On the Correction Method of Test Solutions to the New Model Wave Equation with Two Variable Rates in First Quarter of the Plane

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Abstract: A correction method of test solutions into classical solutions to a new inhomogeneous model wave equation with variable two-rates in the first quarter of the plane was developed to derive the minimum smoothness requirement of its right-hand side. In the case of different rates, it is not possible to derive the minimum smoothness of right-hand side of the inhomogeneous model wave equation in the first quarter of the plane without correcting both test generalized and test classical solutions. In this paper, classical solutions to the inhomogeneous two-rate model wave equation and the smoothness criterion of their right-hand side are obtained. We need them to find explicit unique and stable classical solutions and Hadamard correctness criteria to mixed (initial-boundary) problems for this wave equation using developed by the author of the ”implicit characteristics method”.

Keywords: Two-rate model wave equation, correction method, test solution, classical solution, implicit characteristics method, smoothness criterion, correctness criterion

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1. Introduction

For the first time, a correction method of test solutions was proposed for a model wave equation with two constant coefficients-rates for rectilinear first quarter in [1] and for curvilinear first quarter in [2] of the plane. The developed correction method is used to obtain classical (twice continuously differentiable) solutions for which are twice continuously differentiable on the critical characteristics for the obtained classical solutions. A smoothness criterion of the right-hand side of this equation in the first quarter of the plane is derived. It is proved that if the right-hand side of the two-rate model wave equation depends only on one of two independent variables, then its continuity in one of these variables is necessary and sufficient for these solutions to be classical solutions of this equation in quarter of the plane.

2. Classical Solutions of a Inhomogeneous Model Wave Equation With Variable Two-rates

Using the generalization of the correcting Goursat problem from [1] to two variable coefficients-rates $a_1(x,t)$ and $a_2(x,t)$ by the new ”implicit characteristics method” it was calculated classical solutions of new model wave equation

$$u_{tt}(x,t)+(a_1-a_2)u_{tx}(x,t)-a_1a_2u_{xx}(x,t)-$$
$$-a_2^{-1}(a_2)_t u_t(x,t)-a_1(a_2)_x u_x(x,t)=f(x,t),$$
$$(x,t)\in \dot{G}_\infty =]0,+\infty[\times]0,+\infty[, \tag{1}$$

where $f(x,t),\ a_1(x,t),\ a_2(x,t)$ are given real functions of the variables x and t.

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The characteristic equations $dx=(-1)^i a_{3-i}(x,t)dt$, $i=1,2$, corresponds to equation (1). to the which have implicit general integrals $g_i(x,t)=C_i$, $C_i\in R$, $i=1,2$. If the coefficients a_{3-i} are strictly positive, i. e. $a_{3-i}(x,t)\geq a^{(0)}_{3-i}>0$, $(x,t)\in G_\infty$, then variable t on the characteristics $g_1(x,t)=C_1$, $C_1\in R$, strictly decreases on characteristics $g_2(x,t)=C_2$, $C_2\in R$, and strictly increases with the growth of the variable x. Therefore, the implicit functions $y_i=g_i(x,t)=C_i$, $x\in R$, $t\geq 0$, have strictly monotone implicit inverse functions $x=h_i\{y_i,t\}$, $t\geq 0$, and $t=h^{(i)}[x,y_i]$, $x\in R$, $i=1,2$. By the definition of inverse mappings on G_∞, the following inversion identities hold:

$$g_i(h_i\{y_i,t\},t)=y_i,\ t\geq 0;\ h_i\{g_i(x,t),t\}=x,\ x\in R,$$
$$i=1,2, \tag{2}$$

$$g_i(x,h^{(i)}[x,y_i])=y_i,\ x\in R;\ h^{(i)}[x,g_i(x,t)]=t,\ t\geq 0,$$
$$i=1,2, \tag{3}$$

$$h_i\{y_i,h^{(i)}[x,y_i]\}=x,\ x\in R;\ h^{(i)}[h_i\{y_i,t\},y_i]=t,\ t\geq 0,$$
$$i=1,2. \tag{4}$$

If $a_{3-i}(x,t)\geq a^{(0)}_{3-i}>0$, $(x,t)\in G_\infty$, $a_{3-i}\in C^2(G_\infty)$, then the implicit functions g_i, h_i, $h^{(i)}$ are twice continuously differentiable with respect to $x,\ t,\ y_i$, $i=1,2$, on G_∞.

Let $C^k(\Omega)$ be a set of k times continuous differentiable functions on the subset $\Omega\subset\mathbb{R}^2$ and $C^0(\Omega)=C(\Omega)$. The critical characteristic $g_2(x,t)=g_2(0,0)$ divides G_∞ into two sets $G_-=\{(x,t)\in G_\infty : g_2(x,t)>g_2(0,0)\}$ and $G_+=\{(x,t)\in G_\infty : g_2(x,t)\leq g_2(0,0)\}$.

Theorem 1. *Let be* $a_{3-i}(x,t) \geq a_{3-i}^{(0)} > 0$, $(x,t) \in G_\infty = [0,+\infty[\times[0,+\infty[$, $a_{3-i} \in C^2(G_\infty)$, $i=1,2$. *If the function* $f \in C(G_\infty)$, *then on* G_+ *equation (1) has classical solutions*

$$F^{(\epsilon)}(x,t) = \int\limits_0^{g_2(x,t)} \int\limits_{h_1\{g_1(-\varepsilon g_2(x,t),g_2(x,t)),\tau\}}^{h_1\{g_1(x,t),\tau\}} \frac{f(\delta,\tau)}{a_1(\delta,\tau)+a_2(\delta,\tau)} \times$$

$$\times \exp\left\{\int\limits_{g_1(\delta,\tau)}^{g_1(x,t)} E(\widetilde{\delta},\widetilde{\tau})ds\right\} d\delta d\tau +$$

$$+ \int\limits_{g_2(x,t)}^{t} \int\limits_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}} \frac{f(\delta,\tau)}{a_1(\delta,\tau)+a_2(\delta,\tau)} \times$$

$$\times \exp\left\{\int\limits_{g_1(\delta,\tau)}^{g_1(x,t)} E(\widetilde{\delta},\widetilde{\tau})ds\right\} d\delta d\tau,\ \varepsilon = \tilde{\varepsilon}+1 > 0, \quad (5)$$

where in the exponent is the integrand

$$E(\tilde{\delta},\tilde{\tau}) = \frac{a_2^2(\tilde{\delta},\tilde{\tau})\Big(\frac{a_1(\tilde{\delta},\tilde{\tau})}{a_2(\tilde{\delta},\tilde{\tau})}\Big)_{\tilde{\delta}} - a_2(\tilde{\delta},\tilde{\tau})\Big(\frac{a_1(\tilde{\delta},\tilde{\tau})}{a_2(\tilde{\delta},\tilde{\tau})}\Big)_{\tilde{\tau}}}{[a_1(\tilde{\delta},\tilde{\tau})+a_2(\tilde{\delta},\tilde{\tau})]^2\big(g_1(\tilde{\delta},\tilde{\tau})\big)_{\tilde{\delta}}}.$$

Theorem 2. *Let the assumptions of Theorem 1 be true. Then the functions (5) are classical solutions of equation (1) on* G_+ *for necessary smoothness*

$$f \in C(G_\infty),\ \int\limits_0^t f\big(h_1\{g_1(x,t),\tau\},\tau\big)d\tau \in C^1(G_+),$$

$$\int\limits_{-\varepsilon g_2(x,t)}^{h_2\{g_2(x,t),g_2(x,t)\}} f\big(\delta,\, g_2(x,t)\big)d\delta -$$

$$- \int\limits_0^{g_2(x,t)} f\big(h_1\{g_1\big(-\varepsilon g_2(x,t), g_2(x,t)\big),\tau\},\tau\big)d\tau -$$

$$- \int\limits_{g_2(x,t)}^{t} f\big(h_2\{g_2(x,t),\tau\},\tau\big)d\tau \in C^1(G_+). \quad (6)$$

Theorem 3. *In the requirements of Theorem 1, equation (1) has classical solutions*

$$F^{(\vartheta)}(x,t) = \int\limits_0^{g_2(x,t)} \int\limits_{h_1\{g_1(\vartheta g_2(x,t),g_2(x,t)),\tau\}}^{h_1\{g_1(x,t),\tau\}} \frac{f(\delta,\tau)}{a_1(\delta,\tau)+a_2(\delta,\tau)} \times$$

$$\times \exp\left\{\int\limits_{g_1(\delta,\tau)}^{g_1(x,t)} E(\tilde{\delta},\tilde{\tau})ds\right\} d\delta d\tau +$$

$$+ \int\limits_{g_2(x,t)}^{t} \int\limits_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}} \frac{f(\delta,\tau)}{a_1(\delta,\tau)+a_2(\delta,\tau)} \times$$

$$\times \exp\left\{\int\limits_{g_1(\delta,\tau)}^{g_1(x,t)} E(\tilde{\delta},\tilde{\tau})ds\right\} d\delta d\tau,\ \vartheta = \tilde{\vartheta} - 1 \geq 1, \quad (7)$$

on G_- *under the necessary smoothness conditions*

$$f \in C(G_-),\ \int\limits_0^t f\big(h_1\{g_1(x,t),\tau\},\tau\big)d\tau \in C^1(\tilde{G}_-),$$

$$\int\limits_{\vartheta g_2(x,t)}^{h_2\{g_2(x,t),g_2(x,t)\}} f\big(\delta,\, g_2(x,t)\big)d\delta -$$

$$- \int\limits_0^{g_2(x,t)} f\big(h_1\{g_1\big(\vartheta g_2(x,t), g_2(x,t)\big),\tau\},\tau\big)d\tau -$$

$$- \int\limits_{g_2(x,t)}^{t} f\big(h_2\{g_2(x,t),\tau\},\tau\big)d\tau \in C^1(G_-). \quad (8)$$

Corollary 1. *Under the assumptions of Theorem 1, the functions* $F^{(\epsilon)}$ *from (5) on* G_+ *and* $F^{(\vartheta)}$ *from (7) on* G_- *with* $\tilde{\vartheta} = \tilde{\varepsilon}+2$ *are classical solutions of Eq. (1) on the first quarter of the plane* G_∞ *with smoothness criterion (6) and (8) for* $\tilde{\vartheta} = \tilde{\varepsilon}+2$ *of the right-hand side* f *on* G_∞.

Corollary 2. *Let the assumptions of Theorem 1 hold and the right-hand side* f *of Eq. (1) does not depend on* x *or* t *in* G_∞. *Then the continuity of* $f \in [0,+\infty[$ *in* t *or* x, *respectively, is necessary and sufficient for the functions* $F^{(\epsilon)}$ *from (5) and* $F^{(\vartheta)}$ *from (7) with* $\tilde{\vartheta} = \tilde{\varepsilon}+2$ *are classical solutions of the inhomogeneous equation (1) in* G_∞.

Corollary 3. *Let the requirements of Theorem 1 be true and the function* f *depends on* x *and* t. *Then for* $f \in C(G_\infty)$ *the requirement that the integrals from (6) and (8) for* $\tilde{\vartheta} = \tilde{\varepsilon}+2$ *bilong to the space* $C^1(G_\infty)$ *it is equivalent that they belong to the spaces* $C^{(1,0)}(G_\infty)$ *or* $C^{(0,1)}(G_\infty)$. *Here* $C^{(1,0)}(G_\infty)$ *or* $C^{(0,1)}(G_\infty)$ *are respectively, the spaces of continuously differentiable with respect to* x *or* t *and continuous with respect to* t *or* x *functions on* G_∞.

Notes. In the rectilinear [1] and curvilinear [2] first quarter of the plane for the wave equation (1) with constant coefficients $a_{3-i}(x,t) = a_{3-i}^{(0)} > 0,\ i=1,2$, its particular classical solution is constructed on G_∞ for the parameters $\tilde{\varepsilon} = 0$ on G_+ and $\tilde{\vartheta} = 2$ on G_-, which satisfy the equality $\tilde{\vartheta} = \tilde{\varepsilon}+2$ from our Corollary 1. Calculation of classical solutions with minimal smoothness the right-hand side of the wave equations in a quarter plane by correction method in mathematics is a kind of analogue of the relativity theory in physics.

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