A semi-analytical solution for the dynamic analysis of a rectangular viscoelastic plate subjected to a moving inertial load

M. MOFID, M. A. FOYOUZAT
Department of Civil Engineering, Sharif University of Technology, Azadi Ave., Tehran, P.O. Box: 11155-9313,. IRAN

Abstract: A semi-analytical method is developed to determine the response of a thin rectangular plate made of a general viscoelastic material to the excitation of a moving inertial load. The governing equation of the general problem is derived in the Laplace domain, which, for any particular viscoelastic model, is transformable into a system of differential equations in the time domain. Any standard procedure can then be readily employed to solve this system of equations. Using this method, sample response spectra are presented, through which the effect of viscosity, mass and velocity is scrutinized. The results show that, when the viscosity is not large enough, inertial terms cannot be ignored, especially when a heavy load is travelling at a high velocity.

Keywords: Moving inertial load, Plate, Viscoelastic material, Laplace transform.

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1. Introduction

MOVING load problems are commonly encountered in a wide range of engineering applications (a helpful review can be found in [1]). Many of such applications deal with materials that behave viscoelastically under applied loads, that is, they exhibit a combination of both viscous and elastic characteristics in their response. Examples include asphalt roads, concrete airport runways, floating ice sheets, and constrained layer viscoelastic laminated dampers, of which all can be modeled as viscoelastic beams or plates under an external moving load.

One of the earliest contributions to the foregoing problem was made by Kelly [2] who examined the response of a viscoelastic beam acted upon by a moving force. Several studies have been conducted since then, focusing on the analysis of viscoelastic beams, including those by Flügge [3], Lv et al. [4], and Louhghalam et al. [5]. In all of these studies, the inertial effects of the moving object due to convective acceleration terms are neglected in order to further simplify the analysis. However, this assumption may cause a significant error, usually in cases involving a large high-speed moving mass. In addition, the solutions proposed in those studies are restricted to the special case of a Kelvin model for the viscoelastic material, so that they cannot be generalized to cover various types of viscoelastic behavior. Therefore, in the current study, we aim at introducing a new analytical-numerical solution that can be used to evaluate the dynamic response of a rectangular plate made of a general viscoelastic material and subjected to a moving inertial load. In this study, we take advantage of the Laplace transform to derive the governing equation of motion. This equation is then transformed into a system of linear differential equations in the time domain, of which the solution leads to the dynamic response of the plate.

2. Method of Solution

Consider a rectangular plate made of a viscoelastic material and subjected to a moving inertial load m traveling along a rectilinear trajectory at a constant speed v_0 on line $Y_0 = b/2$. The schematic of the plate is shown in Fig. 1. In this figure, a, b, and h represent the length, width and thickness of the plate, respectively, and k and c are the stiffness and damping factors of the supporting foundation. Any type of boundary condition that can guarantee the stability conditions may be assumed for the plate. Considering the equilibrium equation of an element of the plate leads to

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -f(x, y, t) \tag{1}$$

where M_x, M_y and M_{xy} are the internal moments, and

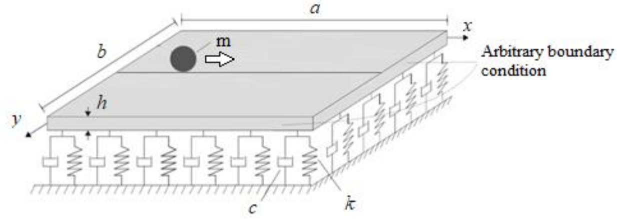


Fig. 1. Viscoelastic plate subjected to a moving inertial load

$$f(x,y,t) = -\rho h \frac{\partial^2 z(x,y,t)}{\partial t^2} + m\left[g - \frac{d^2 z(v_0 t, b/2, t)}{dt^2} \right]$$

$$\delta(x - v_0 t)\delta(y - b/2) - k\, z(x,y,t) - c\frac{\partial z(x,y,t)}{\partial t} \qquad (2)$$

where ρ, g, δ and $z(x, y, t)$ are the mass density of the plate, the acceleration of gravity, the Dirac delta function and the displacement field of the plate, respectively, and

$$\frac{d^2 z\left(v_0 t, b/2, t\right)}{dt^2} = \left[\frac{\partial^2 z}{\partial t^2} + 2v_0 \frac{\partial^2 z}{\partial x \partial t} + v_0^2 \frac{\partial^2 z}{\partial x^2} \right]_{x=v_0 t,\, y=b/2} \qquad (3)$$

is the inertial term of the mass. One can express the stress-strain relation of a viscoelastic material as [3]

$$\sigma_{ij} = \int_0^t \Re_{ijkl}(t-\tau)\frac{d\varepsilon_{kl}}{dt'}\,\mathrm{d}\tau \qquad (4)$$

where σ_{ij}, ε_{kl}, and $\Re_{ijkl}$ are respectively the stress, strain, and relaxation components. The relaxation tensor in the above equation can be represented in terms of relaxation functions $\Re_0(t)$ and $\Re_1(t)$ as $\Re_{ijkl} = \Re_1(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \Re_0(t)\delta_{ij}\delta_{kl}$.

By imposing a Laplace transform on Eq. (1) with respect to time, taking advantage of Eq. (4) along with compatibility equations resulted from the well-known Kirchhoff's hypothesis, one will get

$$\hat{f} = \frac{h^3}{12} s(\hat{\Re}_0 + 2\hat{\Re}_1)\left(\frac{\partial^4 \hat{z}}{\partial x^4} + 2\frac{\partial^4 \hat{z}}{\partial x^2 \partial y^2} + \frac{\partial^4 \hat{z}}{\partial y^4}\right)$$

$$= \frac{h^3}{12} s(\hat{\Re}_0 + 2\hat{\Re}_1)\nabla^4 \hat{z} \qquad (5)$$

where carets denote the Laplace transform. Assuming a solution of the form $z(x,y,t) = \sum_{n=1}^{\infty} \psi_n(x,y) F_n(t)$ for the response, where ψ_n's are the eigenfunctions of the plate, and exploiting the orthogonality property, one will reach

$$\frac{12m}{h^3}\int_A \psi_n(x,y)\,\mathrm{L}\left\{\left(\frac{d^2 z(v_0 t, \frac{b}{2}, t)}{dt^2} - g\right)\delta(x - v_0 t)\delta(y - \frac{b}{2})\right\}\mathrm{d}A$$

$$+\left[s(\hat{\Re}_0 + 2\hat{\Re}_1)\bar{\omega}_n^4 + \frac{12}{h^3}\left(\rho h s^2 + cs + k\right)\right]\hat{F}_n(s) = 0 \qquad (6)$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform, $\bar{\omega}_n^4 = \rho h \omega_n^2 / D$, ω_n are the natural frequencies, and D is the flexural rigidity. Substituting the relaxation functions of the viscoelastic material to Eq. (6) and imposing an inverse Laplace transform on either side of it, the equation is transformed into the time domain. For example, for the Kelvin model with $\Re_1(t) = G + \mu\delta(t)$, $3\Re_0(t) + 2\Re_1(t) = 3K$ (see [3]), where G, K and μ are the shear modulus, bulk modulus, and viscosity coefficient, respectively, application of this method leads to

$$\mathbf{M}_{\mathrm{KV}}(t)\{\ddot{F}_{\mathrm{KV}}\} + \mathbf{C}_{\mathrm{KV}}(t)\{\dot{F}_{\mathrm{KV}}\} + \mathbf{K}_{\mathrm{KV}}(t)\{F_{\mathrm{KV}}\} = \mathbf{f}_{\mathrm{KV}}(t) \quad (7)$$

where

$$\left(M_{\mathrm{KV}}\right)_{ij} = \frac{12}{h^3}(\rho h \delta_{ij} + m\psi_i \psi_j);$$

$$\left(C_{\mathrm{KV}}\right)_{ij} = \frac{4}{3}\mu\Omega_{ii} + \frac{12c}{h^3}\delta_{ij} + \frac{24m}{h^3}v_0 \psi_i \psi_{j,x};$$

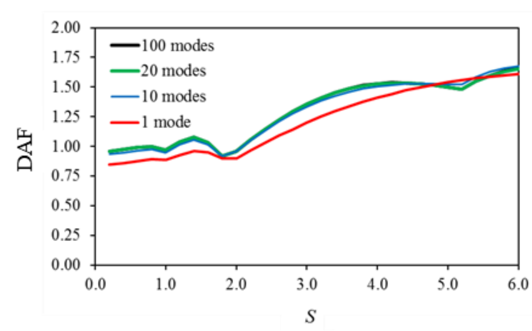
$$\left(K_{\mathrm{KV}}\right)_{ij} = (K + \frac{4}{3}G)\Omega_{ii} + \frac{12k}{h^3}\delta_{ij} + \frac{12m}{h^3}v_0^2 \psi_i \psi_{j,xx};$$

$$\{F_{\mathrm{KV}}\}^{\mathrm{T}} = \{F_1(t),\ F_2(t),\ \ldots,\ F_N(t)\}; \quad \mathbf{\Omega}_{N\times N} = \mathrm{diag}(\bar{\omega}_1^4, \bar{\omega}_2^4, \ldots, \bar{\omega}_N^4) \quad (8)$$

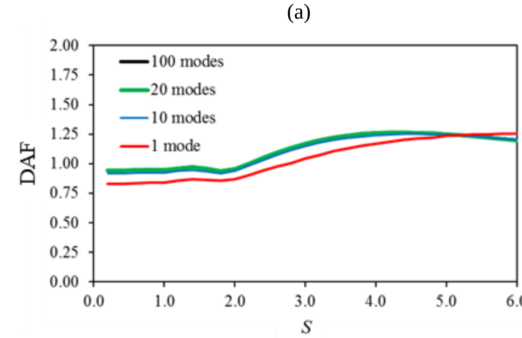
and N is the number of considered modes in the solution. The system of Eqs. (7) can be readily solved, employing any standard procedure.

3. Results

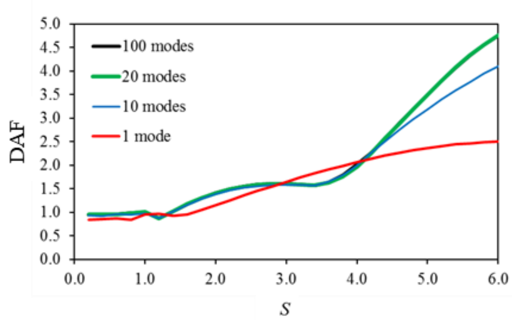
In Fig. 2, the effect of higher modes in the response of the system is examined. It is understood from this figure that the heavier the moving inertial load becomes, the more the higher modes affect the response. The same conclusion also holds true as the viscosity level decreases, or as the speed of the moving object further increases.



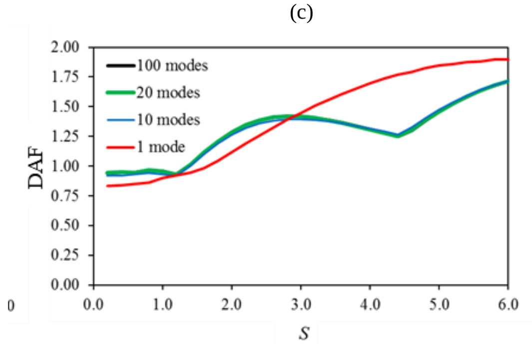
(a)



(b)



(c)



(d)

Fig. 2. Effect of higher modes in the response: (a) $\gamma = 0.1$, $H = 0$; (b) $\gamma = 0.1$, $H = 10^{-2}$; (c) $\gamma = 0.7$, $H = 0$; (d) $\gamma = 0.7$, $H = 10^{-2}$.

4. Conclusion

In the current study, a semi-analytical method was put forward to the dynamic response of a viscoelastic plate to a moving inertial load. The Laplace transform employed in the solution made possible the treatment of any type of viscoelastic material. To verify the solution, the numerical results were compared with those coming from the Moving Least Square Method (MLSM), where an excellent agreement was obtained. A numerical example was also solved to examine the effect of viscosity and inertial terms on the response. It was shown that the inertial terms of the moving object cannot be ignored in low levels of viscosity, especially when a heavy inertial load is travelling at high velocities.

The effect of considering a higher number of modes in the solution was also investigated in the current study. It was shown that as the inertial load or the speed of the moving object rises, the effect of higher modes becomes more crucial. Conversely, at a high viscosity level, the effect of higher modes is proved to be less strong.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Massood Mofid conceived and designed the analysis and supervised the findings of this work as well.

Mohammad Ali Foyouzat performed the analytic calculations and the numerical simulations, and also wrote the manuscript.