{44}$$

where y_i = heat evolved after 180 days of curing measured in calories per gram of cement, X_1 = tricalcium aluminate, X_2 = tricalcium silicate, X_3 = tetra calcium aluminoferrite, and X_4 = β-dicalcium silicate. The variance inflation factors (VIF) are VIF1 = 38.50, VIF2 = 254.42, VIF3 = 46.87, and VIF4 = 282.51. Eigenvalues of $X'X$ matrix are λ1 = 44676.206, λ2 = 5965.422, λ3 = 809.952, and λ4 = 105.419, and the condition number of $X'X$ is approximately 424. The VIFs, eigenvalues, and condition numbers indicate that severe multicollinearity exists. The estimated parameters and the MSE values of the estimators are presented in Table 5.

From Table 5, the proposed estimator (MLKL) performs the best among all other estimators because it gives the smallest MSE value. The OLS estimator did not perform well in the presence of multicollinearity, as it has the highest MSE.

6. Conclusions

This paper proposed a new two-parameter estimator to handle the multicollinearity problem for the linear regression models. The proposed estimator was theoretically compared with five other existing estimators. A simulation study was then conducted to compare the performance of the proposed estimator and the five existing estimators: the OLS, Liu estimator, Ridge estimator, KL estimator, and Modified One-Parameter Liu estimator. It is evident from the theoretical comparison that the proposed estimator performs best among the existing estimators considered in this research work.
A simulation study also supports the theoretical analysis as the proposed estimator performs best among all the existing estimators used in the study. A real-life dataset was also analyzed to bolster the theoretical comparison and simulation study results. The proposed estimator gives the best

result using the Mean Square Error criterion among the existing estimators.
We recommend this new estimator for practitioners and researchers to use whenever there is a multicollinearity problem among their explanatory variables when using linear regression models. This estimator will surely give a suitable and reliable result or output.

LIST OF ABBREVIATIONS

OLS: Ordinary Least Square

MSEM: Mean Square Error Matrix

ORR: Ordinary Ridge Regression

ML: Modified Liu

KL: Kibria-Lukman

MLKL: Modified Liu Kibria-Lukman

VIF: Variance Inflation Factors

DECLARATIONS

Availability of data and materials

The datasets used in the study are available upon reasonable request.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

OJ, JI, and K conceived and designed the idea. OJ, JI, and AT analyzed and interpreted the results. OJ, JI, and AT wrote the first draft of the work. K supervised the work. All the authors reviewed the results and approved the final version of the manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This research received no external funding.

Conflict of Interest

The authors declare that there is no competing interest.

Acknowledgments

Not applicable

Table 1: Estimated MSE when n=50 and p=3.

K	d	Sigma	Rho	OLS	ORR	LIU	K-L	ML	MLKL
0.3	0.2	3	0.8	1.227	1.1802	1.1098	1.1344	1.0535	**0.9748**

			0.9	2.2158	2.0587	1.8412	1.9076	1.6674	**1.4383**
			0.95	4.2136	3.6579	2.9999	3.1428	2.4725	**1.8549**
			0.99	20.175	10.956	6.5009	4.5913	2.5317	**0.6497**
		5	0.8	3.4082	3.2782	3.0825	3.1509	2.9261	**2.7072**
			0.9	6.1551	5.7185	5.1142	5.2987	4.6314	**3.9947**
			0.95	11.704	10.161	8.333	8.73	6.868	**5.1526**
			0.99	56.042	30.434	18.058	12.754	7.0326	**1.8048**
		10	0.8	13.633	13.113	12.329	12.603	11.704	**10.827**
			0.9	24.62	22.874	20.457	21.194	18.525	**15.978**
			0.95	46.818	40.644	33.332	34.92	27.472	**20.61**
			0.99	224.17	121.73	72.233	51.015	28.131	**7.2194**
	0.5	3	0.8	1.227	1.1802	1.153	1.1344	1.0124	**0.9369**
			0.9	2.2158	2.0587	1.9774	1.9076	1.543	**1.3318**
			0.95	4.2136	3.6579	3.4302	3.1428	2.1117	**1.5878**
			0.99	20.175	10.956	10.733	4.5913	0.8096	**0.2479**
		5	0.8	3.4082	3.2782	3.2026	3.1509	2.8117	**2.6019**
			0.9	6.1551	5.7185	5.4928	5.2987	4.2857	**3.6988**
			0.95	11.704	10.161	9.5284	8.73	5.8658	**4.4104**
			0.99	56.042	30.434	29.813	12.754	2.2488	**0.6885**
		10	0.8	13.633	13.113	12.81	12.603	11.246	**10.406**
			0.9	24.62	22.874	21.971	21.194	17.142	**14.794**
			0.95	46.818	40.644	38.114	34.92	23.464	**17.642**
			0.99	224.17	121.73	119.25	51.015	8.9955	**2.7542**
	0.8	3	0.8	1.227	1.1802	1.1971	1.1344	0.9721	**0.8999**
			0.9	2.2158	2.0587	2.1188	1.9076	1.4237	**1.2297**
			0.95	4.2136	3.6579	3.8903	3.1428	1.7807	**1.3426**
			0.99	20.175	10.956	16.04	4.5913	0.1629	**0.0795**
		5	0.8	3.4082	3.2782	3.3251	3.1509	2.6997	**2.4988**
			0.9	6.1551	5.7185	5.8855	5.2987	3.9542	**3.415**
			0.95	11.704	10.161	10.806	8.73	4.9464	**3.7292**
			0.99	56.042	30.434	44.554	12.754	0.4524	**0.2206**
		10	0.8	13.633	13.113	13.3	12.603	10.797	**9.993**
			0.9	24.62	22.874	23.542	21.194	15.816	**13.658**
			0.95	46.818	40.644	43.226	34.92	19.786	**14.917**
			0.99	224.17	121.73	178.22	51.015	1.8099	**0.8825**
0.6	0.2	3	0.8	1.227	1.1362	1.1098	1.0491	1.0535	**0.9023**
			0.9	2.2158	1.9183	1.8412	1.6431	1.6674	**1.2416**
			0.95	4.2136	3.208	2.9999	2.3436	2.4725	**1.3927**
			0.99	20.175	6.9114	6.5009	0.721	2.5317	**0.1499**
		5	0.8	3.4082	3.1559	3.0825	2.9139	2.9261	**2.5055**
			0.9	6.1551	5.3284	5.1142	4.5638	4.6314	**3.448**
			0.95	11.704	8.9111	8.333	6.5101	6.868	**3.8686**
			0.99	56.042	19.198	18.058	2.0029	7.0326	**0.4162**
		10	0.8	13.633	12.623	12.329	11.655	11.704	**10.02**
			0.9	24.62	21.313	20.457	18.255	18.525	**13.79**

Olasunkanmi James Oladapo, Janet Iyabo Idowu,
Abiola Timothy Owolabi, Kayode Ayinde

			0.95	46.818	35.645	33.332	26.041	27.472	**15.474**
			0.99	224.17	76.794	72.233	8.0117	28.131	**1.6647**
	0.5	3	0.8	1.227	1.1362	1.153	1.0491	1.0124	**0.8675**
			0.9	2.2158	1.9183	1.9774	1.6431	1.543	**1.1504**
			0.95	4.2136	3.208	3.4302	2.3436	2.1117	**1.1953**
			0.99	20.175	6.9114	10.733	0.721	0.8096	**0.086**
		5	0.8	3.4082	3.1559	3.2026	2.9139	2.8117	**2.4086**
			0.9	6.1551	5.3284	5.4928	4.5638	4.2857	**3.1948**
			0.95	11.704	8.9111	9.5284	6.5101	5.8658	**3.3199**
			0.99	56.042	19.198	29.813	2.0029	2.2488	**0.2387**
		10	0.8	13.633	12.623	12.81	11.655	11.246	**9.6317**
			0.9	24.62	21.313	21.971	18.255	17.142	**12.777**
			0.95	46.818	35.645	38.114	26.041	23.464	**13.28**
			0.99	224.17	76.794	119.25	8.0117	8.9955	**0.9548**
	0.8	3	0.8	3.3789	3.1035	3.2884	2.8414	2.6147	**2.2138**
			0.9	6.3296	5.406	6.0293	4.5647	3.9195	**2.8913**
			0.95	12.278	9.1533	11.277	6.5421	4.9438	**2.8608**
			0.99	59.959	20.078	47.489	3.3132	1.8153	**0.2175**
		5	0.8	3.4082	3.1559	3.3251	2.9139	2.6997	**2.3137**
			0.9	6.1551	5.3284	5.8855	4.5638	3.9542	**2.9518**
			0.95	11.704	8.9111	10.806	6.5101	4.9464	**2.816**
			0.99	56.042	19.198	44.554	2.0029	0.4524	**0.1489**
		10	0.8	13.633	12.623	13.3	11.655	10.797	**9.2518**
			0.9	24.62	21.313	23.542	18.255	15.816	**11.805**
			0.95	46.818	35.645	43.226	26.041	19.786	**11.264**
			0.99	224.17	76.794	178.22	8.0117	1.8099	**0.5956**

	0.2	3	0.8	1.227	1.0947	1.1098	0.9706	1.0535	**0.8355**
			0.9	2.2158	1.7923	1.8412	1.4156	1.6674	**1.0722**
			0.95	4.2136	2.8383	2.9999	1.7424	2.4725	**1.044**
			0.99	20.175	4.7705	6.5009	0.2033	2.5317	**0.064**
		5	0.8	3.4082	3.0407	3.0825	2.6954	2.9261	**2.3195**
			0.9	6.1551	4.9784	5.1142	3.9317	4.6314	**2.9773**
			0.95	11.704	7.8841	8.333	4.84	6.868	**2.8997**
			0.99	56.042	13.251	18.058	0.5647	7.0326	**0.1775**
0.9		10	0.8	13.633	12.162	12.329	10.78	11.704	**9.2753**
			0.9	24.62	19.913	20.457	15.726	18.525	**11.907**
			0.95	46.818	31.537	33.332	19.36	27.472	**11.599**
			0.99	224.17	53.006	72.233	2.2591	28.131	**0.7097**
	0.5	3	0.8	1.227	1.0947	1.153	0.9706	1.0124	**0.8035**
			0.9	2.2158	1.7923	1.9774	1.4156	1.543	**0.9943**
			0.95	4.2136	2.8383	3.4302	1.7424	2.1117	**0.8988**
			0.99	20.175	4.7705	10.733	0.2033	0.8096	**0.0512**

		5	0.8	3.4082	3.0407	3.2026	2.6954	2.8117	**2.2303**
			0.9	6.1551	4.9784	5.4928	3.9317	4.2857	**2.7607**
			0.95	11.704	7.8841	9.5284	4.84	5.8658	**2.4962**
			0.99	56.042	13.251	29.813	0.5647	2.2488	**0.1419**
		10	0.8	13.633	12.162	12.81	10.78	11.246	**8.9179**
			0.9	24.62	19.913	21.971	15.726	17.142	**11.04**
			0.95	46.818	31.537	38.114	19.36	23.464	**9.9845**
			0.99	224.17	53.006	119.25	2.2591	8.9955	**0.5672**
	0.8	3	0.8	1.227	1.0947	1.1971	0.9706	0.9721	**0.7721**
			0.9	2.2158	1.7923	2.1188	1.4156	1.4237	**0.9195**
			0.95	4.2136	2.8383	3.8903	1.7424	1.7807	**0.7653**
			0.99	20.175	4.7705	16.04	0.2033	0.1629	**0.0473**
		5	0.8	3.4082	3.0407	3.3251	2.6954	2.6997	**2.143**
			0.9	6.1551	4.9784	5.8855	3.9317	3.9542	**2.5528**
			0.95	11.704	7.8841	10.806	4.84	4.9464	**2.1252**
			0.99	56.042	13.251	44.554	0.5647	0.4524	**0.1308**
		10	0.8	13.633	12.162	13.3	10.78	10.797	**8.5681**
			0.9	24.62	19.913	23.542	15.726	15.816	**10.208**
			0.95	46.818	31.537	43.226	19.36	19.786	**8.5006**
			0.99	224.17	53.006	178.22	2.2591	1.8099	**0.5227**

NOTE: Minimum MSE value is bolded in each row.

Table 2: Estimated MSE when n=100 and p=3

K	d	sigma	rho	OLS	ORR	LIU	K-L	ML	MLKL
			0.8	0.5596	0.5499	0.5345	0.5403	0.5223	**0.5044**
		3	0.9	1.0178	0.9839	0.9322	0.9506	0.8909	**0.8326**
			0.95	1.9464	1.8211	1.6447	1.7002	1.5038	**1.3157**
			0.99	9.4002	6.9	4.8107	4.7956	3.0985	**1.6185**
			0.8	1.5543	1.5275	1.4848	1.5009	1.4507	**1.4012**
	0.2	5	0.9	2.8273	2.7332	2.5896	2.6407	2.4748	**2.3127**
			0.95	5.4066	5.0586	4.5687	4.7227	4.1773	**3.6546**
			0.99	26.112	19.167	13.363	13.321	8.607	**4.4957**
			0.8	6.2173	6.11	5.9393	6.0036	5.8028	**5.6045**
0.3		10	0.9	11.309	10.933	10.358	10.563	9.8992	**9.2507**
			0.95	21.626	20.235	18.275	18.891	16.709	**14.618**
			0.99	104.45	76.667	53.452	53.285	34.428	**17.983**
			0.8	0.5596	0.5499	0.5438	0.5403	0.5132	**0.4957**
		3	0.9	1.0178	0.9839	0.9639	0.9506	0.8606	**0.8044**
			0.95	1.9464	1.8211	1.7548	1.7002	1.4025	**1.2276**
	0.5		0.99	9.4002	6.9	6.3497	4.7956	2.0693	**1.0957**
			0.8	1.5543	1.5275	1.5107	1.5009	1.4254	**1.3768**
		5	0.9	2.8273	2.7332	2.6774	2.6407	2.3906	**2.2343**
			0.95	5.4066	5.0586	4.8743	4.7227	3.8957	**3.4099**

			0.99	26.112	19.167	17.638	13.321	5.7479	**3.0434**
			0.8	6.2173	6.11	6.0427	6.0036	5.7016	**5.5071**
		10	0.9	11.309	10.933	10.71	10.563	9.5622	**8.9371**
			0.95	21.626	20.235	19.497	18.891	15.583	**13.639**
			0.99	104.45	76.667	70.552	53.285	22.991	**12.173**
			0.8	0.5596	0.5499	0.5532	0.5403	0.5042	**0.487**
		3	0.9	1.0178	0.9839	0.9961	0.9506	0.8309	**0.7767**
			0.95	1.9464	1.8211	1.8685	1.7002	1.3049	**1.1428**
			0.99	9.4002	6.9	8.1072	4.7956	1.2584	**0.6814**
			0.8	1.5543	1.5275	1.5368	1.5009	1.4004	**1.3527**
	0.8	5	0.9	2.8273	2.7332	2.7668	2.6407	2.3079	**2.1574**
			0.95	5.4066	5.0586	5.1902	4.7227	3.6245	**3.1741**
			0.99	26.112	19.167	22.52	13.321	3.4955	**1.8927**
			0.8	6.2173	6.11	6.1472	6.0036	5.6014	**5.4106**
		10	0.9	11.309	10.933	11.067	10.563	9.2314	**8.6293**
			0.95	21.626	20.235	20.761	18.891	14.498	**12.696**
			0.99	104.45	76.667	90.08	53.285	13.982	**7.5703**

			0.8	0.5596	0.5405	0.5345	0.5218	0.5223	**0.4873**
		3	0.9	1.0178	0.9518	0.9322	0.8881	0.8909	**0.7783**
			0.95	1.9464	1.708	1.6447	1.4859	1.5038	**1.1518**
			0.99	9.4002	5.2929	4.8107	2.3853	3.0985	**0.8317**
			0.8	1.5543	1.5014	1.4848	1.4495	1.4507	**1.3534**
		5	0.9	2.8273	2.6439	2.5896	2.4669	2.4748	**2.1618**
	0.2		0.95	5.4066	4.7444	4.5687	4.1273	4.1773	**3.1991**
			0.99	26.112	14.702	13.363	6.6258	8.607	**2.3101**
			0.8	6.2173	6.0056	5.9393	5.7978	5.8028	**5.4135**
		10	0.9	11.309	10.575	10.358	9.8677	9.8992	**8.6468**
0.6			0.95	21.626	18.978	18.275	16.509	16.709	**12.796**
			0.99	104.45	58.81	53.452	26.503	34.428	**9.2401**
			0.8	0.5596	0.5405	0.5438	0.5218	0.5132	**0.4789**
		3	0.9	1.0178	0.9518	0.9639	0.8881	0.8606	**0.7521**
			0.95	1.9464	1.708	1.7548	1.4859	1.4025	**1.0752**
			0.99	9.4002	5.2929	6.3497	2.3853	2.0693	**0.5737**
	0.5		0.8	1.5543	1.5014	1.5107	1.4495	1.4254	**1.33**
		5	0.9	2.8273	2.6439	2.6774	2.4669	2.3906	**2.0889**
			0.95	5.4066	4.7444	4.8743	4.1273	3.8957	**2.9864**
			0.99	26.112	14.702	17.638	6.6258	5.7479	**1.5934**
		10	0.8	6.2173	6.0056	6.0427	5.7978	5.7016	**5.3197**

			0.9	11.309	10.575	10.71	9.8677	9.5622	**8.355**
			0.95	21.626	18.978	19.497	16.509	15.583	**11.945**
			0.99	104.45	58.81	70.552	26.503	22.991	**6.3732**
		3	0.8	0.5596	0.5405	0.5532	0.5218	0.5042	**0.4706**
			0.9	1.0178	0.9518	0.9961	0.8881	0.8309	**0.7263**
			0.95	1.9464	1.708	1.8685	1.4859	1.3049	**1.0014**
			0.99	9.4002	5.2929	8.1072	2.3853	1.2584	**0.3679**
	0.8	5	0.8	1.5543	1.5014	1.5368	1.4495	1.4004	**1.3068**
			0.9	2.8273	2.6439	2.7668	2.4669	2.3079	**2.0173**
			0.95	5.4066	4.7444	5.1902	4.1273	3.6245	**2.7815**
			0.99	26.112	14.702	22.52	6.6258	3.4955	**1.0216**
		10	0.8	6.2173	6.0056	6.1472	5.7978	5.6014	**5.2268**
			0.9	11.309	10.575	11.067	9.8677	9.2314	**8.0686**
			0.95	21.626	18.978	20.761	16.509	14.498	**11.125**
			0.99	104.45	58.81	90.08	26.503	13.982	**4.0859**

0.9	0.2	3	0.8	0.5596	0.5314	0.5345	0.504	0.5223	**0.4708**
			0.9	1.0178	0.9213	0.9322	0.8299	0.8909	**0.7277**
			0.95	1.9464	1.6055	1.6447	1.2988	1.5038	**1.0086**
			0.99	9.4002	4.1956	4.8107	1.1091	3.0985	**0.408**
		5	0.8	1.5543	1.476	1.4848	1.3999	1.4507	**1.3075**
			0.9	2.8273	2.5591	2.5896	2.3051	2.4748	**2.0212**
			0.95	5.4066	4.4597	4.5687	3.6077	4.1773	**2.8013**
			0.99	26.112	11.654	13.363	3.0808	8.607	**1.1331**
		10	0.8	6.2173	5.9041	5.9393	5.5996	5.8028	**5.2295**
			0.9	11.309	10.236	10.358	9.2202	9.8992	**8.0842**
			0.95	21.626	17.839	18.275	14.431	16.709	**11.205**
			0.99	104.45	46.617	53.452	12.323	34.428	**4.5317**
	0.5	3	0.8	0.5596	0.5314	0.5438	0.504	0.5132	**0.4627**
			0.9	1.0178	0.9213	0.9639	0.8299	0.8606	**0.7033**
			0.95	1.9464	1.6055	1.7548	1.2988	1.4025	**0.9421**
			0.99	9.4002	4.1956	6.3497	1.1091	2.0693	**0.2901**
		5	0.8	1.5543	1.476	1.5107	1.3999	1.4254	**1.2849**
			0.9	2.8273	2.5591	2.6774	2.3051	2.3906	**1.9533**
			0.95	5.4066	4.4597	4.8743	3.6077	3.8957	**2.6165**
			0.99	26.112	11.654	17.638	3.0808	5.7479	**0.8054**
		10	0.8	6.2173	5.9041	6.0427	5.5996	5.7016	**5.1392**
			0.9	11.309	10.236	10.71	9.2202	9.5622	**7.8126**
			0.95	21.626	17.839	19.497	14.431	15.583	**10.465**
			0.99	104.45	46.617	70.552	12.323	22.991	**3.2209**
	0.8	3	0.8	0.5596	0.5314	0.5532	0.504	0.5042	**0.4547**

			0.9	1.0178	0.9213	0.9961	0.8299	0.8309	**0.6794**
			0.95	1.9464	1.6055	1.8685	1.2988	1.3049	**0.8779**
			0.99	9.4002	4.1956	8.1072	1.1091	1.2584	**0.195**
		5	0.8	1.5543	1.476	1.5368	1.3999	1.4004	**1.2626**
			0.9	2.8273	2.5591	2.7668	2.3051	2.3079	**1.8867**
			0.95	5.4066	4.4597	5.1902	3.6077	3.6245	**2.4384**
			0.99	26.112	11.654	22.52	3.0808	3.4955	**0.5414**
		10	0.8	6.2173	5.9041	6.1472	5.5996	5.6014	**5.0497**
			0.9	11.309	10.236	11.067	9.2202	9.2314	**7.546**
			0.95	21.626	17.839	20.761	14.431	14.498	**9.7527**
			0.99	104.45	46.617	90.08	12.323	13.982	**2.1649**

NOTE: Minimum MSE value is bolded in each row.

Table 3: Estimated MSE when n=50 and p=7

K	d	sigma	rho	OLS	ORR	LIU	K-L	ML	MLKL
0.3	0.2	3	0.8	3.3789	3.2362	3.0254	3.097	2.8569	**2.6242**
			0.9	6.3296	5.8369	5.1772	5.3669	4.6497	**3.9699**
			0.95	12.278	10.524	8.5534	8.9216	6.9701	**5.1847**
			0.99	59.959	31.635	19.093	13.216	7.675	**2.4063**
		5	0.8	9.3859	8.9894	8.4038	8.6027	7.9359	**7.2892**
			0.9	17.582	16.214	14.381	14.908	12.916	**11.028**
			0.95	34.106	29.232	23.759	24.782	19.361	**14.402**
			0.99	166.55	87.876	53.037	36.712	21.319	**6.684**
		10	0.8	37.544	35.958	33.615	34.411	31.744	**29.157**
			0.9	70.329	64.855	57.524	59.633	51.664	**44.11**
			0.95	136.42	116.93	95.037	99.129	77.444	**57.607**
			0.99	666.21	351.5	212.15	146.85	85.278	**26.736**
	0.5	3	0.8	3.3789	3.2362	3.1553	3.097	2.7343	**2.513**
			0.9	6.3296	5.8369	5.5941	5.3669	4.2755	**3.658**
			0.95	12.278	10.524	9.8629	8.9216	5.9046	**4.4285**
			0.99	59.959	31.635	31.601	13.216	3.055	**1.1837**
		5	0.8	9.3859	8.9894	8.7648	8.6027	7.5951	**6.9804**
			0.9	17.582	16.214	15.539	14.908	11.876	**10.161**
			0.95	34.106	29.232	27.397	24.782	16.402	**12.301**
			0.99	166.55	87.876	87.781	36.712	8.486	**3.2881**
		10	0.8	37.544	35.958	35.059	34.411	30.38	**27.921**
			0.9	70.329	64.855	62.157	59.633	47.505	**40.645**
			0.95	136.42	116.93	109.59	99.129	65.606	**49.205**
			0.99	666.21	351.5	351.12	146.85	33.944	**13.152**
	0.8	3	0.8	3.3789	3.2362	3.2884	3.097	2.6147	**2.4047**
			0.9	6.3296	5.8369	6.0293	5.3669	3.9195	**3.3611**
			0.95	12.278	10.524	11.277	8.9216	4.9438	**3.7436**
			0.99	59.959	31.635	47.489	13.216	1.8153	**0.5532**

		5	0.8	9.3859	8.9894	9.1346	8.6027	7.263	**6.6794**
			0.9	17.582	16.214	16.748	14.908	10.887	**9.3363**
			0.95	34.106	29.232	31.325	24.782	13.733	**10.399**
			0.99	166.55	87.876	131.91	36.712	5.0426	**1.5366**
		10	0.8	37.544	35.958	36.538	34.411	29.052	**26.717**
			0.9	70.329	64.855	66.992	59.633	43.55	**37.345**
			0.95	136.42	116.93	125.3	99.129	54.93	**41.594**
			0.99	666.21	351.5	527.66	146.85	20.17	**6.1463**

0.6	0.2	3	0.8	3.3789	3.1035	3.0254	2.8414	2.8569	**2.4129**
			0.9	6.3296	5.406	5.1772	4.5647	4.6497	**3.4002**
			0.95	12.278	9.1533	8.5534	6.5421	6.9701	**3.8933**
			0.99	59.959	20.078	19.093	3.3132	7.675	**0.7876**
		5	0.8	9.3859	8.6207	8.4038	7.8928	7.9359	**6.7022**
			0.9	17.582	15.017	14.381	12.68	12.916	**9.4448**
			0.95	34.106	25.426	23.759	18.172	19.361	**10.815**
			0.99	166.55	55.772	53.037	9.2035	21.319	**2.1879**
		10	0.8	37.544	34.483	33.615	31.571	31.744	**26.808**
			0.9	70.329	60.067	57.524	50.719	51.664	**37.779**
			0.95	136.42	101.7	95.037	72.689	77.444	**43.257**
			0.99	666.21	223.09	212.15	36.814	85.278	**8.7515**
	0.5	3	0.8	3.3789	3.1035	3.1553	2.8414	2.7343	**2.3121**
			0.9	6.3296	5.406	5.5941	4.5647	4.2755	**3.1396**
			0.95	12.278	9.1533	9.8629	6.5421	5.9046	**3.3527**
			0.99	59.959	20.078	31.601	3.3132	3.055	**0.4449**
		5	0.8	9.3859	8.6207	8.7648	7.8928	7.5951	**6.422**
			0.9	17.582	15.017	15.539	12.68	11.876	**8.721**
			0.95	34.106	25.426	27.397	18.172	16.402	**9.3128**
			0.99	166.55	55.772	87.781	9.2035	8.486	**1.2356**
		10	0.8	37.544	34.483	35.059	31.571	30.38	**25.688**
			0.9	70.329	60.067	62.157	50.719	47.505	**34.884**
			0.95	136.42	101.7	109.59	72.689	65.606	**37.251**
			0.99	666.21	223.09	351.12	36.814	33.944	**4.9425**
	0.8	3	0.8	3.3789	3.1035	3.2884	2.8414	2.6147	**2.2138**
			0.9	6.3296	5.406	6.0293	4.5647	3.9195	**2.8913**
			0.95	12.278	9.1533	11.277	6.5421	4.9438	**2.8608**
			0.99	59.959	20.078	47.489	3.3132	1.8153	**0.2175**
		5	0.8	9.3859	8.6207	9.1346	7.8928	7.263	**6.1489**
			0.9	17.582	15.017	16.748	12.68	10.887	**8.0312**

			0.95	34.106	25.426	31.325	18.172	13.733	**7.9462**
			0.99	166.55	55.772	131.91	9.2035	5.0426	**0.6041**
		10	0.8	37.544	34.483	36.538	31.571	29.052	**24.595**
			0.9	70.329	60.067	66.992	50.719	43.55	**32.125**
			0.95	136.42	101.7	125.3	72.689	54.93	**31.784**
			0.99	666.21	223.09	527.66	36.814	20.17	**2.4163**

0.9	0.2	3	0.8	3.3789	2.9798	3.0254	2.6094	2.8569	**2.2207**
			0.9	6.3296	5.0261	5.1772	3.8921	4.6497	**2.9197**
			0.95	12.278	8.0565	8.5534	4.8238	6.9701	**2.942**
			0.99	59.959	14.03	19.093	2.4148	7.675	**0.3235**
		5	0.8	9.3859	8.2772	8.4038	7.2481	7.9359	**6.1681**
			0.9	17.582	13.962	14.381	10.811	12.916	**8.11**
			0.95	34.106	22.379	23.759	13.399	19.361	**8.1719**
			0.99	166.55	38.973	53.037	6.7076	21.319	**0.8984**
		10	0.8	37.544	33.109	33.615	28.992	31.744	**24.672**
			0.9	70.329	55.846	57.524	43.245	51.664	**32.44**
			0.95	136.42	89.516	95.037	53.597	77.444	**32.687**
			0.99	666.21	155.89	212.15	26.831	85.278	**3.5936**
	0.5	3	0.8	3.3789	2.9798	3.1553	2.6094	2.7343	**2.1292**
			0.9	6.3296	5.0261	5.5941	3.8921	4.2755	**2.7016**
			0.95	12.278	8.0565	9.8629	4.8238	5.9046	**2.5544**
			0.99	59.959	14.03	31.601	2.4148	3.055	**0.1571**
		5	0.8	9.3859	8.2772	8.7648	7.2481	7.5951	**5.9139**
			0.9	17.582	13.962	15.539	10.811	11.876	**7.5043**
			0.95	34.106	22.379	27.397	13.399	16.402	**7.095**
			0.99	166.55	38.973	87.781	6.7076	8.486	**0.4363**
		10	0.8	37.544	33.109	35.059	28.992	30.38	**23.655**
			0.9	70.329	55.846	62.157	43.245	47.505	**30.017**
			0.95	136.42	89.516	109.59	53.597	65.606	**28.379**
			0.99	666.21	155.89	351.12	26.831	33.944	**1.7452**
	0.8	3	0.8	3.3789	2.9798	3.2884	2.6094	2.6147	**2.0399**
			0.9	6.3296	5.0261	6.0293	3.8921	3.9195	**2.4936**
			0.95	12.278	8.0565	11.277	4.8238	4.9438	**2.1998**
			0.99	59.959	14.03	47.489	2.4148	1.8153	**0.1334**
		5	0.8	9.3859	8.2772	9.1346	7.2481	7.263	**5.6659**

			0.9	17.582	13.962	16.748	10.811	10.887	**6.9264**
			0.95	34.106	22.379	31.325	13.399	13.733	**6.1101**
			0.99	166.55	38.973	131.91	6.7076	5.0426	**0.3703**
			0.8	37.544	33.109	36.538	28.992	29.052	**22.663**
			0.9	70.329	55.846	66.992	43.245	43.55	**27.705**
		10	0.95	136.42	89.516	125.3	53.597	54.93	**24.439**
			0.99	666.21	155.89	527.66	26.831	20.17	**1.4812**

NOTE: Minimum MSE value is bolded in each row.

Table 4: Estimated MSE when n=100 and p=7

K	d	sigma	rho	OLS	ORR	LIU	K-L	ML	MLKL
			0.8	1.5582	1.5296	1.4842	1.5013	1.448	**1.3954**
			0.9	2.9335	2.8321	2.6781	2.7326	2.555	**2.3815**
			0.95	5.7032	5.3277	4.8022	4.9658	4.3823	**3.8239**
		3	0.99	27.908	20.44	14.275	14.191	9.2139	**4.8628**
			0.8	4.3282	4.2488	4.1228	4.1701	4.0221	**3.876**
	0.2		0.9	8.1487	7.867	7.4391	7.5906	7.0971	**6.6152**
			0.95	15.842	14.799	13.34	13.794	12.173	**10.622**
		5	0.99	77.524	56.777	39.653	39.418	25.594	**13.508**
			0.8	17.313	16.995	16.491	16.681	16.088	**15.504**
			0.9	32.595	31.468	29.756	30.363	28.389	**26.461**
			0.95	63.369	59.197	53.358	55.175	48.693	**42.488**
		10	0.99	310.09	227.11	158.61	157.67	102.38	**54.031**
0.3		3	0.8	1.5582	1.5296	1.5117	1.5013	1.4211	**1.3695**
			0.9	2.9335	2.8321	2.7724	2.7326	2.4647	**2.2977**
			0.95	5.7032	5.3277	5.1305	4.9658	4.0808	**3.5631**
			0.99	27.908	20.44	18.839	14.191	6.1861	**3.3254**
		5	0.8	4.3282	4.2488	4.1992	4.1701	3.9474	**3.8042**
	0.5		0.9	8.1487	7.867	7.7012	7.5906	6.8463	**6.3825**
			0.95	15.842	14.799	14.251	13.794	11.336	**9.8975**
			0.99	77.524	56.777	52.33	39.418	17.184	**9.2372**
		10	0.8	17.313	16.995	16.797	16.681	15.79	**15.217**
			0.9	32.595	31.468	30.805	30.363	27.385	**25.53**
			0.95	63.369	59.197	57.006	55.175	45.342	**39.59**
			0.99	310.09	227.11	209.32	157.67	68.735	**36.949**
	0.8	3	0.8	1.5582	1.5296	1.5395	1.5013	1.3945	**1.344**
			0.9	2.9335	2.8321	2.8685	2.7326	2.3761	**2.2156**

			0.95	5.7032	5.3277	5.4703	4.9658	3.7908	**3.3121**
			0.99	27.908	20.44	24.061	14.191	3.8168	**2.1058**
		5	0.8	4.3282	4.2488	4.2764	4.1701	3.8735	**3.7332**
			0.9	8.1487	7.867	7.9681	7.5906	6.6003	**6.1543**
			0.95	15.842	14.799	15.195	13.794	10.53	**9.2002**
			0.99	77.524	56.777	66.837	39.418	10.602	**5.8493**
		10	0.8	17.313	16.995	17.105	16.681	15.494	**14.933**
			0.9	32.595	31.468	31.872	30.363	26.401	**24.617**
			0.95	63.369	59.197	60.781	55.175	42.12	**36.801**
			0.99	310.09	227.11	267.35	157.67	42.409	**23.397**

0.6	0.2	3	0.8	1.5582	1.5018	1.4842	1.4466	1.448	**1.3448**
			0.9	2.9335	2.7362	2.6781	2.5462	2.555	**2.2204**
			0.95	5.7032	4.99	4.8022	4.3272	4.3823	**3.3394**
			0.99	27.908	15.687	14.275	7.1158	9.2139	**2.5384**
		5	0.8	4.3282	4.1717	4.1228	4.0182	4.0221	**3.7355**
			0.9	8.1487	7.6006	7.4391	7.0728	7.0971	**6.1678**
			0.95	15.842	13.861	13.34	12.02	12.173	**9.276**
			0.99	77.524	43.574	39.653	19.766	25.594	**7.0511**
		10	0.8	17.313	16.687	16.491	16.073	16.088	**14.942**
			0.9	32.595	30.402	29.756	28.291	28.389	**24.671**
			0.95	63.369	55.444	53.358	48.079	48.693	**37.104**
			0.99	310.09	174.3	158.61	79.065	102.38	**28.204**
	0.5	3	0.8	1.5582	1.5018	1.5117	1.4466	1.4211	**1.32**
			0.9	2.9335	2.7362	2.7724	2.5462	2.4647	**2.1427**
			0.95	5.7032	4.99	5.1305	4.3272	4.0808	**3.1135**
			0.99	27.908	15.687	18.839	7.1158	6.1861	**1.7693**
		5	0.8	4.3282	4.1717	4.1992	4.0182	3.9474	**3.6665**
			0.9	8.1487	7.6006	7.7012	7.0728	6.8463	**5.9518**
			0.95	15.842	13.861	14.251	12.02	11.336	**8.6487**
			0.99	77.524	43.574	52.33	19.766	17.184	**4.9146**
		10	0.8	17.313	16.687	16.797	16.073	15.79	**14.666**
			0.9	32.595	30.402	30.805	28.291	27.385	**23.807**
			0.95	63.369	55.444	57.006	48.079	45.342	**34.595**
			0.99	310.09	174.3	209.32	79.065	68.735	**19.658**
	0.8	3	0.8	1.5582	1.5018	1.5395	1.4466	1.3945	**1.2954**
			0.9	2.9335	2.7362	2.8685	2.5462	2.3761	**2.0664**
			0.95	5.7032	4.99	5.4703	4.3272	3.7908	**2.8962**
			0.99	27.908	15.687	24.061	7.1158	3.8168	**1.1502**

		5	0.8	4.3282	4.1717	4.2764	4.0182	3.8735	**3.5983**
			0.9	8.1487	7.6006	7.9681	7.0728	6.6003	**5.74**
			0.95	15.842	13.861	15.195	12.02	10.53	**8.0449**
			0.99	77.524	43.574	66.837	19.766	10.602	**3.195**
		10	0.8	17.313	16.687	17.105	16.073	15.494	**14.393**
			0.9	32.595	30.402	31.872	28.291	26.401	**22.96**
			0.95	63.369	55.444	60.781	48.079	42.12	**32.179**
			0.99	310.09	174.3	267.35	79.065	42.409	**12.78**

0.9	0.2	3	0.8	1.5582	1.4748	1.4842	1.394	1.448	**1.2962**
			0.9	2.9335	2.6454	2.6781	2.373	2.555	**2.0707**
			0.95	5.7032	4.6849	4.8022	3.7723	4.3823	**2.9175**
			0.99	27.908	12.451	14.275	3.3905	9.2139	**1.2694**
		5	0.8	4.3282	4.0968	4.1228	3.8721	4.0221	**3.6004**
			0.9	8.1487	7.3482	7.4391	6.5917	7.0971	**5.7518**
			0.95	15.842	13.014	13.34	10.479	12.173	**8.1041**
			0.99	77.524	34.587	39.653	9.4181	25.594	**3.5262**
		10	0.8	17.313	16.387	16.491	15.488	16.088	**14.401**
			0.9	32.595	29.393	29.756	26.367	28.389	**23.007**
			0.95	63.369	52.055	53.358	41.914	48.693	**32.416**
			0.99	310.09	138.35	158.61	37.672	102.38	**14.105**
	0.5	3	0.8	1.5582	1.4748	1.5117	1.394	1.4211	**1.2723**
			0.9	2.9335	2.6454	2.7724	2.373	2.4647	**1.9985**
			0.95	5.7032	4.6849	5.1305	3.7723	4.0808	**2.7219**
			0.99	27.908	12.451	18.839	3.3905	6.1861	**0.9046**
		5	0.8	4.3282	4.0968	4.1992	3.8721	3.9474	**3.5341**
			0.9	8.1487	7.3482	7.7012	6.5917	6.8463	**5.5514**
			0.95	15.842	13.014	14.251	10.479	11.336	**7.5609**
			0.99	77.524	34.587	52.33	9.4181	17.184	**2.5127**
		10	0.8	17.313	16.387	16.797	15.488	15.79	**14.136**
			0.9	32.595	29.393	30.805	26.367	27.385	**22.205**
			0.95	63.369	52.055	57.006	41.914	45.342	**30.243**
			0.99	310.09	138.35	209.32	37.672	68.735	**10.051**
	0.8	3	0.8	1.5582	1.4748	1.5395	1.394	1.3945	**1.2487**
			0.9	2.9335	2.6454	2.8685	2.373	2.3761	**1.9277**
			0.95	5.7032	4.6849	5.4703	3.7723	3.7908	**2.5336**

		0.99	27.908	12.451	24.061	3.3905	3.8168	**0.6059**
	5	0.8	4.3282	4.0968	4.2764	3.8721	3.8735	**3.4685**
		0.9	8.1487	7.3482	7.9681	6.5917	6.6003	**5.3548**
		0.95	15.842	13.014	15.195	10.479	10.53	**7.0378**
		0.99	77.524	34.587	66.837	9.4181	10.602	**1.6829**
	10	0.8	17.313	16.387	17.105	15.488	15.494	**13.874**
		0.9	32.595	29.393	31.872	26.367	26.401	**21.419**
		0.95	63.369	52.055	60.781	41.914	42.12	**28.151**
		0.99	310.09	138.35	267.35	37.672	42.409	**6.7316**

NOTE: The bolded MSE value is the minimum in each row.

Table 5: Results of regression coefficients and the corresponding MSE values

Coef	OLS	ORR	LIU	KL	ML	MLKL
$\hat{\beta}_1$	-52.9936	-51.7281	-49.7842	-50.4627	-48.1795	43.81268
$\hat{\beta}_2$	0.071073	0.070811	0.070406	0.070548	0.070072	48.46854
$\hat{\beta}_3$	-0.4142	-0.412	-0.4086	-0.4098	-0.40579	14.06693
$\hat{\beta}_4$	-0.42347	-0.42653	-0.43124	-0.42959	-0.43513	0.867867
$\hat{\beta}_5$	-0.57257	-0.57329	-0.57439	-0.57401	-0.5753	0.376839
$\hat{\beta}_6$	48.41787	48.27776	48.06159	48.13766	47.88346	-4.94995
MSE	17095.15	16356.98	15261.43	15638.54	14355.67	**13192.21**
k/d		0.3	0.2	0.3	0.2	0.3/0.2

References

[1] D. N. Gujarati, Basic Econometrics, New York: McGraw Hill, 2003.

[2] A. E. Hoerl and R. W. Kennard, " Ridge regression: Ridge regression: biased estimation for nonorthogonal problems.," *Technometrics,* vol. 12, pp. 55-67, 1970.

[3] K. Liu, "A new class of biased estimate in linear regression," *Communications in Statistics- Theory and Methods,* vol. 22, p. pp. 393–402, 1993.

[4] C. Stein, "Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability," *University California Press,* pp. 197-206, 1956.

[5] B. Swindel, "Good ridge estimators based on prior information," *Communications in Statistics - Theory and Methods,* vol. A5, pp. 1065 - 1075, 1976.

[6] S. Ahmad and M. Aslam, "Another proposal about the new two-parameter estimator for linear regression model with correlated regressors," *Communications in Statistics - Simulation and Computation,* 2020.

[7] A. F. Lukman, K. Ayinde, S. Binuomote and O. A. Clement, "Modified ridge-type estimator to combat multicollinearity: application to chemical data," *Journal of Chemometrics,* vol. 33(5), p. e3125, 2019b.

[8] Kibria, B.M.G. and Lukman, A.F., "A New Ridge-Type Estimator for the Linear Regression Model: Simulations and Applications," *Scientifica,* p. 1–16, 2020.

[9] Efron, B., Hastie, T., Johnstone, I. and Tibshirani, R., " Least Angle Regression," *The Annals of Statistics,* vol. 32, p. 407–499, 2004.

[10] Draper, N.R. and Smith, H., Applied Regression Analysis, John Willey & Sons, 1981.

[11] Mansson, K., Shukur, G. and Kibria, B. M. G., "On some ridge regression estimators: A Monte Carlo simulation study under different error variances," *Journal of Statistics,* vol. 17, no. 1, pp. 1-22, 2010.

[12] Dempster, A. P., Schatzoff, M. and Wermuth, N., "A simulation study of alternatives to ordinary least squares," *Journal of the American Statistical Association,* vol. 72, no. 357, pp. 77-91, 1977.

[13] Akdeniz, F. and Roozbeh, M., "Generalized difference-based weighted mixed almost unbiased ridge estimator in partially linear models," *Statistical Papers,* vol. 60, no. 5, p. 1717–1739, 2019.

[14] Muniz, G., Kibria, B. M. G., Mansson, K. and Shukur, G., "On developing ridge regression parameters: A graphical investigation," *Statistics and Operations Research Transactions,* vol. 36, no. 2, pp. 115-138, 2012.

[15] Arashi, M. and Valizadeh, T., "Performance of Kibria’s methods in partial linear ridge regression model," *Statistical Papers,* vol. 56, no. 1, pp. 231-246, 2015.

[16] Ayinde, K., Lukman, A. F., Alabi, O. O and Bello, H. A., "A New Approach of Principal Component Regression Estimator with Applications to Collinear Data," *International Journal of Engineering Research and Technology,* vol. 13, 2020.

[17] H. Yang and X. Chang, "A new two-parameter estimator in linear regression," *Communications in Statistics — Theory and Methods,* vol. 39(6), p. 923–934, 2010.

[18] Owolabi, A. T., Ayinde, K. and Alabi, O. O., "A new ridge-type estimator for the linear regression model with correlated regressors," *Concurrency Computat Pract Exper.,* vol. e6933, 2022.

[19] Owolabi, A. T., Ayinde, K., Idowu, J. I., Oladapo, O. J. and Lukman, A. F., "A New Two-Parameter Estimator in the Linear Regression Model with Correlated Regressors," *Journal of Statistics Applications & Probability,* vol. 11, no. 2, pp. 185-201, 2022.

[20] Oladapo, O.J., Owolabi, A.T., Idowu, J.I. and Ayinde, K. , "A New Modified Liu Ridge-Type Estimator for the Linear Regression Model: Simulation and Application," *Int J Clin Biostat Biom,* vol. 8, no. 048, 2022.

[21] Idowu, J. I., Oladapo, O. J., Owolabi, A. T. and Ayinde, K., "On the biased Two-Parameter Estimator to Combat Multicollinearity in Linear Regression Model," *African Scientific Reports,* vol. 1, p. 188–204, 2022.

[22] Lukman, A. F., Kibria, B. M. G., Ayinde, K. and Jegede, S. L., "Modified One-Parameter Liu Estimator for the Linear Regression Model," *Modelling and Simulation in Engineering,* vol. 2020, no. 9574304, 2020.

[23] K. Liu, "Using Liu Type Estimator to Combat Collinearity," *Communications in Statistics-Theory and Methods,* vol. 32, p. 1009–1020, 2003.

[24] R. Farebrother, "Further results on the mean square error of ridge regression," *J R Stat Soc.* , vol. B38, p. 248- 250, 1976.

[25] Trenkler, G. and Toutenburg, H., "Mean squared error matrix comparisons between biased estimators-an overview of recent

results," *Stat Pap.,* vol. 31, no. 1, p. 165-179, 1990.

[26] Hoerl, A. E., Kennard, R. W. and Baldwin, K. F., "Ridge regression: Some simulations," *Communications in Statistics,* vol. 4, no. 2, p. 105–23, 1975.

[27] E. Wencheko, "Estimation of the signal-to-noise in the linear regression model," *Statistical Papers,* vol. 41, no. 3, p. 327–43, 2000.

[28] B. M. G. Kibria and S. Banik "Some ridge regression estimators and their performances," *Journal of Modern Applied Statistical Methods,* vol. 15(1), pp. 206-238, 2016.

[29] Khalaf, G. and Shukur, G., "Choosing ridge parameter for regression problems," *Commun. Stat. Theory Methods,* vol. 21, p. 2227–2246, 2005.

[30] Owolabi, A. T., Ayinde, K. and Alabi, O. O., "A Modified Two Parameter Estimator with Different Forms of Biasing Parameters in the Linear Regression Model," *African Scientific Reports,* vol. 1, p. 212–228, 2022.

[31] Wichern, D. and Churchill, G., "A Comparison of Ridge Estimators," *Technometrics,* vol. 20, p. 301–311, 1978.

[32] Clark, A. E. and Troskie, C. G., "Ridge regression–a simulation study. Communications in Statistics," *Simulation and Computation,* vol. 35, no. 3, p. 605–19, 2006.

[33] McDonald, G. C. and Galarneau, D. I., "A Monte Carlo evaluation of some ridge-type estimators," *Journal of the American Statistical Association,* vol. 70, pp. 407-416, 1975.

[34] D. G. Gibbons, "A simulation study of some ridge estimators," *Journal of the American Statistical Association,* vol. 76, no. 373, p. 131–139, 1981.

[35] Newhouse, J. P., and Oman, S. D., "An evaluation of ridge estimators. " *A report prepared for the United States Air Force project RAND,* 1971.

[36] Woods, H., Steinour, H. H. and Starke, H. R., "Effect of composition of Portland cement on heat evolved during hardening," *Industrial & Engineering Chemistry,* vol. 24, p. 1207–1214, 1932.

[37] Li, Y. and Yang, H., "A new Liu-type estimator in linear regression model," *Stat. Pap.* , vol. 53, p. 427–437, 2012.

[38] K. Ayinde, A. F. Lukman, O. O. Samuel, and S. Ajiboye, "Some new adjusted ridge estimators of linear regression model," *Int J Civ Eng Technol,* vol. 9(11), p. 2838- 2852, 2018.