Approximate Series Solution For Two-Point Fuzzy Boundary Value Problems

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Abstract: - In this work, we have used double decomposition method to find approximate analytical solutions for the two-point fuzzy boundary value problems. This method is based on the standard Adomian decomposition method, which is a highly efficient method for solving fuzzy and non-fuzzy differential equations. This method allows for the solution to be calculated in the form of convergent series, in which the obtained are accurate solutions and very close to the exact analytical solutions. Some numerical results have been given to illustrate the efficiency of the used method.

Key-Words: - double decomposition method, Adomian polynomials, two-point fuzzy boundary value problems, fuzzy series solution.

Received: October 29, 2022. Revised: May 13, 2023. Accepted: June 17, 2023. Published: July 12, 2023.

1 Introduction

Many methods have been developed so far for solving fuzzy differential equations (FDEs). And since the FDEs have many important applications in various types of sciences, medicine and engineering, these proposed methods included all kinds of the numerical solutions, exact-analytical solutions and semi-analytical (series) solutions. Finding different types of solutions gives more freedom in dealing with the FDEs, because the exact-analytical solution may be difficult or non-existent, and then resort to the numerical solution or the semi-analytical solution. The topic of semi-analytical methods (series methods) for solving FDEs has been rapidly growing in recent years, whereas the series solutions of FDEs have been studied by several authors during the past few years.

One of the powerful semi-analytical methods that tackle numerous functional equations successfully: the Adomian decomposition method (ADM). This method has an amazing efficacy and has been endorsed by various researchers in mathematical physics. ADM is a powerful decomposition methodology for the practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, etc. The method provides the solution in a rapidly convergent series with components that can be computed iteratively. Because of its high efficiency in approximating the exact-analytical solution, many students and researchers have used this method to solve FDEs (For more details, see [1, 2, 4, 5, 7, 8, 10, 11, 14].

the main objective of this paper is to employ the double decomposition method to solve two-point fuzzy boundary value problems featuring linear and nonlinear ordinary differential equations. Recall that Adomian and Rach, in 1993, initiated the double decomposition method to improve the proficiency of the standard ADM. Further, Aminataei and Hosseini compared the double decomposition method with the standard ADM on certain boundary-value problems of the second order. Their finding was that the double decomposition method has more virtues, including higher accuracy and faster convergence, against the standard ADM (For more details, see [3, 13]).

during this work, we need many fundamental concepts in the fuzzy set theory, such as fuzzy number, fuzzy function and fuzzy derivative. These concepts can be found in detail in [6, 9, 12].

2 Two-Point Fuzzy Boundary Value Problems

The general form of the two-point fuzzy boundary value problems for the ordinary differential equations is [9]:

$$u''(x) = \ (x,\ u \quad u'(x)\ ),\ x \in [a, b] \qquad (1)$$

with the fuzzy boundary conditions:

$a\ = A\ ,\quad b\ = B$

where:

u is a fuzzy function of the crisp variable x,

$(x,\ u \quad u'(x)\ )$ is a fuzzy function of the crisp variable x and the fuzzy variable u,

$u'(x)$ is the first order fuzzy derivative of x, u(x),

$u''(x)$ is the second order fuzzy derivative of $x,\ u \quad u'(x)$,

a and b are real numbers,

A and B are fuzzy numbers.

The general idea of solving the fuzzy differential equation is based on transforming this equation into a system of non-fuzzy (crisp) differential equations.

Thus, problem (1) can be written as [12, 14]:

$$\underline{u''(x)} = \underline{f}\ (x,\ u,\ u') = H(x,\ \underline{u}, \underline{u}',\ \overline{u}\ ,\ \overline{u}'\ ) \qquad (2)$$

With the boundary conditions:

$\_\ a\ = \underline{A}\ ,\ \_\ b\ = \underline{B}$

$$\overline{u''(x)} = \ \overline{f}\ (x,\ u,\ u') = \ (x,\ \underline{u}, \underline{u}',\ \overline{u}\ ,\ \overline{u}'\ ) \qquad (3)$$

With the boundary conditions:

$\overline{u}\ (a) = \overline{A}\ ,\ \overline{u}\ (b) = \overline{B}$

Where:

$$(x,\ \underline{u}, \underline{u}',\ \overline{u}\ ,\ \overline{u}'\ ) = \mathrm{Min}\{\ f\ (x\ , z) : z \in [\ \underline{u},\ \underline{u}',\ \overline{u}\ ,\ \overline{u}']\ \} \qquad (4)$$

$$(x,\ \underline{u}, \underline{u}',\ \overline{u}\ ,\ \overline{u}'\ ) = \mathrm{Max}\{\ f\ (x\ , z) : z \in [\ \underline{u}\ ,\ \underline{u}',\ \overline{u}\ ,\ \overline{u}']\ \} \qquad (5)$$

The parametric form of the system (4-5) is given by:

$$\underline{u''(x,\ r)} \quad (x,\ \underline{u}(x\ ,\quad ,\underline{u}'(x\ ,\quad ,\ \overline{u}\ (x\ ,\quad ,\ \overline{u}'(x\ ,\quad) \qquad (6)$$

With the boundary condition:

$\_\ a,\ r\ = \underline{A(r)}\ ,\ \_\ b,\ r\ = \underline{B(r)}$

$$\overline{u''(x,\ r)} \quad (x,\ \underline{u}(x\ ,\quad ,\underline{u}'(x\ ,\quad ,\ \overline{u}\ (x\ ,\quad ,\ \overline{u}'(x\ ,\quad) \qquad (7)$$

With the boundary conditions:

$\overline{u}\ (a, r) = \overline{A(r)}\ ,\ \overline{u}\ (b, r) = \overline{B(r)}$

In order to illustrate the above, we give the following example:

If we consider the second order fuzzy differential equation:

$$u''(x) = 6u'(x) - 9u(x) + [1 + r,\ 3 - r\ ]x^2 \qquad (8)$$

With the fuzzy boundary conditions:

$u(0) = [2 \quad ,\ 4 - \ ],\ u(1) = [5 \quad ,\ 7 - \ ]$ and $r \in \quad .$

To convert problem (8) into a system of the second order crisp (non-fuzzy) ordinary differential equations, we apply the following steps:

$$[\ u''(x)\ ]\ = [6u'(x)\ - 9u(x)\ ]\ + [1 + r,\ 3 - r\ ]x^2 \qquad (9)$$

With the fuzzy boundary conditions:

$[u(0)]\ = [\ 2 \quad ,\ 4 - \ ]\ ,\ [u(1)\ ]\ = [\ 5 \quad ,\ 7 - \ ]$

Then, we can get:

$$[u''(x)]\ = 6[u'(x)]\ - 9[u(x)]\ + [1 + r,\ 3 - r\ ]x^2 \qquad (10)$$

With the fuzzy boundary conditions:

$[u(0)] = [\ 2 \quad ,\ 4 - \ ]\ , [u(1)\ ]\ = [\ 5 \quad ,\ 7 - \ ]$

Then, we have:

$[\ [u''(x)]^L\ ,\ [u''(x)]^U\ ] =$

$$[\, 6[u'(x)]^L - 9[u(x)]^L + (1 + r)x^2 ,\; 6[u'(x)]^L - 9[u(x)]^L + (3 - r)x^2 \,] \quad (11)$$

With the fuzzy boundary conditions:

$$[\, [u(0)]^L ,\; [u(0)]^U \,] = [\, 2 \quad , 4 - \quad]$$

$$[\, [u(1)]^L ,\; [u(1)]^U \,] = [\, 5 \quad , 7 - \quad]$$

Then, we get the following system of second order crisp ordinary differential equations:

$$[u''(x)]^L = 6[u'(x)]^L - 9[u(x)]^L + (1 + r)x^2 \quad (12)$$

With the boundary conditions:

$$[u(0)]^L = 2 \quad , [u(1)]^L = 5$$

$$[u''(x)]^U = 6[u'(x)]^U - 9[u(x)]^U + (3 - r)x^2 \quad (13)$$

With the boundary conditions:

$$[u(0)]^U = 4 - \quad [u(1)]^U = 7 -$$

This gives the unique crisp solutions:

$$[u(x)]^L = \left(\frac{52+25r}{27}\right) e^{3x} + \left(\frac{(126\, e^{-3} - 52) + (18\, e^{-3} - 25)r}{27}\right) x\, e^{3x} + \left(\frac{r+1}{9}\right)x^2 + \left(\frac{4r+4}{27}\right)x + \left(\frac{2r+2}{27}\right) \quad (14)$$

$$[u(x)]^U = \left(\frac{102-25r}{27}\right) e^{3x} + \left(\frac{(162\, e^{-3} - 102) + (-18\, e^{-3} + 25)r}{27}\right) x\, e^{3x} + \left(\frac{-r+3}{9}\right)x^2 + \left(\frac{-4r+12}{27}\right)x + \left(\frac{-2r+6}{27}\right) \quad (15)$$

Then, the unique fuzzy solution of problem (8) is:

$$[u(x)] = [[u(x)]_r^L , [u(x)]_r^U]$$

$$[u(x)] = \Big[\left(\frac{52+25r}{27}\right) e^{3x} + \left(\frac{(126\, e^{-3} - 52) + (18\, e^{-3} - 25)r}{27}\right) x\, e^{3x} + \left(\frac{r+1}{9}\right)x^2 + \left(\frac{4r+4}{27}\right)x + \left(\frac{2r+2}{27}\right),\; \left(\frac{102-25r}{27}\right) e^{3x} + \left(\frac{(162\, e^{-3} - 102) + (-18\, e^{-3} + 25)r}{27}\right) x\, e^{3x} + \left(\frac{-r+3}{9}\right)x^2 + \left(\frac{-4r+12}{27}\right)x + \left(\frac{-2r+6}{27}\right) \Big] \quad (16)$$

3 Double Decomposition Method

To understand the double decomposition method, we consider the nonlinear two-point crisp boundary value problem [13]:

$$Lu(x) + Ru(x) + Nu(x) = g(x) , x \in [\alpha_1 , \alpha_2] \quad (17)$$

With the boundary conditions:

$$u(\alpha_1) = \beta_1 , u(\alpha_2) = \beta_2 \quad (18)$$

Where:

$L = \frac{d^2}{dx^2}$ is the second order linear differential operator that is considered to be effortlessly invertible,

R is also a linear operator that follows same assumptions with L but with order less than that of L ,

N is the nonlinear operator,

$g(x)$ is a given continuous function,

α_1 , α_2 , β_1 and β_2 are real numbers.

By applying the inverse linear differential operator L^{-1} to the both sides of Equation (17), we will obtain:

$$u(x) = \theta(x) + L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x) \quad (19)$$

Where:

$$L^{-1}(*) = \iint (*)dxdx \quad (20)$$

$\theta(x)$ denotes the terms emanating as a result of application of L^{-1} , that is, integrating. Therefore:

$$\theta(x) = a + bx \quad (21)$$

Where a and b are real constants.

The Adomian approach is based on decomposing the unknown function $u(x)$ of any equation and the nonlinear term $Nu(x)$ into a sum of an infinite number of components defined by the decomposition series:

$$u = \sum_{n=0}^{\infty} u_n \quad (22)$$

$$Nu = \sum_{n=0}^{\infty} A_n \tag{23}$$

Where:

The components $u_n(x), n \geq 0$ are to be determined in a recursive manner. The Adomian approach concerns itself by finding the components $u_0(x),\ u_1(x),\ u_2(x), \dots$ individually. The determination of these components can be achieved in an easy way through a recursive relation that involves simple integrals.

The components $A_n, n \geq 0$ depending on $u_0(x),\ u_1(x),\ u_2(x), \dots\ u_n(x)$ are called the Adomian polynomials, and are obtained for the nonlinearity $Nu = f(u(x))$ as following [3, 14]:

$$A_0 = f(u_0) \tag{24}$$

$$A_1 = u_1 f'(u_0) \tag{25}$$

$$A_2 = u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0) \tag{26}$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0) \tag{27}$$

$$A_4 = u_4 f'(u_0) + (u_1 u_3 + \frac{u_2^2}{2!}) f''(u_0) + \frac{u_1^2 u_2}{2!} f^{(3)}(u_0) + \frac{u_1^4}{4!} f^{(4)}(u_0) \tag{28}$$

$$A_5 = u_5 f'(u_0) + (u_2 u_3 + u_1 u_4) f''(u_0) + (\frac{u_1 u_2^2}{2!} + \frac{u_3 u_1^2}{2!}) f^{(3)}(u_0) + \frac{u_2 u_1^3}{3!} f^{(4)}(u_0) + \frac{u_1^5}{5!} f^{(5)}(u_0) \tag{29}$$

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$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \mu^n} \left[f\left(\sum_{k=0}^{\infty} \mu^k u_k \right) \right]_{\mu=0}, \quad n = 0, 1, 2, \dots \tag{30}$$

where μ is a grouping parameter of convenience.

Now, we decompose the term $\theta(x)$ in equation (19) into a sum of an infinite series as follows:

$$\theta = \sum_{n=0}^{\infty} \theta_n \tag{31}$$

Therefore, the equation (19) can be rewritten as follows:

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \theta_n + L^{-1} g - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n \tag{32}$$

Where:

$$\theta_0 = a_0 + b_0 x \tag{33}$$

$$\theta_1 = a_1 + b_1 x \tag{34}$$

$$\theta_2 = a_2 + b_2 x \tag{35}$$

$$\theta_3 = a_3 + b_3 x \tag{36}$$

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$$\theta_n = a_n + b_n x \tag{37}$$

where a_n and b_n , $n \geq 0$ are real constants.

It is necessary to note that equation (37) is a special case of equation (21).

The solution steps of the double decomposition method can be derived from equation (32) as follows:

$$u_0 = \theta_0 + L^{-1} g(x) = a_0 + b_0 x + L^{-1} g(x) \tag{38}$$

$$u_1 = \theta_1 - L^{-1} R(u_0) - L^{-1} A_0 = a_1 + b_1 x - L^{-1} R(u_0) - L^{-1} A_0 \tag{39}$$

$$u_2 = \theta_2 - L^{-1} R(u_1) - L^{-1} A_1 = a_2 + b_2 x - L^{-1} R(u_1) - L^{-1} A_1 \tag{40}$$

$$u_3 = \theta_3 - L^{-1} R(u_2) - L^{-1} A_2 = a_3 + b_3 x - L^{-1} R(u_2) - L^{-1} A_2 \tag{41}$$

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$$u_n = \theta_n - L^{-1} R(u_{n-1}) - L^{-1} A_{n-1} = a_n + b_n x - L^{-1} R(u_{n-1}) - L^{-1} A_{n-1}, \quad n \geq 1 \tag{42}$$

It is necessary to note that the real constants a_n and b_n , $n \geq 0$ will be computed for every case of n by using the boundary conditions (equation 18).

Then, we have the approximate solution as follows:

$$\gamma_1(x) = u_0(x) \tag{43}$$

$$\gamma_2(x) = \gamma_1(x) + u_1(x) = u_0(x) + u_1(x) \tag{44}$$

$$\gamma_3(x) = \gamma_2(x) + u_2(x) = u_0(x) + u_1(x) + u_2(x) \tag{45}$$

$$\gamma_4(x) = \gamma_3(x) + u_3(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) \quad (46)$$

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$$\gamma_{n+1}(x) = \gamma_n(x) + u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots + u_{n-1}(x) + u_n(x), \ n \geq 0 \quad (47)$$

This means, if we consider the first terms (say m) from the solution series (equation 22), then the approximate solution of problem (17) is:

$$u(x) \approx \gamma_m(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_{m-1}(x) \quad (48)$$

It is important to note that with regard to the two-point fuzzy boundary value problem that we explained in the second section, we first convert this equation into two non-fuzzy equations (as we explained in the second section) and then apply the double decomposition method to each equation separately, to finally get the fuzzy solution.

4 Applied Examples

In this section, we will solve three fuzzy problems to illustrate the efficiency of the double decomposition method. To show the accuracy of the results, we will give a numerical comparison between the exact analytical solution and the series solution. We test the accuracy by computing the absolute errors:

$$[error]_r^L = | [u_{exact}]_r^L - [u_{series}]_r^L |$$

$$[error]_r^U = | [u_{exact}]_r^U - [u_{series}]_r^U |$$

Example 1: Consider the linear two-point fuzzy boundary value problem:

$$u''(x) = x - u(x) \ , \ x \in [0,1]$$

With the fuzzy boundary conditions:

$$[u(0)]_r = [3 + r \ , \ 5 - r] \ ,$$

$$[u(1)]_r = [r \ , \ 2 - r] \ ; \ r \in [0,1].$$

The fuzzy exact-analytical solution for this problem is:

$$[u(x)] = [[u(x)]_r^L , [u(x)]_r^U]$$

Where:

$$[u(x)]_r^L = x + (3 + r)cosx + (r - 1)csc1sinx - (3 + r)cot1sinx$$

$$[u(x)]_r^U = x + (5 - r)cosx + (1 - r)csc1sinx - (5 - r)cot1sinx$$

We will find the fuzzy series solution if $r = 0.5$, of course we can find the solution for every $r \in [0,1]$.

Lower bound of the fuzzy solution:

$u''(x) = x - u(x)$; u(0)=3.5 , u(1)=0.5

$$u(x) = \theta_n(x) + L^{-1}(x) - L^{-1}(u(x)) \ ;$$

$$\theta_n(x) = a_n + b_n x$$

$$u_0(x) = \theta_0(x) + L^{-1}(x)$$

$$u_n(x) = \theta_n(x) - L^{-1}(u_{n-1}(x)) \ , n \geq 1$$

$$\gamma_1(x) = u_0(x) \ ; \ u_0(x) = a_0 + b_0 x + \frac{x^3}{6}$$

By using the boundary conditions, we get:

$$a_0 = \frac{7}{2} \ , \ b_0 = -\frac{19}{6}$$

Therefore, we have:

$$u_0(x) = \frac{7}{2} - \frac{19}{6}x + \frac{x^3}{6}$$

$$\gamma_1(x) = \frac{7}{2} - \frac{19}{6}x + \frac{x^3}{6}$$

$$\gamma_2(x) = \gamma_1(x) + u_1(x) \ ;$$

$$u_1(x) = \theta_1(x) - L^{-1}(u_0(x))$$

$$u_1(x) = a_1 + b_1 x - L^{-1}\left(\frac{7}{2} - \frac{19}{6}x + \frac{x^3}{6}\right)$$

$$u_1(x) = a_1 + b_1 x - \frac{7}{4}x^2 + \frac{19}{36}x^3 - \frac{1}{120}x^5$$

$$\gamma_2(x) = a_1 + b_1 x + \frac{7}{2} - \frac{19}{6}x - \frac{7}{4}x^2 + \frac{25}{36}x^3 - \frac{1}{120}x^5$$

By using the boundary conditions, we get:

$$a_1 = 0 \ , \ b_1 = \frac{443}{360}$$

Therefore, we have:

$u_1(x) = \frac{443}{360}x - \frac{7}{4}x^2 + \frac{19}{36}x^3 - \frac{1}{120}x^5$

$\gamma_2(x) = \frac{7}{2} - \frac{697}{360}x - \frac{7}{4}x^2 + \frac{25}{36}x^3 - \frac{1}{120}x^5$

$\gamma_3(x) = \gamma_2(x) + u_2(x)$;

$u_2(x) = \theta_2(x) - L^{-1}(u_1(x))$

$u_2(x) = a_2 + b_2x - L^{-1}(\frac{443}{360}x - \frac{7}{4}x^2 + \frac{19}{36}x^3 - \frac{1}{120}x^5)$

$u_2(x) = a_2 + b_2x - \frac{443}{2160}x^3 + \frac{7}{48}x^4 - \frac{19}{720}x^5 + \frac{1}{5040}x^7$

$\gamma_3(x) = a_2 + b_2x + \frac{7}{2} - \frac{697}{360}x - \frac{7}{4}x^2 + \frac{1057}{2160}x^3 + \frac{7}{48}x^4 - \frac{5}{144}x^5 + \frac{1}{5040}x^7$

By using the boundary conditions, we get:

$a_2 = 0$, $b_2 = \frac{323}{3780}$

Therefore, we have:

$u_2(x) = \frac{323}{3780}x - \frac{443}{2160}x^3 + \frac{7}{48}x^4 - \frac{19}{720}x^5 + \frac{1}{5040}x^7$

$\gamma_3(x) = \frac{7}{2} - \frac{13991}{7560}x - \frac{7}{4}x^2 + \frac{1057}{2160}x^3 + \frac{7}{48}x^4 - \frac{5}{144}x^5 + \frac{1}{5040}x^7$

$\gamma_4(x) = \gamma_3(x) + u_3(x)$;

$u_3(x) = \theta_3(x) - L^{-1}(u_2(x))$

$u_3(x) = a_3 + b_3x - L^{-1}(\frac{323}{3780}x - \frac{443}{2160}x^3 + \frac{7}{48}x^4 - \frac{19}{720}x^5 + \frac{1}{5040}x^7)$

$u_3(x) = a_3 + b_3x - \frac{323}{22680}x^3 + \frac{443}{43200}x^5 - \frac{7}{1440}x^6 + \frac{19}{30240}x^7 - \frac{1}{362880}x^9$

$\gamma_4(x) = a_3 + b_3x + \frac{7}{2} - \frac{13991}{7560}x - \frac{7}{4}x^2 + \frac{21551}{45360}x^3 + \frac{7}{48}x^4 - \frac{1057}{43200}x^5 - \frac{7}{1440}x^6 + \frac{5}{6048}x^7 - \frac{1}{362880}x^9$

By using the boundary conditions, we get:

$a_3 = 0$, $b_3 = \frac{4973}{604800}$

Therefore, we have:

$u_3(x) = \frac{4973}{604800}x - \frac{323}{22680}x^3 + \frac{443}{43200}x^5 - \frac{7}{1440}x^6 + \frac{19}{30240}x^7 - \frac{1}{362880}x^9$

$\gamma_4(x) = \frac{7}{2} - \frac{1114307}{604800}x - \frac{7}{4}x^2 + \frac{21551}{45360}x^3 + \frac{7}{48}x^4 - \frac{1057}{43200}x^5 - \frac{7}{1440}x^6 + \frac{5}{6048}x^7 - \frac{1}{362880}x^9$

$\gamma_5(x) = \gamma_4(x) + u_4(x)$;

$u_4(x) = \theta_4(x) - L^{-1}(u_3(x))$

$u_4(x) = a_4 + b_4x - L^{-1}(\frac{4973}{604800}x - \frac{323}{22680}x^3 + \frac{443}{43200}x^5 - \frac{7}{1440}x^6 + \frac{19}{30240}x^7 - \frac{1}{362880}x^9)$

$u_4(x) = a_4 + b_4x - \frac{4973}{3628800}x^3 + \frac{323}{453600}x^5 - \frac{443}{1814400}x^7 + \frac{1}{11520}x^8 - \frac{19}{2177280}x^9 + \frac{1}{39916800}x^{11}$

$\gamma_5(x) = a_4 + b_4x + \frac{7}{2} - \frac{1114307}{604800}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$

By using the boundary conditions, we get:

$a_4 = 0$, $b_4 = \frac{94021}{114048000}$

Therefore, we have:

$u_4(x) = \frac{94021}{114048000}x - \frac{4973}{3628800}x^3 + \frac{323}{453600}x^5 - \frac{443}{1814400}x^7 + \frac{1}{11520}x^8 - \frac{19}{2177280}x^9 + \frac{1}{39916800}x^{11}$

$\gamma_5(x) = \frac{7}{2} - \frac{79392263}{43110144}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$

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Therefore, the lower bound of the fuzzy series solution is:

$[u(x)]_r^L \approx \gamma_5(x)$

$[u(x)]_r^L = \frac{7}{2} - \frac{79392263}{43110144}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$

Upper bound of the fuzzy solution:

$u''(x) = x - u(x)$; u(0)=4.5 , u(1)=1.5

In the same manner that we followed in the first part of this example, we can find:

$\gamma_1(x) = \frac{9}{2} - \frac{19}{6}x + \frac{x^3}{6}$

$\gamma_2(x) = \frac{9}{2} - \frac{517}{360}x - \frac{9}{4}x^2 + \frac{25}{36}x^3 - \frac{1}{120}x^5$

$\gamma_3(x) = \frac{9}{2} - \frac{1237}{945}x - \frac{9}{4}x^2 + \frac{877}{2160}x^3 + \frac{9}{48}x^4 - \frac{25}{720}x^5 + \frac{1}{5040}x^7$

$\gamma_4(x) = \frac{9}{2} - \frac{784187}{604800}x - \frac{9}{4}x^2 + \frac{1091}{2835}x^3 + \frac{9}{48}x^4 - \frac{877}{43200}x^5 - \frac{9}{1440}x^6 + \frac{5}{6048}x^7 - \frac{1}{362880}x^9$

$\gamma_5(x) = \frac{9}{2} - \frac{239327713}{184757760}x - \frac{9}{4}x^2 + \frac{1388987}{3628800}x^3 + \frac{9}{48}x^4 - \frac{1091}{56700}x^5 - \frac{9}{1440}x^6 + \frac{877}{1814400}x^7 + \frac{9}{80640}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$

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Therefore, the upper bound of the fuzzy series solution is:

$[u(x)]_r^U \approx \gamma_5(x)$

$[u(x)]_r^U = \frac{9}{2} - \frac{239327713}{184757760}x - \frac{9}{4}x^2 + \frac{1388987}{3628800}x^3 + \frac{9}{48}x^4 - \frac{1091}{56700}x^5 - \frac{9}{1440}x^6 + \frac{877}{1814400}x^7 + \frac{9}{80640}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$

Then, we have the following fuzzy series solution:

$[u(x)] = [[u(x)]_r^L, [u(x)]_r^U]$

$[u(x)] = [\frac{7}{2} - \frac{79392263}{43110144}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}, \frac{9}{2} - \frac{239327713}{184757760}x - \frac{9}{4}x^2 + \frac{1388987}{3628800}x^3 + \frac{9}{48}x^4 - \frac{1091}{56700}x^5 - \frac{9}{1440}x^6 + \frac{877}{1814400}x^7 + \frac{9}{80640}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}]$

Numerical results for this problem can be found in table 1.

Example 2: Consider the nonlinear two-point fuzzy boundary value problem:

$u''(x) = -(u'(x))^2$, $x \in [0, 2]$

With the fuzzy boundary conditions:

$[u(0)]_r = [r, 2 - r]$,

$[u(2)]_r = [1 + r, 3 - r]$; $r \in [0, 1]$.

The fuzzy exact-analytical solution for this problem is:

$[u(x)] = [[u(x)]_r^L, [u(x)]_r^U]$

Where:

$[u(x)]_r^L = ln\left(x + \frac{2}{e-1}\right) + r - ln(\frac{2}{e-1})$

$[u(x)]_r^U = ln\left(x + \frac{2}{e-1}\right) + 2 - r - ln(\frac{2}{e-1})$

We will find the fuzzy series solution if $r = 0.5$, of course we can find the solution for every $r \in [0, 1]$.

First, we find the Adomian polynomials for the function $(u'(x))^2$ *as follows*:

$A_0 = (u_0'(x))^2$

$A_1 = 2u_0'(x)u_1'(x)$

$A_2 = 2u_0'(x)u_2'(x) + (u_1'(x))^2$

$A_3 = 2u_0'(x)u_3'(x) + 2u_1'(x)u_2'(x)$

$A_4 = 2u_0'(x)u_4'(x) + (u_2'(x))^2 + 2u_1'(x)u_3'(x)$

$A_5 = 2u_0'(x)u_5'(x) + 2u_2'(x)u_3'(x) + 2u_1'(x)u_4'(x)$

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Lower bound of the fuzzy solution:

$u''(x) = -\left(u'(x)\right)^2$; u(0)=0.5 , u(2)=1.5

$u(x) = \theta_n(x) - L^{-1}(\left(u'(x)\right)^2)$;

$\theta_n(x) = a_n + b_n x$

$u_0(x) = \theta_0(x)$

$u_n(x) = \theta_n(x) - L^{-1}(A_{n-1})$, $n \geq 1$

$\gamma_1(x) = u_0(x)$; $u_0(x) = a_0 + b_0 x$

By using the boundary conditions, we get:

$a_0 = \frac{1}{2}$, $b_0 = \frac{1}{2}$

Therefore, we have:

$u_0(x) = \frac{1}{2} + \frac{1}{2}x$

$\gamma_1(x) = \frac{1}{2} + \frac{1}{2}x$

$\gamma_2(x) = \gamma_1(x) + u_1(x)$;

$u_1(x) = \theta_1(x) - L^{-1}(A_0)$

$u_1(x) = a_1 + b_1 x - L^{-1}((u_0'(x))^2)$

$u_1(x) = a_1 + b_1 x - \frac{1}{8}x^2$

$\gamma_2(x) = a_1 + b_1 x + \frac{1}{2} + \frac{1}{2}x - \frac{1}{8}x^2$

By using the boundary conditions, we get:

$a_1 = 0$, $b_1 = \frac{1}{4}$

Therefore, we have:

$u_1(x) = \frac{1}{4}x - \frac{1}{8}x^2$

$\gamma_2(x) = \frac{1}{2} + \frac{3}{4}x - \frac{1}{8}x^2$

$\gamma_3(x) = \gamma_2(x) + u_2(x)$;

$u_2(x) = \theta_2(x) - L^{-1}(A_1)$

$u_2(x) = a_2 + b_2 x - L^{-1}(\,2u_0'(x)u_1'(x))$

$u_2(x) = a_2 + b_2 x - \frac{1}{8}x^2 + \frac{1}{24}x^3$

$\gamma_3(x) = a_2 + b_2 x + \frac{1}{2} + \frac{3}{4}x - \frac{1}{4}x^2 + \frac{1}{24}x^3$

By using the boundary conditions, we get:

$a_2 = 0$, $b_2 = \frac{1}{12}$

Therefore, we have:

$u_2(x) = \frac{1}{12}x - \frac{1}{8}x^2 + \frac{1}{24}x^3$

$\gamma_3(x) = \frac{1}{2} + \frac{5}{6}x - \frac{1}{4}x^2 + \frac{1}{24}x^3$

$\gamma_4(x) = \gamma_3(x) + u_3(x)$;

$u_3(x) = \theta_3(x) - L^{-1}(A_2)$

$u_3(x) = a_3 + b_3 x - L^{-1}(\,2u_0'(x)u_2'(x) + (u_1'(x))^2)$

$u_3(x) = a_3 + b_3 x - \frac{7}{96}x^2 + \frac{1}{16}x^3 - \frac{1}{64}x^4$

$\gamma_4(x) = a_3 + b_3 x + \frac{1}{2} + \frac{5}{6}x - \frac{31}{96}x^2 + \frac{5}{48}x^3 - \frac{1}{64}x^4$

By using the boundary conditions, we get:

$a_3 = 0$, $b_3 = \frac{1}{48}$

Therefore, we have:

$u_3(x) = \frac{1}{48}x - \frac{7}{96}x^2 + \frac{1}{16}x^3 - \frac{1}{64}x^4$

$\gamma_4(x) = \frac{1}{2} + \frac{41}{48}x - \frac{31}{96}x^2 + \frac{5}{48}x^3 - \frac{1}{64}x^4$

$\gamma_5(x) = \gamma_4(x) + u_4(x)$;

$u_4(x) = \theta_4(x) - L^{-1}(A_3)$

$u_4(x) = a_4 + b_4 x - L^{-1}(\,2u_0'(x)u_3'(x) + 2u_1'(x)u_2'(x)\,)$

$u_4(x) = a_4 + b_4 x - \frac{1}{32}x^2 + \frac{5}{96}x^3 - \frac{1}{32}x^4 + \frac{1}{160}x^5$

$\gamma_5(x) = a_4 + b_4 x + \frac{1}{2} + \frac{41}{48}x - \frac{17}{48}x^2 + \frac{5}{32}x^3 - \frac{3}{64}x^4 + \frac{1}{160}x^5$

By using the boundary conditions, we get:

$a_4 = 0 \quad , \quad b_4 = \frac{1}{240}$

Therefore, we have:

$u_4(x) = \frac{1}{240}x - \frac{1}{32}x^2 + \frac{5}{96}x^3 - \frac{1}{32}x^4 + \frac{1}{160}x^5$

$\gamma_5(x) = \frac{1}{2} + \frac{103}{120}x - \frac{17}{48}x^2 + \frac{5}{32}x^3 - \frac{3}{64}x^4 + \frac{1}{160}x^5$

$\gamma_6(x) = \gamma_5(x) + u_5(x) \quad ;$

$u_5(x) = \theta_5(x) - L^{-1}(A_4)$

$u_5(x) = a_5 + b_5 x - L^{-1}(\, 2u_0'(x)u_4'(x) + (u_2'(x))^2 + 2u_1'(x)u_3'(x))$

$u_5(x) = a_5 + b_5 x - \frac{31}{2880}x^2 + \frac{1}{32}x^3 - \frac{13}{384}x^4 + \frac{1}{64}x^5 - \frac{1}{384}x^6$

$\gamma_6(x) = a_5 + b_5 x + \frac{1}{2} + \frac{103}{120}x - \frac{1051}{2880}x^2 + \frac{3}{16}x^3 - \frac{31}{384}x^4 + \frac{7}{320}x^5 - \frac{1}{384}x^6$

By using the boundary conditions, we get:

$a_5 = 0 \quad , \quad b_5 = \frac{1}{1440}$

Therefore, we have:

$u_5(x) = \frac{1}{1440}x - \frac{31}{2880}x^2 + \frac{1}{32}x^3 - \frac{13}{384}x^4 + \frac{1}{64}x^5 - \frac{1}{384}x^6$

$\gamma_6(x) = \frac{1}{2} + \frac{1237}{1440}x - \frac{1051}{2880}x^2 + \frac{3}{16}x^3 - \frac{31}{384}x^4 + \frac{7}{320}x^5 - \frac{1}{384}x^6$

$\gamma_7(x) = \gamma_6(x) + u_6(x) \quad ;$

$u_6(x) = \theta_6(x) - L^{-1}(A_5)$

$u_6(x) = a_6 + b_6 x - L^{-1}(\, 2u_0'(x)u_5'(x) + 2u_2'(x)u_3'(x) + \, 2u_1'(x)u_4'(x))$

$u_6(x) = a_6 + b_6 x - \frac{1}{320}x^2 + \frac{43}{2880}x^3 - \frac{5}{192}x^4 + \frac{1}{48}x^5 - \frac{1}{128}x^6 + \frac{1}{896}x^7$

$\gamma_7(x) = a_6 + b_6 x + \frac{1}{2} + \frac{1237}{1440}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$

By using the boundary conditions, we get:

$a_6 = 0 \quad , \quad b_6 = \frac{1}{10080}$

Therefore, we have:

$u_6(x) = \frac{1}{10080}x - \frac{1}{320}x^2 + \frac{43}{2880}x^3 - \frac{5}{192}x^4 + \frac{1}{48}x^5 - \frac{1}{128}x^6 + \frac{1}{896}x^7$

$\gamma_7(x) = \frac{1}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$

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Therefore, the lower bound of the fuzzy series solution is:

$[u(x)]_r^L \approx \gamma_7(x)$

$[u(x)]_r^L = \frac{1}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$

Upper bound of the fuzzy solution:

$u''(x) = -\left(u'(x)\right)^2$; u(0)=1.5 , u(2)=2.5

In the same manner that we followed in the first part of this example, we can find:

$\gamma_1(x) = \frac{3}{2} + \frac{1}{2}x$

$\gamma_2(x) = \frac{3}{2} + \frac{3}{4}x - \frac{1}{8}x^2$

$\gamma_3(x) = \frac{3}{2} + \frac{5}{6}x - \frac{1}{4}x^2 + \frac{1}{24}x^3$

$\gamma_4(x) = \frac{3}{2} + \frac{41}{48}x - \frac{31}{96}x^2 + \frac{5}{48}x^3 - \frac{1}{64}x^4$

$\gamma_5(x) = \frac{3}{2} + \frac{103}{120}x - \frac{17}{48}x^2 + \frac{5}{32}x^3 - \frac{3}{64}x^4 + \frac{1}{160}x^5$

$\gamma_6(x) = \frac{3}{2} + \frac{1237}{1440}x - \frac{1051}{2880}x^2 + \frac{3}{16}x^3 - \frac{31}{384}x^4 + \frac{7}{320}x^5 - \frac{1}{384}x^6$

$\gamma_7(x) = \frac{3}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$

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Therefore, the upper bound of the fuzzy series solution is:

$[u(x)]_r^U \approx \gamma_7(x)$

$[u(x)]_r^U = \frac{3}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$

Then, we have the following fuzzy series solution:

$[u(x)] = [[u(x)]_r^L, [u(x)]_r^U]$

$[u(x)] = [\frac{1}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7, \frac{3}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7]$

Numerical results for this problem can be found in table 2.

Example 3: Consider the fuzzy Painleve Equation I:

$u''(x) = x + 6\big(u(x)\big)^2, \; x \in [0, 1]$

With the fuzzy boundary conditions:

$[u(0)]_r = [0.5r + 1, -0.5r + 2], [u(1)]_r = [0.5r + 3, -0.5r + 4]; r \in [0, 1]$

First, we apply the double decomposition method to get:

$u(x) = \theta_n(x) + L^{-1}(x) + L^{-1}\left(6\big(u(x)\big)^2\right);$

$\theta_n(x) = a_n + b_n x$

$u_0(x) = \theta_0(x) + L^{-1}(x) = \theta_0(x) + \frac{1}{6}x^3$

$u_n(x) = \theta_n(x) + L^{-1}(A_{n-1}) \quad, n \geq 1$

Now, we find the Adomian polynomials for the function $6\big(u(x)\big)^2$ *as follows*:

$A_0 = 6(u_0(x))^2$

$A_1 = 12u_0(x)u_1(x)$

$A_2 = 12u_0(x)u_2(x) + 6(u_1(x))^2$

$A_3 = 12u_0(x)u_3(x) + 12u_1(x)u_2(x)$

$A_4 = 12u_0(x)u_4(x) + 6(u_2(x))^2 + 12u_1(x)u_3(x)$

$A_5 = 12u_0(x)u_5(x) + 12u_2(x)u_3(x) + 12u_1(x)u_4(x)$

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In the same manner that we followed in the previous examples, we can find the fuzzy approximate solution for this problem if $r = 0.5$, as follows:

$[u(x)] = [[u(x)]_r^L, [u(x)]_r^U]$

Where:

$[u(x)]_r^L = \frac{5}{4} + \frac{68989740030}{3487787229}x + \frac{75}{16}x^2 - \frac{1465}{63}x^3 - \frac{786017}{60480}x^4 + \frac{279}{32}x^5 + \frac{270409}{75600}x^6 + \frac{434}{378}x^7 + \frac{167}{672}x^8 + \frac{121}{1440}x^9 + \frac{33}{10080}x^{10} + \frac{781}{277200}x^{11} + \frac{1}{26208}x^{13}$

$[u(x)]_r^U = \frac{7}{4} + \frac{8317770101}{215694943}x + \frac{147}{16}x^2 - \frac{19787}{360}x^3 - \frac{875771}{60480}x^4 + \frac{2723}{160}x^5 + \frac{365287}{75600}x^6 + \frac{2656}{1890}x^7 + \frac{233}{672}x^8 + \frac{121}{1440}x^9 + \frac{33}{7200}x^{10} + \frac{781}{277200}x^{11} + \frac{1}{26208}x^{13}$

Table 1: Numerical results for example 1.

x	$[u_{series}(x)]_r^L$	$[error]_r^L$	$[u_{series}(x)]_r^U$	$[error]_r^U$
0	3.500000000000000	0	4.500000000000000	0
0.1	3.298826638405800	9.12e-6	4.348365372544997	1.38e-5
0.2	3.065692461813599	1.73e-5	4.154283696960543	2.62e-5
0.3	2.803926760812489	2.39e-5	3.920694434856567	3.61e-5
0.4	2.517144752079745	2.81e-5	3.650931567389507	4.24e-5
0.5	2.209211400275634	2.95e-5	3.348690198407133	4.46e-5
0.6	1.884202765178947	2.80e-5	3.017989580334067	4.24e-5
0.7	1.546365263519024	2.38e-5	2.663132937253816	3.61e-5
0.8	1.200073253413404	1.73e-5	2.288664488096421	2.62e-5
0.9	0.849785363711897	9.11e-6	1.899324097232523	1.38e-5
1	0.500000000927856	9.28e-10	1.500000000154643	1.55e-10

Table 2: Numerical results for example 2.

x	$[u_{series}(x)]_r^L$	$[error]_r^L$	$[u_{series}(x)]_r^U$	$[error]_r^U$
0	0.500000000000000	0	1.500000000000000	0
0.2	0.658564800000000	2.79e-7	1.658564800000000	2.79e-7
0.4	0.795380622222222	1.39e-5	1.795380622222222	1.39e-5
0.6	0.915729933333333	5.29e-6	1.915729933333333	5.29e-6
0.8	1.023155200000000	1.80e-5	2.023155200000000	1.80e-5
1	1.120138888888889	2.44e-5	2.120138888888889	2.44e-5
1.2	1.208519466666666	6.40e-6	2.208519466666666	6.40e-6
1.4	1.289715400000000	1.26e-5	2.289715400000000	1.26e-5
1.6	1.364829155555555	1.06e-5	2.364829155555555	1.06e-5
1.8	1.434703200000000	1.54e-6	2.434703200000000	1.54e-6
2	1.500000000000000	0	2.500000000000000	0

5 Conclusions

In this work, we have studied the series solutions of the two-point fuzzy boundary value problems, we have used double decomposition method to find these solutions. Based on the numerical results that we obtained, double decomposition method is a highly efficient method in solving and gives accurate results. The accuracy of this method varies from one problem to another, and this depends on the type of problem, whether it is linear or non-linear. Also, the accuracy of the results in this method depends on the number of solution series terms that we calculate.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.