Grüss type inequalities for variational fractional integral

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Abstract: Grüss type inequalities for variational fractional integral with general kernels are established in a rather general form, thus closing the series of papers. Our result includes the corresponding ones in Baleanu et al. (2015), Choi and Purohit (2015), Wang et al. (2014), Tariboon et al. (2014) and Dahmani et al. (2010). Our method is much simpler than those in the literature. Finally, an open problem for further investigations is given.

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1. Introduction

Fractional integrals can be applied in various field of science and engineering [17]. For example, they are useful in solving a series of various problems in differential equations, probability, statistics, chemical engineering, underground water, population dynamics and so forth [3], [14]–[17], [19], [21].

Recently, a new generalized fractional integral, including the left and the right Riemann Liouville fractional integrals, the Riesz fractional integral, has been introduced by Agrawal [5] in 2010.

Definition 1.1: [5] The generalized variational fractional integral operator $\mathbb{S}_P^\alpha$ of order α for function $f(t)$ is defined as:

$$\begin{aligned} \mathbb{S}^\alpha_{\langle a,t,b,p,q\rangle} f(t) &= p\int_a^t k_\alpha(t,s)f(s)ds \\ &+ q\int_t^b k_\alpha(s,t)f(s)ds = \mathbb{S}_P^\alpha f(t), \end{aligned}$$

where $t\in(a,b)$, $p,q\in\mathbb{R}$, $P=\langle a,t,b,p,q\rangle$ is a parameter set and $k_\alpha(t,s)$ is a non-negative kernel which may depend on a parameter α.

Remark 1.2: (I) Clearly, for $k_\alpha(t,s) := \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$, $P=\langle a,t,b,p,q\rangle$,

- if $P=P_1=\langle a,t,b,1,0\rangle$, then the left Riemann Liouville fractional integral
$$\mathbb{S}^\alpha_{P_1} f(t) = \int_a^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}f(s)ds = \mathbb{I}^\alpha_{a+}f(t),$$
is obtained.
- if $P=P_2=\langle a,t,b,0,1\rangle$, then we obtain the right Riemann Liouville fractional integral
$$\mathbb{S}^\alpha_{P_2} f(t) = \int_t^b \frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1}f(s)ds = \mathbb{I}^\alpha_{b-}f(t).$$
- if $P=P_3=\langle a,t,b,\frac{1}{2},\frac{1}{2}\rangle$, then we obtain the Riesz fractional integral of $f(t)$ of order α
$$\mathbb{S}^\alpha_{P_3} f(t) = \frac{1}{2}\mathbb{S}^\alpha_{P_1}f(t)+\frac{1}{2}\mathbb{S}^\alpha_{P_2}f(t) = \frac{1}{2}\mathbb{I}^\alpha_{a+}f(t)+\frac{1}{2}\mathbb{I}^\alpha_{b-}f(t).$$
is given.

(II) for $k_\alpha(t,s) := \frac{\beta t^{-\beta(\eta+\alpha)}\tau^{\beta(\eta+1)-1}}{\Gamma(\alpha)}(t^\beta-\tau^\beta)^{\alpha-1}$, $P=P_1=\langle a,t,b,1,0\rangle$, we obtain the Erdélyi-Kober fractional integral [6]

$$\mathbb{I}^{\eta,\alpha}_\beta f(t) = \frac{\beta t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{\beta(\eta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}f(\tau)d\tau.$$

(III) Let $\alpha>0$, $\mu>-1, \beta,\eta\in\mathbb{R}$. For $k_\alpha(t,s) := \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^\mu(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right)$ and $P_1=\langle a,t,b,1,0\rangle$, we obtain a generalized fractional integral (or the generalized Saigo fractional integral) [9], [18] of order α for a real-valued continuous f,

$$\begin{aligned} &\mathbb{I}_t^{\alpha,\beta,\eta,\mu} f(t) = \\ &\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^t \tau^\mu(t-\tau)^{\alpha-1}\times \\ &{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right)f(\tau)d\tau, \quad \text{(I.1)} \end{aligned}$$

where the function ${}_2F_1(.)$ appearing as a kernel for the operator (1.5) is the familiar Gaussian hypergeometric function.

Integral inequalities play important roles in nonlinear analysis [1]–[3], [19], [20], [23]. The Grüss inequality [13] is a well-known inequality in mathematics which has been discussed by many researchers [7]–[13], [20], [23], [24], [26]–[28].

Theorem 1.3: Let f and g be two continuous functions defined on $[a,b]$ such that $m\le f(t)\le M,\ p\le g(t)\le P$ for all $t\in[a,b]$ and some real constants m,M,p,P. Then the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt\int_a^b g(t)\,dt\right| \le \frac{1}{4}(M-m)(P-p),$$

holds.

Further information concerning the history and applications of some inequalities in fractional calculus can be found in [2],

[7], [8], [10], [25], [25]–[28]. Recently, some inequalities of Grüss type for several kinds of fractional integrals have been established [7], [8], [10], [26]–[28]. For example, in 2010, Dahmani *et al.* [10] proposed the following version of Grüss inequality for Riemann-Liouville fractional integral $\mathbb{I}_{0+}^{\alpha}[.]$.

Theorem 1.4: Let f and g be two integrable functions on $[0,\infty)$ such that $m \le f(t) \le M,\ p \le g(t) \le P$ for all $t \in [0,\infty)$ and some real constants m, M, p, P. Then the following inequality

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}\mathbb{I}_{0+}^{\alpha}[f(t)g(t)] - \mathbb{I}_{0+}^{\alpha}[f(t)]\,\mathbb{I}_{a+}^{\alpha}[g(t)]\right| \le \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{2}(M-m)(P-p).$$

In 2014, Wang *et al.* [28] proved Grüss inequality for generalized fractional integral $\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[.]$, thus generalizing the results of [10].

Theorem 1.5: Let f and g be two integrable functions on $[0,\infty)$ such that $m \le f(t) \le M,\ p \le g(t) \le P$ for all $t \in [0,\infty)$ and some real constants m, M, p, P. Then the following inequality

$$\begin{aligned}&|\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[f(t)g(t)]\\ &-\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[f(t)]\,\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[g(t)]\,|\\ &\le \left(\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right)^{2}\times\\ &(M-m)(P-p).\end{aligned}$$

In 2015, Choi and Purohit [8] obtained the general version of Theorem 1.5 which generalized the pervious results of Baleanu *et al.* [7], Wang *et al.* [28] and Tariboon *et al.* [26].

Theorem 1.6: Assume that f and g are two integrable functions on $[0,\infty)$ and $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are four integrable functions on $[0,\infty)$ such that

$$\varphi_1(t) \le f(t) \le \varphi_2(t),\ \ \psi_1(t) \le g(t) \le \psi_2(t), \tag{I.2}$$

for all $t \in [0,\infty)$. Then for $t>0$, $\alpha > \max\{0,-\beta-\mu\}$, $\mu>-1, \beta<1$ and $\beta-1<\eta<0$, the following inequality

$$\begin{aligned}&|\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[f(t)g(t)]\\ &-\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[f(t)]\,\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[g(t)]\,|\\ &\le \sqrt{\mathcal{T}(f,\varphi_1,\varphi_2)\,\mathcal{T}(g,\psi_1,\psi_2)},\end{aligned}$$

holds where

$$\begin{aligned}&\mathcal{T}(u,\upsilon,\omega) =\\ &\left(\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\omega(t)] - \mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[u(t)]\right)\times\\ &\left(\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[u(t)] - \mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\upsilon(t)]\right)\\ &+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\upsilon(t)u(t)]\\ &-\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\upsilon(t)]\,\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[u(t)]\\ &+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\omega(t)u(t)]\\ &-\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\omega(t)]\,\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[u(t)]\\ &+\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\upsilon(t)]\,\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\omega(t)]\\ &-\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\mathbb{I}_{t}^{\alpha,\beta,\eta,\mu}[\upsilon(t)\omega(t)].\end{aligned}$$

In this paper, an Agrawal fractional integral inequality of Grüss type is established. Our results generalize the corresponding ones in the literature [7], [8], [10], [26]–[28].

The presentation of the paper is as follows. In Section 3, we will focus on the Grüss inequality for the Agrawal fractional integral with a general kernel, thus generalizing the results of [7], [8], [10], [26]–[28]. In Section 4, we give more general versions of inequalities in previous section. Finally, some conclusions and problems for further investigations are given.

2. Gr¿ss Type Inequality for Variational Fractional Integral

This section provides general inequalities related to Grüss type for variational fractional integral. Now, our results can be stated as follows. Throughout Sections 2 and 3, $P = \langle a,t,b,p,q\rangle$ is a parameter set where $a<t<b$ and $p,q \ge 0$.

Theorem 2.1: Let f and g be two integrable functions on $[0,\infty[$ satisfying the following condition

$$m \le f(x) \le M,\ \ n \le g(x) \le N \tag{II.1}$$

for any $m, M, n, N \in \mathbb{R}, x \in [a,b]$. Then for $t>0,\ \alpha>0$, we have:

$$\begin{aligned}&|(\mathbb{S}_{P}^{\alpha}(1)\,\mathbb{S}_{P}^{\alpha}(fg)(t) - \mathbb{S}_{P}^{\alpha}f(t)\,\mathbb{S}_{P}^{\alpha}g(t))|\\ &\le \frac{1}{4}(M-m)(P-p)\times\\ &\left[\left(\mathbb{S}_{\langle a,t,b,p,0\rangle}^{\alpha}(1)(t)\right)^{4} + \left(\mathbb{S}_{\langle a,t,b,0,q\rangle}^{\alpha}(1)(t)\right)^{4}\right]^{\frac{1}{2}}\\ &\le \frac{1}{4}(M-m)(P-p)\left(\mathbb{S}_{\langle a,t,b,p,q\rangle}^{\alpha}(1)(t)\right)^{2}.\end{aligned}$$

Remark 2.2: Using Theorem (2.1) for $P = P_1 = \langle a,t,b,1,0\rangle$,

- if $k_\alpha(t,s) = \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$, we have the result of Dahmani *et al.* [10, 2010] which is a generalization of the classical Grüss inequality.
- for $k_\alpha(t,s) := \frac{\beta t^{-\beta(\eta+\alpha)}\tau^{\beta(\eta+1)-1}}{\Gamma(\alpha)}(t^\beta - \tau^\beta)^{\alpha-1}$, we obtain Grüss type inequality for the Erdélyi-Kober fractional integral.

- for $k_\alpha(t,s) := \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right)$, $\alpha>0$, $\mu>-1,\beta,\eta\in\mathbb{R}$, we have Theorem 1.6 obtained by Wang *et al.* [28, 2014].

Theorem 2.3: Let f and g be two integrable functions on $[0,\infty[$ and let $\varphi_1,\varphi_2,\psi_1$ and ψ_2 be four integrabel functions on $[0,\infty[$ satisfying the condition (I.2) on $[0,\infty[$. Then for $t>0$, $\alpha>0$, we have:

$$\left|\mathbb{S}_P^\alpha(1)\,\mathbb{S}_P^\alpha(fg)(t)-\mathbb{S}_P^\alpha f(t)\,\mathbb{S}_P^\alpha g(t)\right| \tag{II.2}$$
$$\leq\left(\begin{array}{c}\mathcal{H}_{\langle a,b,t,p,0\rangle}(f,\varphi_1,\varphi_2)\mathcal{H}_{\langle a,b,t,p,0\rangle}(g,\psi_1,\psi_2)+\\ \mathcal{H}_{\langle a,b,t,0,q\rangle}(f,\varphi_1,\varphi_2)\mathcal{H}_{\langle a,b,t,0,q\rangle}(g,\psi_1,\psi_2)\end{array}\right)^{\frac{1}{2}},$$

where $\mathcal{H}_{\langle a,b,t,p,q\rangle}(u,v,w)$ is defined by

$$\begin{aligned}
&\mathcal{H}_{\langle a,b,t,p,q\rangle}(u,v,w)=\\
&\left(\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}w(t)-\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}u(t)\right)\times\\
&\left(\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}u(t)-\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}v(t)\right)\\
&+\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}(1)\,\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}vu(t)\\
&-\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}v(t)\,\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}u(t)\\
&+\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}(1)\,\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}wu(t)\\
&-\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}w(t)\,\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}u(t)\\
&+\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}v(t)\,\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}w(t)\\
&-\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}(1)\,\mathbb{S}^\alpha_{\langle a,b,t,p,q\rangle}vw\,(t)\,.
\end{aligned}$$

Remark 2.4: Using Theorem (2.3) for $P=P_1=\langle a,t,b,1,0\rangle$,

- for
$$\begin{aligned}
&k_\alpha(t,s):=\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(t-\tau)^{\alpha-1}\times\\
&{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right),\\
&\alpha>0,\ \mu>-1,\beta,\eta\in\mathbb{R},
\end{aligned}$$
we obtain Theorem 1.6 which obtained by Choi and Purohit [8], thus generalizing the pervious results of Baleanu *et al.* [7], Wang *et al.* [28] and Tariboon *et al.* [26].

Theorem 2.5: Let f and g be two integrable functions on $[0,\infty[$ and let $\varphi_1,\varphi_2,\psi_1$ and ψ_2 be four integrabel functions satisfying the condition (I.2) on $[0,\infty[$. Then for $t>0$ and $\alpha>0$, the following inequalities hold:

$$\begin{aligned}
&\mathbb{S}_P^\alpha\psi_1(t)\,\mathbb{S}_P^\alpha f(t)+\mathbb{S}_P^\alpha\varphi_2(t)\,\mathbb{S}_P^\alpha g(t)\geq\\
&\mathbb{S}_P^\alpha\psi_1(t)\,\mathbb{S}_P^\alpha\varphi_2(t)+\mathbb{S}_P^\alpha f(t)\,\mathbb{S}_P^\alpha g(t).
\end{aligned}$$

Proof. From the condition (I.2), we have for $t\in[0,\infty[$ that

$$\left(\varphi_2(\tau)-f(\tau)\right)\left(g(\rho)-\psi_1(\rho)\right)\geq 0.$$

Then

$$\varphi_2(\tau)g(\rho)+\psi_1(\rho)f(\tau)\geq\psi_1(\rho)\varphi_2(\tau)+f(\tau)g(\rho). \tag{II.3}$$

Multiplying both sides of (II.3) by $pk_\alpha(t,\tau)$, and then integrating over (a,t), we get

$$\begin{aligned}
&p\int_a^t k_\alpha(t,\tau)\varphi_2(\tau)g(\rho)d\tau+p\int_a^t k_\alpha(t,\tau)\psi_1(\rho)f(\tau)d\tau\\
\geq\ &p\int_a^t k_\alpha(t,\tau)\psi_1(\rho)\varphi_2(\tau)d\tau+p\int_a^t k_\alpha(t,\tau)f(\tau)g(\rho)d\tau.
\end{aligned} \tag{II.4}$$

Multiplying both sides of (II.3) by $qk_\alpha(\tau,t)$, and then integrating over (t,b), we obtain

$$\begin{aligned}
&q\int_t^b k_\alpha(\tau,t)\varphi_2(\tau)g(\rho)d\tau+q\int_t^b k_\alpha(\tau,t)\psi_1(\rho)f(\tau)d\tau\\
\geq\ &q\int_t^b k_\alpha(\tau,t)\psi_1(\rho)\varphi_2(\tau)d\tau+q\int_t^b k_\alpha(\tau,t)f(\tau)g(\rho)d\tau.
\end{aligned} \tag{II.5}$$

Adding the inequalities (II.4) and (II.5), we have

$$g(\rho)\,\mathbb{S}_P^\alpha\varphi_2(t)+\psi_1(\rho)\,\mathbb{S}_P^\alpha f(t)\geq\psi_1(\rho)\,\mathbb{S}_P^\alpha\varphi_2(t)+g(\rho)\,\mathbb{S}_P^\alpha f(t). \tag{II.6}$$

Multiplying both sides of (II.6) by $pk_\alpha(t,\rho)$, and then integrating over (a,t), we obtain

$$\begin{aligned}
&\mathbb{S}_P^\alpha\varphi_2(t)\,p\int_a^t k_\alpha(t,\rho)g(\rho)d\rho\\
&+\mathbb{S}_P^\alpha f(t)\,p\int_a^t k_\alpha(t,\rho)\psi_1(\rho)d\rho\\
&\geq\mathbb{S}_P^\alpha\varphi_2(t)\,p\int_a^t k_\alpha(t,\rho)\psi_1(\rho)d\rho\\
&+\mathbb{S}_P^\alpha f(t)\,p\int_a^t k_\alpha(t,\rho)g(\rho)d\rho.
\end{aligned} \tag{II.7}$$

Multiplying both sides of (II.6) by $qk_\alpha(\rho,t)$, and then integrating over (t,b), we get

$$\begin{aligned}
&\mathbb{S}_P^\alpha\varphi_2(t)\,q\int_t^b k_\alpha(\rho,t)g(\rho)d\rho\\
&+\mathbb{S}_P^\alpha f(t)\,q\int_t^b k_\alpha(\rho,t)\psi_1(\rho)d\rho\\
&\geq\mathbb{S}_P^\alpha\varphi_2(t)\,q\int_t^b k_\alpha(\rho,t)\psi_1(\rho)d\rho\\
&+\mathbb{S}_P^\alpha f(t)\,q\int_t^b k_\alpha(\rho,t)g(\rho)d\rho.
\end{aligned} \tag{II.8}$$

Adding the inequalities (II.7) and (II.8) we obtain

$$\begin{aligned}
&\mathbb{S}_P^\alpha\psi_1(t)\,\mathbb{S}_P^\alpha f(t)+\mathbb{S}_P^\alpha\varphi_2(t)\,\mathbb{S}_P^\alpha g(t)\geq\\
&\mathbb{S}_P^\alpha\psi_1(t)\,\mathbb{S}_P^\alpha\varphi_2(t)+\mathbb{S}_P^\alpha f(t)\,\mathbb{S}_P^\alpha g(t),
\end{aligned}$$

and this ends the proof.

Remark 2.6: Using Theorem (2.5) for $P=\langle a,t,b,1,0\rangle$,
(I) if $k_\alpha(t,s)=\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$, we have [26, Theorem 5].
(II) $k_\alpha(t,s):=\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$, we obtain [27, Theorem 2].

Corollary 2.7: Let f and g be two integrable functions on $[0,\infty[$. Assume that there exist real constants m,M,n,N

such that

$$m \leq f(t) \leq M, \quad n \leq g(t) \leq N, \quad \forall\, t \in [0, \infty[.$$

Then for $t > 0$, $\alpha, \beta > 0$, we have

$$n\, \mathbb{S}_P^\beta(1)\, \mathbb{S}_P^\alpha f(t) + M\, \mathbb{S}_P^\alpha(1)\, \mathbb{S}_P^\beta g(t) \geq nM\, \mathbb{S}_P^\alpha(1)\, \mathbb{S}_P^\beta(1) + \mathbb{S}_P^\alpha f(t)\, \mathbb{S}_P^\beta g(t).$$

Specially, when $f = g$ in Theorem (2.5), we get the following corollary.

Corollary 2.8: Let f be an integrable function on $[0, \infty[$. Assume that there exist two integrable functions φ_1, φ_2 on $[0, \infty[$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \qquad \forall t \in [0, \infty).$$

Then, for $t > 0$, $\alpha > 0$, one has:

$$\mathbb{S}_P^\alpha \varphi_1(t)\, \mathbb{S}_P^\alpha f(t) + \mathbb{S}_P^\alpha \varphi_2(t)\, \mathbb{S}_P^\alpha f(t) \geq \mathbb{S}_P^\alpha \varphi_2(t)\, \mathbb{S}_P^\alpha \varphi_1(t) + \left(\mathbb{S}_P^\alpha f(t)\right)^2.$$

Example 2.9: Let f be a function satisfying $t \leq f(t) \leq t+1$ for $t \in [0, \infty[$. Then for $t > 0$, $\alpha > 0$, we have

$$\Big(2S_P^\alpha(t) + \mathbb{S}_P^\alpha(1)\Big)\mathbb{S}_P^\alpha f(t) \geq \mathbb{S}_P^\alpha(t+1)\, \mathbb{S}_P^\alpha(t) + (\mathbb{S}_P^\alpha f(t))^2.$$

3. Further Discussion

In this section, we give more general versions of Theorem 2.1 and Theorem 2.5.

Theorem 3.1: Let f and g be two integrable functions on $[0, \infty[$ and satisfying the condition (II.1) on $[0, \infty[$ and let x be a nonnegative continuous function on $[0, \infty[$. Then for $t > 0$, $\alpha > 0$, we have:

$$\begin{aligned}
&|(\mathbb{S}_P^\alpha x(t)\, \mathbb{S}_P^\alpha (xfg)(t) - \mathbb{S}_P^\alpha (xf)(t)\, \mathbb{S}_P^\alpha (xg)(t))| \\
&\leq \frac{1}{4}\,(M-m)\,(P-p) \times \\
&\left[\left(\mathbb{S}_{\langle a,t,b,p,0\rangle}^\alpha\,(x)\,(t)\right)^4 + \left(\mathbb{S}_{\langle a,t,b,0,q\rangle}^\alpha\,(x)\,(t)\right)^4\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\,(M-m)\,(P-p)\left(\mathbb{S}_{\langle a,t,b,p,q\rangle}^\alpha\,(x)\,(t)\right)^2.
\end{aligned}$$

Theorem 3.2: Let f and g be two integrable functions on $[0, \infty[$ and let $\varphi_1, \varphi_2, \psi_1$ and ψ_2 be four integrabel functions on $[0, \infty[$ satisfying the condition (I.2) on $[0, \infty[$. Then for $t > 0$, $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$, the following inequalities hold:

$$\begin{aligned}
&\mathbb{S}_P^\beta \psi_1(t)\, \mathbb{S}_P^\alpha f(t) \\
&+\mathbb{S}_P^\alpha \varphi_2(t)\, \mathbb{S}_P^\beta g(t) \geq \mathbb{S}_P^\beta \psi_1(t)\, \mathbb{S}_P^\alpha \varphi_2(t) + \mathbb{S}_P^\alpha f(t)\, \mathbb{S}_P^\beta g(t).
\end{aligned}$$

Proof. Multiplying both sides of (II.6) by $pk_\beta(t,\rho)$, and then integrating over (a,t), we obtain

$$\begin{aligned}
&\mathbb{S}_P^\alpha \varphi_2(t)\, p \int_a^t k_\beta(t,\rho) g(\rho) d\rho \\
&+\mathbb{S}_P^\alpha f(t)\, p \int_a^t k_\beta(t,\rho)\psi_1(\rho) d\rho \\
\geq\ &\mathbb{S}_P^\alpha \varphi_2(t)\, p \int_a^t k_\beta(t,\rho)\psi_1(\rho) d\rho \\
&+\mathbb{S}_P^\alpha f(t)\, p \int_a^t k_\beta(t,\rho) g(\rho) d\rho. \qquad \text{(III.1)}
\end{aligned}$$

Multiplying both sides of (II.6) by $qk_\beta(\rho,t)$, and then integrating over (t,b), we get

$$\begin{aligned}
&\mathbb{S}_P^\alpha \varphi_2(t)\, q \int_t^b k_\beta(\rho,t) g(\rho) d\rho \\
&+\mathbb{S}_P^\alpha f(t)\, q \int_t^b k_\beta(\rho,t)\psi_1(\rho) d\rho \\
\geq\ &\mathbb{S}_P^\alpha \varphi_2(t)\, q \int_t^b k_\beta(\rho,t)\psi_1(\rho) d\rho \\
&+\mathbb{S}_P^\alpha f(t)\, q \int_t^b k_\beta(\rho,t) g(\rho) d\rho. \qquad \text{(III.2)}
\end{aligned}$$

Adding the inequalities (III.1) and (III.2), we obtain

$$\mathbb{S}_P^\beta \psi_1(t)\, \mathbb{S}_P^\alpha f(t) + \mathbb{S}_P^\alpha \varphi_2(t)\, \mathbb{S}_P^\beta g(t) \geq \mathbb{S}_P^\beta \psi_1(t)\, \mathbb{S}_P^\alpha \varphi_2(t) + \mathbb{S}_P^\alpha f(t)\, \mathbb{S}_P^\beta g(t),$$

and this ends the proof.

Remark 3.3: Using Theorem (3.2) for $\alpha = \beta$, we have Theorem (2.5).

Corollary 3.4: Let f and g be two integrable functions on $[0, \infty[$ satisfying the condition (II.1) for any $m, M, n, N \in \mathbb{R}$.Then for $t > 0$, $\alpha, \beta > 0$, $p, q > 0$, we have

$$n\, \mathbb{S}_P^\beta(1)\, \mathbb{S}_P^\alpha f(t) + M\, \mathbb{S}_P^\alpha(1)\, \mathbb{S}_P^\beta g(t) \geq nM\, \mathbb{S}_P^\alpha(1)\, \mathbb{S}_P^\beta(1) + \mathbb{S}_P^\alpha f(t)\, \mathbb{S}_P^\beta g(t).$$

If $f = g$, in Theorem (3.2), we get the following result.

Corollary 3.5: Let f be an integrable function on $[0, \infty[$. Assume that there exist two integrable functions φ_1, φ_2 on $[0, \infty[$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \qquad \forall t \in [0, \infty).$$

Then, for $t > 0$, $\alpha > 0$ and $\beta > 0$, one has:

$$\mathbb{S}_P^\beta \varphi_1(t)\, \mathbb{S}_P^\alpha f(t) + \mathbb{S}_P^\alpha \varphi_2(t)\, \mathbb{S}_P^\beta f(t) \geq \mathbb{S}_P^\alpha \varphi_2(t)\, \mathbb{S}_P^\beta \varphi_1(t) + \mathbb{S}_P^\alpha f(t)\, \mathbb{S}_P^\beta f(t).$$

4. Concluding Remarks

We have investigated general versions of Grüss type inequality for the variational fractional integral. Some well-known results on fractional Grüss' inequality [7], [8], [10], [26]–[28] are shown to be special cases of our results. Recently, Neamaty *et al.* [22] introduced the concept of variational fractional quantum integral (q-integral) with general kernels, which generalizes several types of fractional integrals known from the literature. As open problems for future research, it would be interesting to extend the Grüss type inequality for the variational fractional integral to quantum calculus.

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