Direct and Transform Methods to Higher Derivatives of Ki(x)

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Abstract: - Higher derivatives and associated polynomials of the standard Nield-Kuznetsov function of the second kind are investigated in this work. Two approaches are introduced in this work. The first, is the direct method of differentiation and generalization of the nth derivative. This approach is dependent on higher derivatives of the Nield-Kuznetsov function of the first kind. The second is the transform method in which integral transforms associated with the Nield-Kuznetsov function of the second kind are introduce first, and higher derivatives are then obtained. The transform method is independent of the direct higher derivatives of the Nield-Kuznetsov function of the first kind. Both approaches are important in practical and theoretical mathematical analysis, and both give rise to associated Airy polynomials, discussed in this work.

Key-Words: - Higher derivatives, Ki(x), Airy's polynomials.

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1 Introduction

In a previous article, Hamdan *et.al.* [1] provided analysis of the higher derivatives of the function $Ni(x)$, and the associated Airy's polynomials that arise with, and define its nth derivative. The function $Ni(x)$, better-known as the Nield-Kuznetsov function of the first kind, arises in the general and particular solutions to inhomogeneous equation when homogeneity is due to a constant forcing function, [2,3].

For the sake of clarity, Airy's inhomogeneous equation takes the form, [4,5]

$$\frac{d^2y}{dx^2} - xy = R \tag{1}$$

where R is a constant, and its general solution is of the form

$$y = c_1 A_i(x) + c_2 B_i(x) - \pi R N_i(x) \tag{2}$$

c_1, c_2 are arbitrary constants, and $N_i(x)$ is defined as:

$$Ni(x) = Ai(x) \int_0^x Bi(t)dt - Bi(x) \int_0^x Ai(t)dt \tag{3}$$

where $Ai(x)$ and $Bi(x)$ are Airy's functions of the first and second kinds, respectively, [5,6].

Studies of higher derivatives of Airy's functions, [7], and of the Nield-Kuznetsov functions, [1], are imperative from both practical applications and theoretical implications. While a knowledge of higher derivatives and associated polynomials might lead to further applications in mathematical physics, quantum theory, and systems theory, [8], they also further our understanding of infinite series representations of the said functions and polynomials.

Success of studies of Airy's functions higher derivatives and polynomials, [7], and of $Ni(x)$, [1], motivate the current work in which higher derivatives and associated polynomials of the Nield-Kuznetsov function of the second kind, $Ki(x)$, are investigated. This function arises in the particular and general solutions of Airy's equation (1), with the right-hand-side, R, replaced by a continuously differentiable function $f(x)$, [2]. This general solution takes the form:

$$y = e_1 A_i(x) + e_2 B_i(x) + \pi K_i(x) - \pi f(x) N_i(x) \tag{4}$$

The function $Ki(x)$ is defined in either of the forms, [2]:

$$Ki(x) = f(x)Ni(x) - \left\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\right\} \tag{5}$$

$$Ki(x) = Ai(x)\int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt - B_i(x)\int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t)dt \tag{6}$$

To this end, the n^{th} derivative of $Ki(x)$ is obtained in two ways: a direct method in which (5) is differentiated and a generalization is obtained for the nth derivative, and a transform method in which integral transforms are developed for $Ki(x)$ then the nth derivative is obtained.

2 Higher Derivatives of $Ni(x)$

Equation (5) shows that $Ki(x)$ is defined in terms of $Ni(x)$ and Airy's functions $Ai(x)$ and $Bi(x)$. Consequently, derivatives of $Ki(x)$ must dpend on derivatives of $Ni(x)$, $Ai(x)$ and $Bi(x)$. Higher derivatives of $Ni(x)$ are discussed in what follows.

In Hamdan *et.al.*, [1], the first two derivatives of $Ni(x)$ have been expressed as;

$$N'i(x) = A'i(x)\int_0^x Bi(t)\,dt - B'i(x)\int_0^x Ai(t)\,dt \tag{7}$$

$$N''i(x) = xNi(x) - \frac{1}{\pi} \tag{8}$$

However, third and higher derivatives of $Ni(x)$ have been expressed in terms of the functions $Ni(x)$ and $N'i(x)$, and the Wronskian of $Ai(x)$ and $Bi(x)$, $W\big(Ai(x), Bi(x)\big) = \frac{1}{\pi}$.

The n^{th} derivative of $Ni(x)$ can then be expressed as, [1]:

$$Ni^{(n)}(x) = P_n(x)Ni(x) + Z_n(x)N'i(x) - R_n(x)/\pi \tag{9}$$

With the knowledge of the *nth* derivative, the n+1st derivative can be obtained as:

$$Ni^{(n+1)}(x) = [P'_n(x) + xZ_n(x)]Ni(x) + [P_n(x) + Z'_n(x)]N'i(x) - \frac{1}{\pi}[Z_n(x) + R'_n(x)] \tag{10}$$

Equation (10) takes the following form in terms of $Ai(x)$ and $Bi(x)$:

$$N^{(n+1)}(x) = \{[P'_n(x) + xZ_n(x)]Ai(x) + [P_n(x) + Z'_n(x)]A'i(x)\}\int_0^x Bi(t)\,dt - \{[P'_n(x) + xZ_n(x)]Bi(x) + [P_n(x) + Z'_n(x)]B'i(x)\}\int_0^x Ai(t)\,dt - [Z_n(x) + R'(x)]W\big(Ai(x), Bi(x)\big). \tag{11}$$

Using (9) in (10) yields the n+1st derivative as:

$$Ni^{(n+1)}(x) = P_{n+1}(x)Ni(x) + Z_{n+1}(x)N'i(x) - \frac{R_{n+1}(x)}{\pi} \tag{12}$$

Comparing (9) and (10), establishes the following:

$$P_{n+1}(x) = P'_n(x) + xZ_n(x) \tag{13}$$

$$Q_{n+1}(x) = Z'_n(x) + P_n(x) \tag{14}$$

$$R_{n+1}(x) = R'_n(x) + Z_n(x) \tag{15}$$

It is worth noting that polynomials $P_n(x)$, $Z_n(x)$ are the same polynomials that arise in the n^{th} derivatives of $Ai(x)$ and $Bi(x)$, respectively, as obtained by Abramochkin and Razueva, [7].

Clearly, higher derivatives of *Ni(x)* can be expressed in terms of *Ni(x)* and its first derivative *Ni'(x)*, and the Wronskian $W\big(Ai(x), Bi(x)\big) = \frac{1}{\pi}$, whose coefficients are polynomials. Associated with the n^{th} derivative of $Ni(x)$ are the polynomials $P_n(x)$, $Z_n(x)$ and $R_n(x)$, wherein "*n*" denotes the order of the derivative. For instance, coefficients of $Ni(x), Ni'(x)$ and the Wronskian for sample derivatives above are given in **Table 1**, below, for $n \geq 2$, (Hamdan *et.al.* [1])

Table 1. Polynomial Coefficients of $Ni(x)$, $N'i(x)$ and $W\big(Ai(x), Bi(x)\big)$

n	$P_n(x)$	$Z_n(x)$	$R_n(x)$
2	x	0	1
3	*1*	x	0
5	*4x*	x^2	*3*

10	x^5+100x^2	$20x^3+80$	x^4+82x
15	$49x^6 + 4760x^3 + 3640$	$x^7+770x^4 + 8680x$	$48x^5 + 4080x^2$

3 Higher Derivatives of $K_i(x)$

Higher derivatives of $Ki(x)$, defined by (5) and (6), can be obtained in two ways, one of which is following the method used for obtaining higher derivatives of $Ni(x)$, above, and involves derivatives of $Ni(x)$, [9,10], while the second method is independent of $Ni(x)$, but requires the introduction of integral transforms. Both methods are discussed in what follows.

Method 1: The Direct Method

Using (5), the first few derivatives of $Ki(x)$ are obtained as:

$$Ki'(x)=f'(x)Ni(x)+f(x)N'(x)-\{A'i(x)\int_0^x f(t)Bi(t)\,dt-B'i(x)\int_0^x f(t)Ai(t)dt\}. \quad (16)$$

$$Ki''(x)=2f'(x)Ni'(x)+f''(x)Ni(x)+xKi(x). \quad (17)$$

$$Ki'''(x)=3f''(x)Ni'(x)+[f'''(x)+2xf'(x)]Ni(x)+Ki(x)+xKi'(x)-2f'(x)W\big(Ai(x),Bi(x)\big). \quad (18)$$

Continuing in this manner, the n^{th} derivative takes the form:

$$Ki^{(n)}(x)=[f(x)Ni(x)]^{(n)}+p_n(x)\{Ai(x)\int_0^x f(t)Bi(t)\,dt-Bi(x)\int_0^x f(t)Ai(t)\,dt\}+q_n(x)\{Ai'(x)\int_0^x f(t)Bi(t)\,dt-Bi'(x)\int_0^x f(t)Ai(t)dt\}+r_n(x)W(Ai(x),Bi(x)) \quad (19)$$

where $p_n(x), q_n(x)$ and $r_n(x)$ are the polynomial coefficients of the integral terms and of the Wronskian that appear in the *nth* derivative, namely

$p_n(x)$ is coefficient of $\{Ai(x)\int_0^x f(t)Bi(t)\,dt-Bi(x)\int_0^x f(t)Ai(t)\,dt\}$ (20)

$q_n(x)$ is coefficient of $\{Ai'(x)\int_0^x f(t)Bi(t)\,dt-Bi'(x)\int_0^x f(t)Ai(t)dt\}$ (21)

$r_n(x)$ is coefficient of $W\big(Ai(x),Bi(x)\big)=\frac{1}{\pi}$ (22)

Following Alderson and Hamdan, [9], and Jayyousi-Dajani and Hamdan, [10], relationships between polynomials $p_n(x), q_n(x)$ and $r_n(x)$ are given by:

$$p_{n+1}(x)=p'_n(x)+xq_n(x) \quad (23)$$

$$q_{n+1}(x)=q'_n(x)+p_n(x) \quad (24)$$

$$r_{n+1}(x)=r'_n(x)+q_n(x) \quad (25)$$

The *n+1*st derivative of $Ki(x)$, obtained by differentiating (19), takes the form:

$$Ki^{(n+1)}(x)=\sum_{k=0}^{n+1}\binom{n+1}{k}Ni^{(n+1-k)}f^{(k)}(x)+[p_n'(x)+xq_n(x)]\{Ai(x)\int_0^x f(t)Bi(t)\,dt-Bi(x)\int_0^x f(t)Ai(t)\,dt\}+[p_n(x)+q_n'(x)\{Ai'(x)\int_0^x f(t)Bi(t)\,dt-Bi'(x)\int_0^x f(t)Ai(t)dt\}+[r_n'(x)-f(x)q_n(x)]W\big(Ai(x),Bi(x)\big) \quad (26)$$

Polynomials $p_n(x)$, $q_n(x)$ and $r_n(x)$ are associated with the n^{th} derivative of $K_i(x)$, where n refers to the order of the derivative and not the degree of the polynomial. These polynomials are the negatives of the polynomials associated with the n^{th} derivatives of Airy's functions, $Ai(x)$ and $Bi(x)$, and the n^{th} derivative of the standard Nield-Kuznetsov function of the first kind, $Ni(x)$, [1]. Thus, for $n\geq 2$, the following relationships hold:

$$p_n(x)=-P_n(x) \quad (27)$$

$$q_n(x)=-Z_n(x) \quad (28)$$

$$r_n(x)=-R_n(x) \quad (29)$$

Table 2, below, lists the polynomials $p_n(x)$, $q_n(x)$ and $r_n(x)$, for $n=0,1,2,\dots,10$.

Table 2. Coefficient Polynomials and Coefficient Function

$n=0$	$p_n(x)$	$q_n(x)$	$r_n(x)$
0	-1	0	0

1	0	-1	0
2	-x	0	0
3	-1	-x	0
4	$-x^2$	-2	$-x$
5	$-4x$	$-x^2$	-3
6	$-4-x^3$	$-6x$	$-x^2$
7	$-9x^2$	$-10-x^3$	$-8x$
8	$-28x-x^4$	$-12x^2$	$-x^3-18$
9	$-28-16x^3$	$-52x-x^4$	$-15x^2$
10	$-100x^2-x^5$	$-80-20x^3$	$-82x-x^4$

Degrees of the coefficient polynomials may be determined for arbitrary order of derivative, n, and are provided in the following **Table 3** in terms of the floor function.

Table 3. Degrees of Coefficient Polynomials

Polynomial	Degree
$p_n(x)$	$3\left\lfloor\frac{n-2}{2}\right\rfloor-n+3,\ n\geq 2$
$q_n(x)$	$3\left\lfloor\frac{n-3}{2}\right\rfloor-n+4,\ n\geq 3$
$r_n(x)$	$3\left\lfloor\frac{n-4}{2}\right\rfloor-n+5,\ n\geq 4$

Now, using (5), expression (26) can be written in the following form:

$$Ki^{(n+1)}(x)=\sum_{k=0}^{n+1}\binom{n+1}{k}Ni^{(n+1-k)}(x)f(x)^{(k)}$$
$$+[p_n{}'(x)+xq_n(x)]\{f(x)Ni(x)-Ki(x)\}$$
$$+[p_n(x)+q_n{}'(x)]\{f(x)Ni(x)-Ki(x)\}+$$
$$\frac{1}{\pi}[r_n{}'(x)-f(x)q_n(x)] \quad (30)$$

Replacing $n+1$ by n in (23)-(25), the n^{th} derivative of $Ki(x)$, obtained from (30), takes the following form:

$$Ki^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}Ni^{(n-k)}(x)f^{(k)}(x)$$
$$+[p_{n-1}(x)+p'_{n-1}(x)+xq_{n-1}(x)+$$
$$q'_{n-1}(x)]\{f(x)Ni(x)-Ki(x)\}+\frac{1}{\pi}[r'_{n-1}(x)-$$
$$f(x)q_{n-1}(x)];\ n=1,2,3,\ldots \quad (31)$$

The n^{th} derivative of $Ki(x)$ is thus given by (31), and the above discussion furnishes the following Theorem.

Theorem 1:

Let $f(x)\in C^n$ on $x\geq 0$. Then, the Nield-Kuznetsov function of the second kind, defined by

$$Ki(x)=f(x)Ni(x)-\left\{Ai(x)\int_0^x f(t)Bi(t)\,dt-Bi(x)\int_0^x f(t)Ai(t)\,dt\right\}$$

is continuously differentiable with an n^{th} derivative given by

$$Ki^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}Ni^{(n-k)}(x)f^{(k)}(x)$$
$$+[p_{n-1}(x)+p'_{n-1}(x)+xq_{n-1}(x)+q'_{n-1}(x)]\{f(x)Ni(x)-Ki(x)\}$$
$$+\frac{1}{\pi}[r'_{n-1}(x)-f(x)q_{n-1}(x)];\ n=1,2,3,\ldots$$

where $Ni(x)$, $p_n(x)$, $q_n(x)$ and $r_n(x)$ are given by (3), (15)-(17), respectively.

Method 2: The Transform Method

Definition (6) of $Ki(x)$ can be conveniently written in terms of the following transforms.

Define

$$Ti(x) = \int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t) \quad (32)$$

$$Qi(x) = \int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t) \quad (33)$$

then (6) can be written as

$$Ki(x) = Ai(x)Ti(x) - Bi(x)Qi(x) \quad (34)$$

The first few derivatives of (32) and (32) take the form:

$$Ti'(x) = f'(x) \int_0^x Bi(t)dt \quad (35)$$

$$Qi'(x) = f'(x) \int_0^x Ai(t)dt \quad (36)$$

$$Ti''(x) = f''(x) \int_0^x Bi(t)dt + f'(x)Bi(x) \quad (37)$$

$$Qi''(x) = f''(x) \int_0^x Ai(t)dt + f'(x)Ai(x) \quad (38)$$

$$Ti'''(x) = f'''(x) \int_0^x Bi(t)dt + 2 f''(x)Bi(x) + f'(x)Bi'(x) \quad (39)$$

$$Qi'''(x) = f'''(x) \int_0^x Ai(t)dt + 2 f''(x)Ai(x) + f'(x)Ai'(x) \quad (40)$$

$$Ti^{iv}(x) = f^{iv}(x) \int_0^x Bi(t)dt + 3 f'''(x)Bi(x) + 3 f''(x)Bi'(x) + f'(x)Bi''(x) \quad (41)$$

$$Qi^{iv}(x) = f^{iv}(x) \int_0^x Ai(t)dt + 3 f'''(x)Ai(x) + 3 f''(x)Ai'(x) + f'(x)Ai''(x) \quad (42)$$

$$Ti^{v}(x) = f^{v}(x) \int_0^x Bi(t)dt + 4f^{iv}(x)Bi(x) + 6 f'''(x)Bi'(x) + 4 f''(x)Bi''(x) + f'(x)Bi'''(x) \quad (43)$$

$$Qi^{v}(x) = f^{v}(x) \int_0^x Ai(t)dt + 4f^{iv}(x)Ai(x) + 6 f'''(x)Ai'(x) + 4 f''(x)Ai''(x) + f'(x)Ai'''(x) \quad (44)$$

$$Ti^{vi}(x) = f^{vi}(x) \int_0^x Bi(t)dt + 5f^{v}(x)Bi(x) + 10 f^{iv}(x)Bi'(x) + 10f'''(x)Bi''(x) + 5f''(x)Bi'''(x) + f'(x)Bi^{iv}(x) \quad (45)$$

$$Qi^{vi}(x) = f^{vi}(x) \int_0^x Ai(t)dt + 5f^{v}(x)Ai(x) + 10 f^{iv}(x)Ai'(x) + 10f'''(x)Ai''(x) + 5f''(x)Ai'''(x) + f'(x)Ai^{iv}(x) \quad (46)$$

Continuing this pattern, we see that the n^{th} derivatives of $Ti(x)$ and $Qi(x)$ take the forms:

$$Ti^{(n)}(x) = f^{(n)}(x) \int_0^x Bi(t)dt + \sum_{k=1}^{n} \binom{n-1}{k} f^{(n-k)}(x)Bi^{(k-1)}(x) \quad (47)$$

$$Qi^{(n)}(x) = f^{(n)}(x) \int_0^x Ai(t)dt + \sum_{k=1}^{n} \binom{n-1}{k} f^{(n-k)}(x)Ai^{(k-1)}(x) \quad (48)$$

The first few derivatives of (34) are as follows.

$$Ki'(x) = Ai'(x)Ti(x) + Ai(x)Ti'(x) - [Bi'(x)Qi(x) + Bi(x)Qi'(x) \quad (49)$$

$$Ki''(x) = Ai''(x)Ti(x) + 2Ai'(x)Ti'(x) + Ai(x)Ti''(x) - [Bi''(x)Qi(x) + 2Bi'(x)Qi'(x) + Bi(x)Qi''(x) \quad (50)$$

$$Ki'''(x) = Ai'''(x)Ti(x) + 3Ai''(x)Ti'(x) + +3Ai'(x)Ti''(x) + Ai(x)Ti'''(x) - [Bi'''(x)Qi(x) + 3Bi''(x)Qi'(x) + +3Bi'(x)Qi''(x) + Bi(x)Qi'''(x) \quad (51)$$

These derivatives generalize into the following n^{th} derivative of $Ki(x)$:

$$Ki^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \left[Ai^{(n-k)}(x)Ti^{(k)}(x) - Bi^{(n-k)}(x)Qi^{(k)}(x)\right] \quad (52)$$

Using (47) and (48), we write:

$$Ti^{(k)} = f^{(k)}(x) \int_0^x Bi(t)dt + \sum_{m=1}^{k} \binom{k-1}{m} f^{(k-m)}(x)Bi^{(m-1)}(x) \quad (53)$$

$$Qi^{(k)} = f^{(k)}(x) \int_0^x Ai(t)dt + \sum_{m=1}^{k} \binom{k-1}{m} f^{(k-m)}(x)Ai^{(m-1)}(x) \quad (54)$$

Using the following general forms of derivatives of $Ai(x)$and $Bi(x)$, given in Hamdan *et.al.*, [1], and Abramochkin and Razueva, [7]:

$$Ai^{(j)}(x) = P_j(x)Ai(x) + Z_j(x)Ai'(x) \quad (55)$$

$$Bi^{(j)}(x) = P_j(x)Bi(x) + Z_j(x)Bi'(x) \quad (56)$$

we write

$$Ai^{(m-1)}(x) = P_{m-1}(x)Ai(x) + Z_{m-1}(x)Ai'(x) \quad (57)$$

$$Bi^{(m-1)}(x) = P_{m-1}(x)Bi(x) + Z_{m-1}(x)Bi'(x) \quad (58)$$

$$Ai^{(n-k)}(x) = P_{n-k}(x)Ai(x) + Z_{n-k}(x)Ai'(x) \quad (59)$$

$$Bi^{(n-k)}(x) = P_{n-k}(x)Bi(x) + Z_{n-k}(x)Bi'(x) \quad (60)$$

Using (57)-(60) in (52)-(54), we obtain the following form of the n^{th} derivative of $Ki(x)$:

$$Ki^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \left[\{P_{n-k}(x)Ai(x) + Z_{n-k}(x)Ai'(x)\}Ti^{(k)}(x)\right]$$

$$-\sum_{k=0}^{n} \binom{n}{k} [\{P_{n-k}(x)Bi(x) + Z_{n-k}(x)Bi'(x)\}Qi^{(k)}(x) \quad (61)$$

wherein

$$Ti^{(k)} = f^{(k)}(x)\int_0^x Bi(t)dt + \sum_{m=1}^{k} \binom{k-1}{m} f^{(k-m)}(x)[P_{m-1}(x)Bi(x) + Z_{m-1}(x)Bi'(x)] \quad (62)$$

$$Qi^{(k)} = f^{(k)}(x)\int_0^x Ai(t)dt + \sum_{m=1}^{k} \binom{k-1}{m} f^{(k-m)}(x)[P_{m-1}(x)Ai(x) + Z_{m-1}(x)Ai'(x)] \quad (63)$$

With the knowledge of the n^{th} derivative of $Ki(x)$, we can obtain the $n + 1^{st}$ derivative of as:

$$Ki^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left[\{P_{n+1-k}(x)Ai(x) + Z_{n+1-k}(x)Ai'(x)\}Ti^{(k)}(x)\right] - \sum_{k=0}^{n} \binom{n}{k} [\{P_{n+1-k}(x)Bi(x) + Z_{n+1-k}(x)Bi'(x)\}Qi^{(k)}(x) \quad (64)$$

Using (13) and (14) in the form

$$P_{n+1-k}(x) = P_{n-k}(x) + xZ_{n-k}(x) \quad (65)$$

$$Z_{n+1-k}(x) = Z'_{n-k}(x) + P_{n-k}(x) \quad (66)$$

Equation (64) takes the following form:

$$Ki^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left[\{[P_{n-k}(x) + xZ_{n-k}(x)]Ai(x) + [Z'_{n-k}(x) + P_{n-k}(x)]Ai'(x)\}Ti^{(k)}(x)\right]$$
$$-\sum_{k=0}^{n} \binom{n}{k} [\{[P_{n-k}(x) + xZ_{n-k}(x)]Bi(x) + [Z'_{n-k}(x) + P_{n-k}(x)]Bi'(x)\}Qi^{(k)}(x)] \quad (67)$$

The polynomials $P_{n-k}(x)$ and $Z_{n-k}(x)$ appearing in (68) are of course known from the n^{th} derivative of $Ki(x)$. Now, replacing $n + 1$ by n in (67) gives the final form of the n^{th} derivative of $Ki(x)$, and furnishes the following theorem.

Theorem 2:

Let $f(x) \in C^n$ on $x \geq 0$. Then, the Nield-Kuznetsov function of the second kind, defined by

$$Ki(x) = Ai(x)\int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt - B_i(x)\int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t)dt$$

is continuously differentiable with an n^{th} derivative given by

$$Ki^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left[\{[P_{n-1-k}(x) + xZ_{n-1-k}(x)]Ai(x) + [Z'_{n-1-k}(x) + P_{n-1-k}(x)]Ai'(x)\}Ti^{(k)}(x)\right] - \sum_{k=0}^{n-1} \binom{n-1}{k} [\{[P_{n-1-k}(x) + xZ_{n-1-k}(x)]Bi(x) + [Z'_{n-1-k}(x) + P_{n-1-k}(x)]Bi'(x)\}Qi^{(k)}(x)];$$ *n=1,2,3,…*

where $Ti^{(k)}$ and $Qi^{(k)}$ are given by (53) and (54), respectively.

4 Values of the Derivatives at Zero

Although Theorems (1) and Theorem (2) provide equivalent forms of the n^{th} derivative of $Ki(x)$, computations using Theorem 1 are easier to perform. Using Theorem 1, values at $x = 0$ of the n^{th} derivative of $Ki(x)$ are given by:

$$Ki^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} Ni^{(n-k)}(0) f(0)^{(k)} + \frac{1}{\pi}[r_{n-1}'(0) - f(x) q_{n-1}(0)];\ n = 1,2,3,\ldots \quad (68)$$

where $Ni(0) = N'i(0) = 0$, and

$$Ni^{(n)}(0) = P_n(0)Ni(0) + Z_n(0)N'i(0) - \frac{R(0)}{\pi} = -\frac{R(0)}{\pi}. \quad (69)$$

5 Conclusion

In this work, general forms of the n^{th} derivative of the Standard Nield-Kuznetsov Function of the Second Kind, $Ki(x)$ have been obtained using two approaches: the direct approach, which is dependent on the Nield-Kuznetsov function of the first kind, $Ni(x)$, and its higher derivatives, and the second is based on the introduction of integral transforms for $Ki(x)$. Both approaches are viable, yet the first approach is more suitable for evaluation of the derivatives. Airy's polynomials arising for these derivatives have been discussed and quantified, and relationships between them have been investigated.

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Contribution of individual authors

Both authors reviewed the literature, formulated the problem, provided independent analysis, and jointly wrote and revised the manuscript.

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