The sum of reduced harmonic series generated by any number of positive integer factors

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Abstract: This paper is a free continuation of the author's previous papers dealing with the sums of the reduced harmonic series generated of reciprocals of all products generated by all prime divisors of the numbers 2002 and 2022, that were inspired by one task on the sum of a special infinite series on the Berkeley Math Circle. We determined the sums of these series, i.e. the series of all the unit fractions that have denominators with only factors consisting of all prime divisors of the numbers 2022 and 2002, analytically and also by calculation in computer algebra system Maple. In this paper, we generalize our considerations and derive two formulas that obviously hold not only to series generated by n prime numbers, but also to series generated by n positive integers.

Key-Words: CAS Maple 2022, infinite series, geometric series, harmonic series, reduced harmonic series

Received: August 27, 2021. Revised: April 19, 2022. Accepted: May 20, 2022. Published: July 3, 2022.

1 Introduction

In this paper, we generalize the solution of problems from the papers [1], [2] and [3], which concerned the sums of reciprocals of all products generated by prime factors of the numbers 2002 and 2022 based on the decomposition of these numbers into prime numbers. The number 2022 (as well as the year 2022) has three prime divisors $2, 3, 337$, because

$$2022 = 2 \cdot 3 \cdot 337,$$

and the number 2002 (as well as the year 2002) has four prime divisors $2, 7, 11, 13$, because

$$2002 = 2 \cdot 7 \cdot 11 \cdot 13.$$

These prime numbers generate corresponding reduced harmonic series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{337}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{337^2}+ \\ +\frac{1}{2\cdot 3}+\frac{1}{2\cdot 337}+\frac{1}{3\cdot 337}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{337^3}+\cdots$$

and

$$\frac{1}{2}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{2^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+ \\ +\frac{1}{2\cdot 7}+\frac{1}{2\cdot 11}+\frac{1}{2\cdot 13}+\frac{1}{7\cdot 11}+\frac{1}{7\cdot 13}+\frac{1}{11\cdot 13}+ \\ +\frac{1}{2^3}+\frac{1}{7^3}+\frac{1}{11^3}+\frac{1}{13^3}+\frac{1}{2^2\cdot 7}+\frac{1}{2^2\cdot 11}+\cdots$$

The generalization will concern the decomposition into n prime numbers and the derived formula for the sum of the series generated by n prime numbers will be proved by induction. The derived formula will apply not only to the decomposition of numbers into prime numbers, but also to the decomposition into compound numbers.

Because our topic concerns the harmonic series and the so called reduced harmonic series or modified harmonic series, let us now recall the necessary notions from infinite series theory.

2 Basic notions

For any sequence $\{a_k\}$ of numbers the associated *infinite series* or more briefly *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum\limits_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n .$$

The series $\sum\limits_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim\limits_{n\to\infty} s_n = s$. We say that the series $\sum\limits_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$.

The *geometric series* is the sum of an infinite number of terms that have a constant ratio between successive terms. Any geometric series can be written as

$$a + aq + aq^2 + aq^3 + \cdots \ ,$$

where a is the coefficient of each term and q is the common ratio between adjacent terms. Geometric series are among the simplest examples of infinite series and for $|q| < 1$ have the sum

$$s = \frac{a}{1-q}\,. \tag{1}$$

More information about geometric series can be found, for example, on [4].

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the cube numbers is the *Apéry's constant* $\zeta(3)$ which is given by the formula

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \doteq 1.202057\,.$$

The *harmonic series* is the infinite series formed by summing all positive unit fractions:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \ .$$

The harmonic series is divergent. Its divergence was first proven in 1350 by Nicole Oresme. More information about harmonic series can be found, for example, on [5].

The *reduced harmonic series* is defined as the subseries of the harmonic series, which arises by omitting some its terms. As an example of the reduced harmonic series we can take the series formed by reciprocals of primes and number one

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots \ .$$

This reduced harmonic series is divergent. The first proof of its divergence was made by Leonhard Euler (see e.g. book [6]).

3 Six lemmas

Before giving the general formula for the sum of the reduced harmonic series generated by any number of positive integer factors, we present six following useful lemmas.

Lemma 1. Let a be a positive integer. The geometric series in the form of special reduced harmonic series

$$T_a = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \cdots \tag{2}$$

has the sum

$$S_a = \frac{1}{a-1} = \frac{a}{a-1} - 1\,. \tag{3}$$

Proof. Equation (3) follows from the formula (1) for the sum of a convergent geometric series. Because the first term is $1/a$ and common ratio q is also $1/a > 0$, we have

$$S_a = \frac{1/a}{1-1/a} = \frac{1/a}{(a-1)/a} = \frac{1}{a-1} = $$
$$= \frac{a-(a-1)}{a-1} = \frac{a}{a-1} - 1\,.$$

□

Lemma 2. Let $a < b$ be positive integers. The series in the form of special reduced harmonic series

$$\begin{aligned} T_{a\cdot b} &= \frac{1}{ab} + \frac{1}{a^2b} + \frac{1}{b^2a} + \frac{1}{a^3b} + \frac{1}{b^3a} + \\ &+ \frac{1}{a^2b^2} + \frac{1}{a^4b} + \frac{1}{b^4a} + \frac{1}{a^3b^2} + \frac{1}{b^3a^2} + \cdots = \\ &= \frac{1}{ab}\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{ab} + \cdots\right) \end{aligned} \tag{4}$$

has the sum

$$S_{a\cdot b} = \frac{1}{(a-1)(b-1)}\,. \tag{5}$$

Proof. It is clear that we have

$$S_{a\cdot b} = \frac{1}{ab}\big(1 + S_a + S_b + S_{a\cdot b}\big)\,,$$

whence, according to formula (3), we get

$$(ab-1)S_{a\cdot b} = 1 + \frac{1}{a-1} + \frac{1}{b-1}\,,$$

i.e.

$$(ab-1)S_{a\cdot b} = \frac{(a-1)(b-1) + b - 1 + a - 1}{(a-1)(b-1)}\,,$$

i.e.

$$S_{a\cdot b} = \frac{ab-1}{(ab-1)(a-1)(b-1)}\,,$$

whence

$$S_{a\cdot b} = \frac{1}{(a-1)(b-1)}\,.$$

□

Lemma 3. Let $a < b$ be positive integeres. The series in the form of special reduced harmonic series

$$T(a,b) = \frac{1}{a} + \frac{1}{b} + \left(\frac{1}{a^2} + \frac{1}{ab} + \frac{1}{b^2}\right) + \\ + \left(\frac{1}{a^3} + \frac{1}{a^2b} + \frac{1}{b^2a} + \frac{1}{b^3}\right) + \\ + \left(\frac{1}{a^4} + \frac{1}{a^3b} + \frac{1}{b^3a} + \frac{1}{a^2b^2} + \frac{1}{b^4}\right) + \\ + \left(\frac{1}{a^5} + \frac{1}{a^4b} + \frac{1}{b^4a} + \frac{1}{a^3b^2} + \frac{1}{b^3a^2} + \frac{1}{b^5}\right) + \cdots \tag{6}$$

has the sum

$$S(a,b) = \frac{a}{a-1} \cdot \frac{b}{b-1} - 1 \tag{7}$$

and also the sum

$$S(a,b) = \frac{a+b-1}{(a-1)(b-1)} \tag{8}$$

and as well the sum

$$S(a,b) = \frac{1}{a} \cdot \frac{a}{a-1} + \frac{1}{b} \cdot \frac{a}{a-1} \cdot \frac{b}{b-1}. \tag{9}$$

Proof. We gradually determine the sum $S(a,b)$ of the series $T(a,b)$ by rearranging it, appropriately dividing it into suitable subseries and using the well-known formula (1) for the sum of an infinite geometric series.

Assume that its sum $S(a,b)$ is finite and that the series (6) converges. Because all its terms are positive, then the series (6) converges absolutely and so we can rearrange it. For easier determination the sum $S(a,b)$ it is necessary the series (6) rearrange and divide it into three subseries: T_a, T_b and $T_{a\cdot b}$. It is clear that

$$S(a,b) = S_a + S_b + S_{a\cdot b},$$

whence, according to formulas (3) and (5), we get

$$S(a,b) = \frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{(a-1)(b-1)} = \\ = \frac{b-1+a-1+1}{(a-1)(b-1)} = \frac{a+b-1}{(a-1)(b-1)},$$

$$S(a,b) = \frac{a+b-1}{(a-1)(b-1)} = \\ = \frac{ab-ab+a+b-1}{(a-1)(b-1)} = \\ = \frac{ab-(a-1)(b-1)}{(a-1)(b-1)} = \\ = \frac{ab}{(a-1)(b-1)} - 1,$$

$$S(a,b) = \frac{a+b-1}{(a-1)(b-1)} = \frac{ab(b-1)+a^2b}{ab(a-1)(b-1)} = \\ = \frac{ab(b-1)}{ab(a-1)(b-1)} + \frac{a^2b}{ab(a-1)(b-1)} = \\ = \frac{1}{a} \cdot \frac{a}{a-1} + \frac{1}{b} \cdot \frac{a}{a-1} \cdot \frac{b}{b-1}.$$

□

Lemma 4. Let $a < b < c$ be positive integers. The series in the form of special reduced harmonic series

$$T_{a\cdot b|c} = \frac{1}{ab} + \frac{1}{a^2b} + \frac{1}{b^2a} + \frac{1}{abc} + \frac{1}{a^3b} + \frac{1}{b^3a} + \\ + \frac{1}{a^2b^2} + \frac{1}{a^2bc} + \frac{1}{b^2ac} + \frac{1}{c^2ab} + \cdots = \\ = \frac{1}{ab}\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a^2} + \frac{1}{b^2} + \right. \\ \left. + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{c^2} + \cdots\right). \tag{10}$$

has the sum

$$S_{a\cdot b|c} = \frac{c}{(a-1)(b-1)(c-1)}. \tag{11}$$

Proof. Clearly, we have

$$S_{a\cdot b|c} = \frac{1}{ab}\left(1 + S_a + S_b + S_c + S_{a\cdot b|c} + S_{a\cdot c} + S_{b\cdot c}\right),$$

whence, by formulas (3) and (5), we get

$$(ab-1)S_{a\cdot b|c} = 1 + \frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \\ + \frac{1}{(a-1)(c-1)} + \frac{1}{(b-1)(c-1)},$$

i.e.

$$(ab-1)S_{a\cdot b|c} = \frac{1}{(a-1)(b-1)(c-1)} \cdot \\ \cdot \big[(a-1)(b-1)(c-1) + (b-1)(c-1) + \\ + (a-1)(c-1) + (a-1)(b-1) + \\ + (b-1) + (a-1)\big].$$

So, we have

$$S_{a\cdot b|c} = \frac{1}{(a-1)(b-1)(c-1)(ab-1)} \cdot \\ \cdot \big[abc - ab - ac - bc + a + b + c - 1 + bc - b - \\ - c + 1 + ac - a - c + 1 + ab - a - b + 1 + \\ + b - 1 + a - 1\big],$$

i.e.

$$S_{a\cdot b|c} = \frac{abc - c}{(a-1)(b-1)(c-1)}\,,$$

whence we get

$$S_{a\cdot b|c} = \frac{c}{(a-1)(b-1)(c-1)}\,.$$

□

Lemma 5. Let $a < b < c$ be positive integers. The series in the form of special reduced harmonic series

$$\begin{aligned} T(a,b,c) = {} & \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \left(\frac{1}{a^2} + \frac{1}{b^2} + \right. \\ & + \left. \frac{1}{c^2} + \frac{1}{a\cdot b} + \frac{1}{a\cdot c} + \frac{1}{b\cdot c}\right) + \left(\frac{1}{a^3} + \right. \\ & + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{a^2\cdot b} + \frac{1}{a^2\cdot c} + \frac{1}{b^2\cdot a} + \\ & + \left. \frac{1}{b^2\cdot c} + \frac{1}{c^2\cdot a} + \frac{1}{c^2\cdot b} + \frac{1}{a\cdot b\cdot c}\right) + \\ & + \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{a^3\cdot b} + \frac{1}{a^3\cdot c} + \right. \\ & + \frac{1}{b^3\cdot a} + \frac{1}{b^3\cdot c} + \frac{1}{c^3\cdot a} + \frac{1}{c^3\cdot b} + \\ & + \frac{1}{a^2\cdot b\cdot c} + \frac{1}{b^2\cdot a\cdot c} + \frac{1}{c^2\cdot a\cdot b} + \\ & + \left. \frac{1}{a^2\cdot b^2} + \frac{1}{a^2\cdot c^2} + \frac{1}{b^2\cdot c^2}\right) + \cdots \end{aligned} \tag{12}$$

has the sum

$$S(a,b,c) = \frac{a}{a-1}\cdot\frac{b}{b-1}\cdot\frac{c}{c-1} - 1 \tag{13}$$

and also the sum

$$S(a,b,c) = \frac{(a+b-1)(c-1)+ab}{(a-1)(b-1)(c-1)} \tag{14}$$

and as well the sum

$$\begin{aligned} S(a,b,c) = {} & \frac{1}{a}\cdot\frac{a}{a-1} + \frac{1}{b}\cdot\frac{a}{a-1}\cdot\frac{b}{b-1} + \\ & + \frac{1}{c}\cdot\frac{a}{a-1}\cdot\frac{b}{b-1}\cdot\frac{c}{c-1}\,. \end{aligned} \tag{15}$$

Proof. We determine the sum $S(a,b,c)$ of the series $T(a,b,c)$ by rearranging it and using the formula (1).

Assume that its sum $S(a,b,c)$ is finite and that the series (12) converges. Because all its terms are positive, then the series (12) converges absolutely and so we can rearrange it. Now, it is necessary the series (12) rearrange and divide it into six subseries T_a, T_b, T_c, $T_{a\cdot b|c}$, $T_{a\cdot c}$ and $T_{b\cdot c}$. It is clear that

$$S(a,b,c) = S_a + S_b + S_c + S_{a\cdot b|c} + S_{a\cdot c} + S_{b\cdot c}\,,$$

whence, according to formulas (3), (5) and (11), we get

$$\begin{aligned} S(a,b,c) = {} & \frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \\ & + \frac{c}{(a-1)(b-1)(c-1)} + \\ & + \frac{1}{(a-1)(c-1)} + \frac{1}{(b-1)(c-1)} = \\ & = \frac{1}{(a-1)(b-1)(c-1)}\,\cdot \\ & \cdot \big[(b-1)(c-1) + (a-1)(c-1) + \\ & + (a-1)(b-1) + c + b - 1 + a - 1\big] = \\ & = \frac{1}{(a-1)(b-1)(c-1)}\,\cdot \\ & \cdot \big[bc - b - c + 1 + ac - a - c + 1 + \\ & + ab - a - b + 1 + c + b - 1 + a - 1\big] = \\ & = \frac{ab + ac + bc - a - b - c + 1}{(a-1)(b-1)(c-1)} = \\ & = \frac{ac + bc - c - a - b + 1 + ab}{(a-1)(b-1)(c-1)} = \\ & = \frac{(a+b-1)(c-1)+ab}{(a-1)(b-1)(c-1)}\,, \end{aligned}$$

$$\begin{aligned} S(a,b,c) = {} & \frac{(a+b-1)(c-1)+ab}{(a-1)(b-1)(c-1)} = \\ & = \frac{1}{(a-1)(b-1)(c-1)}\,\cdot \\ & \cdot \big[abc - abc + ab + ac + bc - \\ & - a - b - c + 1\big] = \\ & = \frac{1}{(a-1)(b-1)(c-1)}\,\cdot \\ & \cdot \big[abc - ab(c-1) + a(c-1) + \\ & + b(c-1) - (c-1)\big] = \\ & = \frac{abc - (ab - a - b - 1)(c-1)}{(a-1)(b-1)(c-1)} = \\ & = \frac{abc - (a-1)(b-1)(c-1)}{(a-1)(b-1)(c-1)} = \\ & = \frac{abc}{(a-1)(b-1)(c-1)} - 1\,, \end{aligned}$$

$$S(a,b,c) = \frac{(a+b-1)(c-1)+ab}{(a-1)(b-1)(c-1)} =$$
$$= \frac{abc(a+b-1)(c-1)+a^2b^2c}{abc(a-1)(b-1)(c-1)} =$$
$$= \frac{abc(b-1)(c-1)+a^2bc(c-1)+a^2b^2c}{abc(a-1)(b-1)(c-1)} =$$
$$= \frac{1}{a}\cdot\frac{a}{a-1} + \frac{1}{b}\cdot\frac{a}{a-1}\cdot\frac{b}{b-1} +$$
$$+ \frac{1}{c}\cdot\frac{a}{a-1}\cdot\frac{b}{b-1}\cdot\frac{c}{c-1}.$$

□

Remark 1. Let us note that a number $s(n)$ of partial summands making the sum $S(a_1, a_2, \ldots, a_n)$, $n \geq 3$, is

$$s(n) = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} = 2^n - 2,$$

while $s(1) = 1$ and $s(2) = 3$, so

$$s(3) = 2^3 - 2 = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6,$$

according to equations

$$S(a,b) = S_a + S_b + S_{a\cdot b}$$

and

$$S(a,b,c) = S_a + S_b + S_c + S_{a\cdot b|c} + S_{a\cdot c} + S_{b\cdot c}$$

given in Lemmas 3 and 5. For example, the sum $S(2,3,5,7)$ has these

$$s(4) = 2^4 - 2 = \binom{4}{1} + \binom{4}{2} + \binom{4}{3} =$$
$$= 4 + 6 + 4 = 14$$

partial summands:

$$S_2\,, S_3\,, S_5\,, S_7\,,$$
$$S_{2\cdot3}\,, S_{2\cdot5}\,, S_{2\cdot7}\,, S_{3\cdot5}\,, S_{3\cdot7}\,, S_{5\cdot7}\,,$$
$$S_{2\cdot3\cdot5|7}\,, S_{2\cdot3\cdot7}\,, S_{2\cdot5\cdot7}\,, S_{3\cdot5\cdot7}$$

and for example, the sum $S(2,3,5,7,11)$ has these

$$s(5) = 2^5 - 2 = \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} =$$
$$= 5 + 10 + 10 + 5 = 30$$

partial summands:

$$S_2\,, S_3\,, S_5\,, S_7\,, S_{11}\,,$$
$$S_{2\cdot3}\,, S_{2\cdot5}\,, S_{2\cdot7}\,, S_{2\cdot11}\,, S_{3\cdot5}\,,$$
$$S_{3\cdot7}\,, S_{3\cdot11}\,, S_{5\cdot7}\,, S_{5\cdot11}\,, S_{7\cdot11}\,,$$
$$S_{2\cdot3\cdot5}\,, S_{2\cdot3\cdot7}\,, S_{2\cdot3\cdot11}\,, S_{2\cdot5\cdot7}\,, S_{2\cdot5\cdot11}\,,$$
$$S_{2\cdot7\cdot11}\,, S_{3\cdot5\cdot7}\,, S_{3\cdot5\cdot11}\,, S_{3\cdot7\cdot11}\,, S_{5\cdot7\cdot11}\,,$$
$$S_{2\cdot3\cdot5\cdot7|11}\,, S_{2\cdot3\cdot5\cdot11}\,, S_{2\cdot3\cdot7\cdot11}\,, S_{2\cdot5\cdot7\cdot11}\,, S_{3\cdot5\cdot7\cdot11}\,.$$

Lemma 6. For all positive integer n and for any positive integers $a_1, a_2, \ldots, a_n$ it holds the equality

$$\begin{aligned} &1 + \frac{1}{a_1-1} + \frac{a_1}{(a_1-1)(a_2-1)} + \\ &+ \frac{a_1a_2}{(a_1-1)(a_2-1)(a_3-1)} + \cdots \\ &\cdots + \frac{a_1a_2\cdots a_{n-1}}{(a_1-1)(a_2-1)\cdots(a_n-1)} = \\ &= \frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1}\cdot\;\cdots\;\cdot\frac{a_n}{a_n-1}. \end{aligned} \tag{16}$$

Proof. Let us denote the sum on the left-hand side of (16) by L_n, the product on the right-hand side by R_n and use mathematical induction.
1. Base case: The equality (16) holds for $n = 1$:

$$L_1 = 1 + \frac{1}{a_1-1} = \frac{a_1-1+1}{a_1-1} = \frac{a_1}{a_1-1} = R_1.$$

2. Inductive step: Suppose that it holds $L_n = R_n$ for $n \geq 1$ and show that it holds $L_{n+1} = R_{n+1}$. Since

$$L_{n+1} =$$
$$= L_n + \frac{a_1a_2\cdots a_{n-1}a_n}{(a_1-1)(a_2-1)\cdots(a_n-1)(a_{n+1}-1)},$$

then, by the inductive hypothesis, we get

$$L_{n+1} =$$
$$= R_n + \frac{a_1a_2\cdots a_{n-1}a_n}{(a_1-1)(a_2-1)\cdots(a_n-1)(a_{n+1}-1)} =$$
$$= \frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1}\cdot\;\cdots\;\cdot\frac{a_n}{a_n-1} +$$
$$+ \frac{a_1a_2\cdots a_{n-1}a_n}{(a_1-1)(a_2-1)\cdots(a_n-1)(a_{n+1}-1)} =$$
$$= \frac{a_1a_2\cdots a_{n-1}a_n(a_{n+1}-1) + a_1a_2\cdots a_{n-1}a_n}{(a_1-1)(a_2-1)\cdots(a_n-1)(a_{n+1}-1)} =$$
$$= \frac{a_1a_2\cdots a_na_{n+1}}{(a_1-1)(a_2-1)\cdots(a_{n+1}-1)} = R_{n+1}.$$

Therefore the equality (16) holds for all positive integer n and for any positive integers $a_1, a_2, \ldots, a_n$. □

4 Analytical Solution

Now, we determine two general formulas for the sum $S(a_1, a_2, \dots, a_n)$, further denoted briefly by $S[n]$, of the reduced harmonic series

$$T(a_1, a_2, \dots, a_n) = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + $$
$$+ \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \cdots \right.$$
$$\cdots + \frac{1}{a_1 a_n} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_2 a_n} + \cdots$$
$$\left. \cdots + \frac{1}{a_{n-1} a_n}\right) + \left(\frac{1}{a_1^3} + \frac{1}{a_2^3} + \dots + \frac{1}{a_n^3} + \right.$$
$$+ \frac{1}{a_1^2 a_2} + \frac{1}{a_1^2 a_3} + \dots + \frac{1}{a_1^2 a_n} + \frac{1}{a_2^2 a_1} + $$
$$+ \frac{1}{a_2^2 a_3} + \dots + \frac{1}{a_2^2 a_n} + \dots + \frac{1}{a_n^2 a_1} + \frac{1}{a_n^2 a_2} + \cdots$$
$$\cdots + \frac{1}{a_n^2 a_{n-1}} + \frac{1}{a_1 a_2 a_3} + \frac{1}{a_1 a_2 a_4} + \cdots$$
$$\left. \cdots + \frac{1}{a_{n-2} a_{n-1} a_n}\right) + \left(\frac{1}{a_1^4} + \frac{1}{a_2^4} + \dots + \frac{1}{a_n^4} + \right.$$
$$+ \frac{1}{a_1^3 a_2} + \frac{1}{a_1^3 a_3} + \dots + \frac{1}{a_n^3 a_{n-1}} + \cdots$$
$$\left. \cdots + \frac{1}{a_{n-3} a_{n-2} a_{n-1} a_n}\right) + \cdots ,$$

generated obviously not only by n prime numbers $a_1, a_2, \dots, a_n$ but also by n positive integers $a_1, a_2, \dots, a_n$ and prove it by mathematical induction.

By (3) for $S[1] = S(a_1)$ we have

$$S[1] = S_{a_1} = \frac{1}{a_1 - 1},$$

i.e.

$$S[1] = \frac{1}{a_1} \cdot \frac{a_1}{a_1 - 1} = \sum_{i=1}^{1} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right). \quad (17)$$

By (9) for $S[2] = S(a_1, a_2)$ we get

$$S[2] = \frac{1}{a_1} \cdot \frac{a_1}{a_1 - 1} + \frac{1}{a_2} \cdot \frac{a_1}{a_1 - 1} \cdot \frac{a_2}{a_2 - 1},$$

so we have

$$S[2] = \sum_{i=1}^{2} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right). \quad (18)$$

By (15) for $S[3] = S(a_1, a_2, a_3)$ we have

$$S[3] = \frac{1}{a_1} \cdot \frac{a_1}{a_1 - 1} + \frac{1}{a_2} \cdot \frac{a_1}{a_1 - 1} \cdot \frac{a_2}{a_2 - 1} + $$
$$+ \frac{1}{a_3} \cdot \frac{a_1}{a_1 - 1} \cdot \frac{a_2}{a_2 - 1} \cdot \frac{a_3}{a_3 - 1},$$

i.e.

$$S[3] = \sum_{i=1}^{3} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right). \quad (19)$$

So, we can assume that it holds the following theorem:

Theorem 1. For all positive integer n the sum $S[n]$ of the reduced harmonic series generated by positive integers $a_1, a_2, \dots, a_n$ holds the equality

$$S[n] = \sum_{i=1}^{n} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right). \quad (20)$$

Proof. Let us prove the formula (20) by using mathematical induction.
1. Base case: This formula holds for $n = 1$, as was stated in (17):

$$S[1] = \sum_{i=1}^{1} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right).$$

2. Inductive step: Suppose that the formula (20) holds for n and show that it also holds for $n+1$. Since

$$S[n+1] = S[n] + \frac{1}{a_{n+1}} \prod_{j=1}^{n+1} \frac{a_j}{a_j - 1},$$

then by (16) and by the inductive hypothesis we can gradually write

$$S[n+1] = \sum_{i=1}^{n} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right) + $$
$$+ \frac{1}{a_{n+1}} \prod_{j=1}^{n+1} \frac{a_j}{a_j - 1} = $$
$$= \sum_{i=1}^{n} \left[\frac{1}{a_i} \left(1 + \sum_{j=1}^{i} \frac{a_1 a_2 \cdots a_{j-1}}{(a_1 - 1)(a_2 - 1) \cdots (a_j - 1)} \right) \right] + $$
$$+ \frac{1}{a_{n+1}} \left(1 + \sum_{j=1}^{n+1} \frac{a_1 a_2 \cdots a_j}{(a_1 - 1)(a_2 - 1) \cdots (a_{j+1} - 1)} \right) = $$
$$= \sum_{i=1}^{n+1} \left[\frac{1}{a_i} \left(1 + \sum_{j=1}^{i} \frac{a_1 a_2 \cdots a_{j-1}}{(a_1 - 1)(a_2 - 1) \cdots (a_j - 1)} \right) \right] = $$
$$= \sum_{i=1}^{n+1} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right).$$

Therefore the equality (20) holds for all positive integer n and for any positive integers $a_1, a_2, \ldots, a_n$. $\square$

Now, let us prove, also by mathematical induction, another formula for the sum $S[n]$:

Theorem 2. For all positive integer n the sum $S[n]$ of the reduced harmonic series generated by positive integers $a_1, a_2, \ldots, a_n$ holds the equality

$$S[n] = \prod_{i=1}^{n} \frac{a_i}{a_i - 1} - 1. \tag{21}$$

Proof. 1. Base case: It is easy to see that the formula (21) holds for $n = 1$, because by (3) we get

$$S[1] = \frac{a_1}{a_1 - 1} - 1 = \prod_{i=1}^{1} \frac{a_i}{a_i - 1} - 1$$

2. Inductive step: Suppose that the formula (21) holds for n and show that it holds for $n + 1$. Since by (20)

$$S[n+1] = S[n] + \frac{1}{a_{n+1}} \prod_{i=1}^{n+1} \frac{a_i}{a_i - 1},$$

then by the inductive hypothesis we gradually have

$$\begin{aligned}
S[n+1] &= \prod_{i=1}^{n} \frac{a_i}{a_i - 1} - 1 + \frac{1}{a_{n+1}} \prod_{i=1}^{n+1} \frac{a_i}{a_i - 1} = \\
&= \frac{a_{n+1} - 1}{a_{n+1}} \prod_{i=1}^{n+1} \frac{a_i}{a_i - 1} - 1 + \\
&+ \frac{1}{a_{n+1}} \prod_{i=1}^{n+1} \frac{a_i}{a_i - 1} = \\
&= \left(\frac{a_{n+1} - 1}{a_{n+1}} + \frac{1}{a_{n+1}} \right) \prod_{i=1}^{n+1} \frac{a_i}{a_i - 1} - 1 = \\
&= \prod_{i=1}^{n+1} \frac{a_i}{a_i - 1} - 1.
\end{aligned}$$

Therefore for all positive integer n the sum $S[n]$ of the reduced harmonic series generated by positive integers $a_1, a_2, \ldots, a_n$ has also the form (21). $\square$

5 Three simple examples

Now, we calculate the sums of three specific reduced harmonic series generated by two, three and four smallest prime numbers by means of the results of the Theorem 1 and 2.

Example 1. Determine the sum $S[2] = S(2, 3)$ of the reduced harmonic series in the form

$$\begin{aligned}
T(2,3) = {} & \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2 \cdot 3} \right) + \\
& + \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^2 \cdot 3} + \frac{1}{3^2 \cdot 2} \right) + \\
& + \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{2^3 \cdot 3} + \frac{1}{3^3 \cdot 2} + \frac{1}{2^2 \cdot 3^2} \right) + \cdots .
\end{aligned}$$

formed of all the unit fractions that have denominators with only prime factors from the set $\{2, 3\}$.

Solution: For $n = 2$ and $a_1 = 2$, $a_2 = 3$ by the formula (20) from Theorem 1 we have

$$\begin{aligned}
S[2] = S(2,3) &= \sum_{i=1}^{2} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right) = \\
&= \frac{1}{a_1} \cdot \frac{a_1}{a_1 - 1} + \frac{1}{a_2} \cdot \frac{a_1}{a_1 - 1} \cdot \frac{a_2}{a_2 - 1} = \\
&= \frac{1}{2} \cdot \frac{2}{2-1} + \frac{1}{3} \cdot \frac{2}{2-1} \cdot \frac{3}{3-1} = \\
&= \frac{1}{2} \cdot \frac{2}{1} + \frac{1}{3} \cdot \frac{2}{1} \cdot \frac{3}{2} = 1 + 1 = 2.
\end{aligned}$$

By the formula (21) from Theorem 2 we get the same result:

$$\begin{aligned}
S[2] = S(2,3) &= \prod_{i=1}^{2} \frac{a_i}{a_i - 1} - 1 = \\
&= \frac{a_1}{a_1 - 1} \cdot \frac{a_2}{a_2 - 1} - 1 = \\
&= \frac{2}{2-1} \cdot \frac{3}{3-1} - 1 = \frac{2}{1} \cdot \frac{3}{2} - 1 = 2.
\end{aligned}$$

Example 2. Determine the sum $S[3] = S(2, 3, 5)$ of the reduced harmonic series in the form

$$\begin{aligned}
T(2,3,5) = {} & \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \right. \\
& + \left. \frac{1}{5^2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 5} \right) + \left(\frac{1}{2^3} + \right. \\
& + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{2^2 \cdot 3} + \frac{1}{2^2 \cdot 5} + \frac{1}{3^2 \cdot 2} + \\
& + \left. \frac{1}{3^2 \cdot 5} + \frac{1}{5^2 \cdot 2} + \frac{1}{5^2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 5} \right) + \cdots
\end{aligned}$$

formed of all the unit fractions that have denominators with only prime factors from the set $\{2, 3, 5\}$.

Solution: For $n = 3$ and $a_1 = 2$, $a_2 = 3$, $a_3 = 5$ by the formula (20) from Theorem 1 we have

$$S[3] = S(2,3,5) = \sum_{i=1}^{3}\left(\frac{1}{a_i}\prod_{j=1}^{i}\frac{a_j}{a_j-1}\right) =$$
$$= \frac{1}{a_1}\cdot\frac{a_1}{a_1-1} + \frac{1}{a_2}\cdot\frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1} +$$
$$+ \frac{1}{a_3}\cdot\frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1} =$$
$$= \frac{1}{2}\cdot\frac{2}{2-1} + \frac{1}{3}\cdot\frac{2}{2-1}\cdot\frac{3}{3-1} +$$
$$+ \frac{1}{5}\cdot\frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1} =$$
$$= \frac{1}{2}\cdot\frac{2}{1} + \frac{1}{3}\cdot\frac{2}{1}\cdot\frac{3}{2} + \frac{1}{5}\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4} =$$
$$= 1 + 1 + \frac{3}{4} = \frac{11}{4} = 2.75\,.$$

By the formula (21) from Theorem 2 we get the same result:

$$S[3] = S(2,3,5) = \prod_{i=1}^{3}\frac{a_i}{a_i-1} - 1 =$$
$$= \frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1} - 1 =$$
$$= \frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1} - 1 =$$
$$= \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4} - 1 = \frac{11}{4} = 2.75\,.$$

Example 3. Determine the sum $S[4] = S(2,3,5,7)$ of the reduced harmonic series

$$T(2,3,5,7) = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} +$$
$$+ \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{2\cdot 3} + \right.$$
$$\left. + \frac{1}{2\cdot 5} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 7}\right) +$$
$$+ \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{2^2\cdot 3} + \right.$$
$$+ \frac{1}{2^2\cdot 5} + \frac{1}{2^2\cdot 7} + \frac{1}{3^2\cdot 2} + \frac{1}{3^2\cdot 5} +$$
$$+ \frac{1}{3^2\cdot 7} + \frac{1}{5^2\cdot 2} + \frac{1}{5^2\cdot 3} + \frac{1}{5^2\cdot 7} +$$
$$+ \frac{1}{7^2\cdot 2} + \frac{1}{7^2\cdot 3} + \frac{1}{7^2\cdot 5} + \frac{1}{2\cdot 3\cdot 5} +$$
$$\left. + \frac{1}{2\cdot 3\cdot 7} + \frac{1}{2\cdot 5\cdot 7} + \frac{1}{3\cdot 5\cdot 7}\right) + \cdots$$

Solution: For $n = 4$ and $a_1 = 2$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$ by the formula (20) from Theorem 1 we have

$$S[4] = S(2,3,5,7) = \sum_{i=1}^{4}\left(\frac{1}{a_i}\prod_{j=1}^{i}\frac{a_j}{a_j-1}\right) =$$
$$= \frac{1}{a_1}\cdot\frac{a_1}{a_1-1} + \frac{1}{a_2}\cdot\frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1} +$$
$$+ \frac{1}{a_3}\cdot\frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1} +$$
$$+ \frac{1}{a_4}\cdot\frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1}\cdot\frac{a_4}{a_4-1} =$$
$$= \frac{1}{2}\cdot\frac{2}{2-1} + \frac{1}{3}\cdot\frac{2}{2-1}\cdot\frac{3}{3-1} +$$
$$+ \frac{1}{5}\cdot\frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1} +$$
$$+ \frac{1}{7}\cdot\frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1}\cdot\frac{7}{7-1} =$$
$$= \frac{1}{2}\cdot\frac{2}{1} + \frac{1}{3}\cdot\frac{2}{1}\cdot\frac{3}{2} + \frac{1}{5}\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4} +$$
$$+ \frac{1}{7}\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} =$$
$$= 1 + 1 + \frac{3}{4} + \frac{5}{8} = \frac{27}{8} = 3.375\,.$$

By the formula (21) from Theorem 2 we get the same result:

$$S[4] = S(2,3,5,7) = \prod_{i=1}^{4}\frac{a_i}{a_i-1} - 1 =$$
$$= \frac{a_1}{a_1-1}\cdot\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1}\cdot\frac{a_4}{a_4-1} - 1 =$$
$$= \frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1}\cdot\frac{7}{7-1} - 1 =$$
$$= \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} - 1 = \frac{27}{8} = 3.375\,.$$

6 Numerical Solution

We will solve the task to determine the sum S of reciprocals of all products generated by n positive integers, where $n = 2, 3, 4$ numerically by using the basic programming language in the computer algebra system Maple 2022. For $n \geq 5$ we would use the generalization of the following simple procedures `partab`, `partabc` and `partabcd`:

```
partab:=proc(p,a,b)
    local de,i,j,n,s;
    s:=0;
    for n from 1 to p do
        for i from 0 to n do
```

```
            for j from 0 to n-i do
                if i+j>n-1 then
                de:=a^i*b^j;
                s:=s+1/de;
                end if;
            end do;
        end do;
    end do;
    print("sum for integers",a,b,"
    and for",p,"summands is",
    evalf[4](s));
end proc:

partabc:=proc(p,a,b,c)
    local de,i,j,k,n,s;
        ⋮
        for k from 0 to n-i-j do
            if i+j+k>n-1 then
            de:=a^i*b^j*c^k;
            s:=s+1/de;
            end if;
        end do;
        ⋮
    print("sum for integers",a,b,c"
    and for",p,"summands is",
    evalf[4](s));
end proc:

partabcd:=proc(p,a,b,c,d)
    local de,i,j,k,l,n,s;
        ⋮
       for k from 0 to n-i-j do
           for l from 0 to n-i-j-k do
           if i+j+k+l>n-1 then
           de:=a^i*b^j*c^k*d^l;
           s:=s+1/de;
           end if;
       end do;
        ⋮
    print("sum for integers",a,b,c,d"
    and for",p,"summands is",
    evalf[4](s));
end proc:
```

For example, for parametr $p = 10$, the first procedure `partab` generates and sums first 10 numbers of 1-combinations, 2-combinations, 3-combinations, ... with repetitions or also multisubsets of size $n = 1, 2, 3, \dots$ from a set $\{a, b\}$ of size 2. The number of multisubsets of size n is then the number of nonnegative integer solutions of the Diophantine equation

$$i + j = n.$$

So we consider the first 10 summands that have denominators with only factors $a^i b^j$ from the set $\{a, b\}$ until $i + j \le 10$, i.e. these 10 summands:

$$\left(\frac{1}{a^1b^0}+\frac{1}{a^0b^1}\right)+\left(\frac{1}{a^2b^0}+\frac{1}{a^0b^2}+\frac{1}{a^1b^1}\right)+$$
$$+\left(\frac{1}{a^3b^0}+\frac{1}{a^0b^3}+\frac{1}{a^2b^1}+\frac{1}{a^1b^2}\right)+\frac{1}{a^4b^0}$$

Analogously, for $p = 10$, the second procedure `partabc` generates and sums the following 10 summands, which correspond to the solution of the Diophantine equation

$$i + j + k = n,$$

where $n = 1, 2, 3, \dots$:

$$\left(\frac{1}{a^1b^0c^0}+\frac{1}{a^0b^1c^0}+\frac{1}{a^0b^0c^1}\right)+$$
$$+\left(\frac{1}{a^2b^0c^0}+\frac{1}{a^0b^2c^0}+\frac{1}{a^0b^0c^2}+\right.$$
$$\left.+\frac{1}{a^1b^1c^0}+\frac{1}{a^1b^0c^1}+\frac{1}{a^0b^1c^1}\right)+\frac{1}{a^3b^0c^0}$$

and the third procedure `partabcd` generates and sums the following 10 summands, which correspond to the solution of the Diophantine equation

$$i + j + k + l = n,$$

where $n = 1, 2, 3, \dots$:

$$\left(\frac{1}{a^1b^0c^0d^0}+\frac{1}{a^0b^1c^0d^0}+\frac{1}{a^0b^0c^1d^0}+\right.$$
$$\left.+\frac{1}{a^0b^0c^0d^1}\right)+\frac{1}{a^2b^0c^0d^0}+\frac{1}{a^0b^2c^0d^0}+$$
$$+\frac{1}{a^2b^0c^0d^0}+\frac{1}{a^0b^2c^0d^0}+\frac{1}{a^1b^1c^0d^0}+\frac{1}{a^1b^0c^1d^0}$$

The approximate values of 36 sums of the type $S[2] = S(a, b)$, where $a \in \{2, 3, 4, 5, 6, 7, 8, 9\}$, $b \in \{3, 4, 5, 6, 7, 8, 9, 10\}$, $a < b$, i.e. of the sums

$$S(2,3), S(2,4), S(2,5), \dots$$
$$\dots, S(8,9), S(8,10), S(9,10),$$

for parameter $p = 100$, rounded to 6 decimals and obtained by two `for` statements

```
A:={2,3,4,5,6,7,8,9};
B:={3,4,5,6,7,8,9,10};
for a in A do
    for b in B do
        if a < b then
        partab(100,a,b)
        end if;
    end do;
end do;
```

Table 1: 36 approximate sums $S[2] = S(a,b)$ for $a \in \{2,3,\dots,9\}$, $b \in \{3,4,\dots,10\}$, $a < b$.

$S(a,b)$	$b=3$	$b=4$	$b=5$	$b=6$
$a=2$	2.000000	1.666667	1.500000	1.400000
$a=3$	×	1.000000	0.875000	0.800000
$a=4$	×	×	0.666667	0.600000
$a=5$	×	×	×	0.500000
$S(a,b)$	$b=7$	$b=8$	$b=9$	$b=10$
$a=2$	1.333333	1.285714	1.250000	1.222222
$a=3$	0.750000	0.714286	0.687500	0.666667
$a=4$	0.555556	0.523810	0.500000	0.481481
$a=5$	0.458333	0.428571	0.406250	0.388889
$S(a,b)$	$b=7$	$b=8$	$b=9$	$b=10$
$a=6$	0.400000	0.371429	0.350000	0.333333
$a=7$	×	0.333333	0.312500	0.296296
$a=8$	×	×	0.285714	0.269841
$a=9$	×	×	×	0.250000

are written into Table 1.

The approximate values of 36 sums of the type $S[3] = S(2,b,c)$, where $b \in \{3,4,5,6,7,8,9,10\}$ and $c \in \{4,5,6,7,8,9,10,11\}$, $b < c$, i.e. of the sums

$$S(2,3,4), S(2,3,5), S(2,3,6), \dots$$
$$\dots, S(2,9,10), S(2,9,11), S(2,10,11),$$

for parameter $p = 100$, rounded to 6 decimals and obtained by two `for` statements

```
B:={3,4,5,6,7,8,9,10};
C:={4,5,6,7,8,9,10,11};
for b in B do
    for c in C do
        if b < c then
        partabc(100,2,b,c);
        end if;
    end do;
end do;
```

are written into Table 2.

The approximate values of 36 sums of the type $S[4] = S(2,3,c,d)$, where $c \in \{4,5,6,7,8,9,10,11\}$ and $d \in \{5,6,7,8,9,10,11,12\}$, $c < d$, i.e. of the sums

$$S(2,3,4,5), S(2,3,4,6), S(2,3,4,7), \dots$$
$$\dots, S(2,3,10,11), S(2,3,10,12), S(2,3,11,12),$$

for parameter $p = 100$, rounded to 6 decimals and obtained by two `for` statements

Table 2: 36 approximate sums $S[3] = S(2,b,c)$ for $b \in \{3,4,\dots,10\}$, $c \in \{4,5,\dots,11\}$, $b < c$.

$S(2,b,c)$	$c=4$	$c=5$	$c=6$	$c=7$
$b=3$	3.000000	2.750000	2.600000	2.500000
$b=4$	×	2.333333	2.200000	2.111111
$b=5$	×	×	2.000000	1.916667
$b=6$	×	×	×	1.800000
$S(2,b,c)$	$c=8$	$c=9$	$c=10$	$c=11$
$b=3$	2.428571	2.375000	2.333333	2.300000
$b=4$	2.047619	2.000000	1.962963	1.933333
$b=5$	1.857143	1.812500	1.777778	1.750000
$b=6$	1.742857	1.700000	1.666667	1.640000
$S(2,b,c)$	$c=8$	$c=9$	$c=10$	$c=11$
$b=7$	1.666667	1.625000	1.592593	1.566667
$b=8$	×	1.571429	1.539683	1.514286
$b=9$	×	×	1.500000	1.475000
$b=10$	×	×	×	1.444444

```
C:={4,5,6,7,8,9,10,11};
D1:={5,6,7,8,9,10,11,12};
for c in C do
    for d in D1 do
        if c < d then
        partabcd(100,2,3,c,d);
        end if;
    end do;
end do;
```

are written into Table 3.

Table 3: 36 approximate sums $S[4] = S(2,3,c,d)$ for $c \in \{4,5,\dots,11\}$, $d \in \{5,6,\dots,12\}$, $c < d$.

$S(2,3,c,d)$	$d=5$	$d=6$	$d=7$	$d=8$
$c=4$	4.000000	3.800000	3.666667	3.571429
$c=5$	×	3.500000	3.375000	3.285714
$c=6$	×	×	3.200000	3.114286
$c=7$	×	×	×	3.000000
$S(2,3,c,d)$	$d=9$	$d=10$	$d=11$	$d=12$
$c=4$	3.500000	3.444444	3.400000	3.363636
$c=5$	3.218750	3.166667	3.125000	3.090909
$c=6$	3.050000	3.000000	2.960000	2.927273
$c=7$	2.937500	2.888889	2.850000	2.818182
$S(2,3,c,d)$	$d=9$	$d=10$	$d=11$	$d=12$
$c=8$	2.857143	2.809524	2.771429	2.740260
$c=9$	×	2.750000	2.712500	2.681818
$c=10$	×	×	2.666667	2.636364
$c=11$	×	×	×	2.600000

Note that the calculation of these $3 \cdot 36 = 108$ approximate values of sums $S[2]$, $S[3]$ and $S[4]$ took about 2, 58 and 1443 seconds, respectively.

Note also that the numerical results $S(2,3) = 2$, $S(2,3,5) = 2.75$ and $S(2,3,5,7) = 3.375$, obtained by the three above-mentioned procedures, are completely in agreement with the results of Examples 1, 2 and 3.

7 Conclusion

In this paper two general formulas for the sum $S(a_1, a_2, \dots, a_n)$, briefly denoted by $S[n]$, of the convergent reduced harmonic series

$$T(a_1, a_2, \dots, a_n) = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \\ + \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} + \frac{1}{a_1a_2} + \frac{1}{a_1a_3} + \dots \right. \\ \dots + \frac{1}{a_1a_n} + \frac{1}{a_2a_3} + \dots + \frac{1}{a_2a_n} + \dots \\ \left. \dots + \frac{1}{a_{n-1}a_n}\right) + \left(\frac{1}{a_1^3} + \frac{1}{a_2^3} + \dots + \frac{1}{a_n^3} + \dots\right.$$

generated by n positive integers $a_1, a_2, \dots, a_n$ were derived.

The first general formula has the form

$$S[n] = \sum_{i=1}^{n} \left(\frac{1}{a_i} \prod_{j=1}^{i} \frac{a_j}{a_j - 1} \right)$$

and the briefer second one has the form

$$S[n] = \prod_{i=1}^{n} \frac{a_i}{a_i - 1} - 1 .$$

So, it can be said that the convergent reduced harmonic series generated by n positive integers belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically by means of a finite formula.

Area of Further Development

It would be interesting to try to derive a reverse formula to determine a series with a given sum that would not be unambiguous. From Tables 1 to 3 we get, among other, that

$$S(2,5) = S(2,9,10) = 1.5 , \\ S(2,3) = S(2,5,6) = 2 , \\ S(2,3,4) = S(2,3,7,8) = S(2,3,6,10) = 3 .$$

Furthermore, it would be possible to determine whether the formula

$$S(2,3,\dots,n-1,n) = n - 1$$

holds for each integer $n \geq 3$, as indicated by the data in Tables 1 to 3.

Acknowledgement: This research work was supported by the Project for the Development of the Organization „DZRO Military autonomous and robotic systems“.

Conflict of Interest: The author declare no conflict of interest.

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