Connecting Einstein Functions to the Nield-Kuznetsov and Airy's Functions

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Abstract: - In this work, the problem of obtaining particular and general solutions to Airy's inhomogeneous equation when the forcing function is one of Einstein's functions is examined. Expressions for the particular solutions provide connections between the Nield-Kuznetsov and Einstein functions. Computations have been carried out using *Wolfram Alpha.*

Key-Words: - Einstein functions, Nield-Kuznetsov functions, Airy's inhomogeneous equation

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1 Introduction

Interest in special functions stems in part from their direct use in solving applied problems and problems in mathematical physics; their connections to other elementary and special functions; and their contribution to the creation and expansion of our mathematical horizon, (*cf.* [1-3] and the references therein).

Of particular interest to the current work is a class of functions referred to as Einstein functions, which are combinations of exponential and logarithmic functions. Einstein functions have been the subject matter of various studies and tabulations due to their applications in the study of distributions, the determination of physical and chemical material constants, and in the study of Einstein's field equations. For these and many other applications of Einstein functions, one is referred to the works of Abramowitz and Stegun, [1], Hilsenrath and Ziegler, [4], Cezairliyan, [5], and the references therein.

Computations, series representations and some properties of Einstein functions have been illustrated in Wolfram's Mathworld, [6].

Noteworthy in the study of Einstein functions is their connection to polylogarithmic functions, [7,8], which bridge a gap in our mathematical knowledge between Airy's inhomoheneous ordinary differential equation (ODE) with homogeneities due to special functions, such as the sigmoid logistic function. This connection was recently studied and established by Roach and Hamdan, [9], and Hamdan and Roach, [10], whose work underscored the importance of connections between Airy's functions, special functions and the Nield-Kuznetsov integral functions. Their work has inevitably lead to the current work where a connection is being sought between the Einstein functions, the Nield-Kuznetsov functions and the classic Airy's functions, [11,12].

The objective of this work is to provide particular and general solutions to Airy's inhomogeneous ordinary differential equation (1), below, when the inhomogeneity is due to Einstein functions. As already has been established in the literature, solutions to Airy's inhomogeneous equation are expressed in terms of Airy's functions of the first and second kind, and the standard Nield-Kuznetsov functions of the first and second kind, [13]. In order to put this in perspective, the interest is in the following form of Airy's inhomogeneous ODE:

$$y'' - xy = f(x) \tag{1}$$

wherein "prime" notation denotes ordinary differentiation with respect to the independent variable, and the forcing function $f(x)$ is chosen in this work to be one of Einstein's functions, [1].

Solutions to the inhomogeneous ODE (1) are rare. In their elegant analysis, Miller and Musri, [14], introduced a specific-purpose method for solving (1); alas, the method is hardly practical and imposes restrictions on $f(x)$. A general-purpose approach was introduced by Hamdan and Kamel,[13], and can easily provide the general solution to (1) as long as $f(x)$ is a differentiable function. Hamdan and Kamel, [13], expressed the general solution to (1) as:

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi K_i(x) - \pi f(x) N_i(x) \tag{2}$$

where $N_i(x)$ in the Standard Nield-Kuznetsov function of the first kind, defined by, [15]:

$$N_i(x) = A_i(x) \int_0^x B_i(t)dt - B_i(x) \int_0^x A_i(t)dt \tag{3}$$

and $K_i(x)$ is the Standard Nield-Kuznetsov function of the second kind. The integral function, $K_i(x)$, is defined by the following equivalent forms, [13]:

$$K_i(x) = A_i(x) \int_0^x \left\{\int_0^\tau B_i(t)dt\right\} f'(\tau)d\tau - B_i(x) \int_0^x \left\{\int_0^\tau A_i(t)dt\right\} f'(\tau)d\tau \tag{4}$$

$$K_i(x) = f(x)N_i(x) - \left\{A_i(x) \int_0^x f(t)B_i(t)\, dt - B_i(x) \int_0^x f(t)A_i(t)\, dt\right\} \tag{5}$$

The particular solution to (1) can thus be written in one of the equivalent forms:

$$y_p = \pi K_i(x) - \pi f(x) N_i(x) \tag{6}$$

$$y_p = \pi\left\{B_i(x) \int_0^x f(t)A_i(t)\, dt - A_i(x) \int_0^x f(t)B_i(t)\, dt\right\} \tag{7}$$

This approach is robust and versatile, and both forms, (6) and (7), have recently been used to obtain particular solutions to (1) when $f(x) = A_i(x), B_i(x)$, or $N_i(x)$, [16], and when $f(x) = S(x)$ is the sigmoid logistic function, [10].

It is worth noting that the work of Hamdan and Roach, [10], helped to establish a mathematically significant connection between Airy's functions, the sigmoid functions and the dilogarithm function, and between the sigmoid function and the Nield-Kuznetsov functions, $Ni(x)$ and $Ki(x)$. These forms of forcing functions gave rise to interesting integrals involving products of these special functions, and introduced new functions, such as the dilogarithm function, as building blocks of solutions to Airy's ODE.

Those arising integrals enrich our mathematical knowledge and expand applicability of Airy's inhomogeneous ODE to potential and arising subfields of mathematical physics, [17]. They also motivate the current work in which the interest is to solve ODE (1) when $f(x)$ is an Einstein's function. The objective is to establish a connection between Einstein's functions, Airy's functions and the Nield-Kuznetsov functions. The rise of polylogarithm functions is inevitable and helps connect these integral functions using ***Theorem 1*** in this work.

2 Solution to Airy's Inhomogeneous ODE with Einstein Forcing Functions

The general solution to the homogeneous part of Airy's ODE (1) is given by the following complementary function:

$$y_c = c_1 A_i(x) + c_2 B_i(x) \tag{8}$$

where c_1 and c_2 are arbitrary constants, and $A_i(x)$ and $B_i(x)$ are the linearly independent Airy's functions of the first and second kind, respectively, and whose non-zero Wronskian is given by, [1]:

$$W\big(A_i(x), B_i(x)\big) = A_i(x) \frac{dB_i(x)}{dx} - B_i(x) \frac{dA_i(x)}{dx} = \frac{1}{\pi} \tag{9}$$

Now, consider Airy's ODE (1) with $f(x) = Ei(x)$, where $Ei(x)$ is an Einstein function, namely

$$y'' - xy = Ei(x) \tag{10}$$

General solution to (10) is of the form

$$y = c_1 Ai(x) + c_2 Bi(x) + y_p \tag{11}$$

where y_p is given by the following equivalent forms:

$$y_p = \pi Ki(x) - \pi Ei(x)Ni(x) \quad (12)$$

$$y_p = \pi\{Bi(x)\int_0^x Ei(t)Ai(t)\,dt - Ai(x)\int_0^x Ei(t)Bi(t)\,dt\} = \pi(I - J) \quad (13)$$

where

$$I = Bi(x)\int_0^x Ei(t)Ai(t)\,dt = Ai(x)Bi(x)\int_0^x Ei(t)dt - Ai'(x)Bi(x)\int_0^x\{\int_0^\tau Ei(t)dt\}\,d\tau \quad (14)$$

and

$$J = Ai(x)\int_0^x Ei(t)Bi(t)\,dt = Ai(x)Bi(x)\int_0^x Ei(t)dt - Ai(x)Bi'(x)\int_0^x\{\int_0^\tau Ei(t)dt\}\,d\tau \quad (15)$$

Using (14) and (15) in (13) yields

$$y_p = \pi[Ai(x)Bi'(x) - Ai'(x)Bi(x)]\int_0^x\{\int_0^\tau Ei(t)dt\} \quad (16)$$

Using (9) in (16) yields

$$y_p = \int_0^x\{\int_0^\tau Ei(t)dt\} \quad (17)$$

In what follows equation (17) is evaluated for the following four Einstein functions:

$$E_1(x) = \frac{x}{e^x-1} \quad (18)$$

$$E_2(x) = \log(1 - e^{-x}) \quad (19)$$

$$E_3(x) = \frac{x}{e^x-1} - \log(1 - e^{-x}) \quad (20)$$

$$E_4(x) = \frac{x^2e^x}{(e^x-1)^2} \quad (21)$$

Although graphs are provided for the functions representing the particular solutions in intervals around $x = 0$, it is emphasized here that the particular solutions obtained for Airy's inhomogeneous ODE in this work are valid for $x \geq 0$. In the notation below, log refers to the natural logarithm.

Case 1: $\boldsymbol{E1(x) = \frac{x}{e^x-1}}$

Integrating $E1(x)$ yields a convergent improper integral that can be used to obtain y_p, as follows.

$$\int_0^\tau E1(t)dt = \lim_{r\to0^+}\int_r^\tau \frac{t}{e^t-1}dt = \tau log(1 - e^{-\tau}) - L_{i2}(e^{-\tau}) - \frac{\pi^2}{6} \quad (22)$$

$$y_p = \int_0^x\{\int_0^\tau E1(t)dt\}\,d\tau = \lim_{r\to0^+}\int_r^x\left[\tau log(1 - e^{-\tau}) - L_{i2}(e^{-\tau}) - \frac{\pi^2}{6}\right]d\tau \quad (23)$$

Evaluating the limit, y_p takes the following final form:

$$y_p = xL_{i2}e^{-x} + 2L_{i3}e^{-x} - \frac{\pi^2x}{6} - 2\zeta(3) \quad (24)$$

and the following general solution is then obtained:

$$y = c_1Ai(x) + c_2Bi(x) + xL_{i2}e^{-x} + 2L_{i3}e^{-x} - \frac{\pi^2x}{6} - 2\zeta(3) \quad (25)$$

where $\zeta(x)$ is the zeta function and $\zeta(3) = 1.2020569$.

Graph of the particular solution is shown in **Fig. 1**, below.

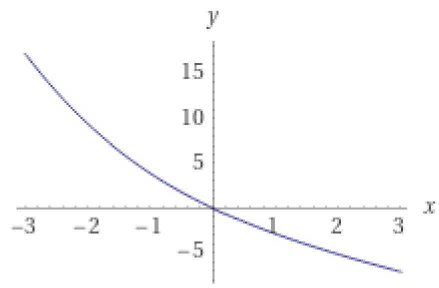

Fig. 1. Graph of $\boldsymbol{y_p}$ when $\boldsymbol{f(x) = E1(x)}$

Case 2: $\boldsymbol{E2(x) = \log(1 - e^{-x})}$

Integrating $E2(x)$ yields a convergent improper integral that can be used to obtain y_p, as follows.

$$\int_0^\tau E2(t)dt = \lim_{r\to0^+}\int_r^\tau \log(1 - e^{-t})\,d\tau = L_{i2}(e^{-\tau}) - \frac{\pi^2}{6} \quad (26)$$

$$y_p = \int_0^x\{\int_0^\tau E1(t)dt\}\,d\tau = \lim_{r\to0^+}\int_r^x\left[L_{i2}(e^{-\tau}) - \frac{\pi^2}{6}\right]d\tau \quad (27)$$

Evaluating the limit, y_p takes the following final form:

$$y_p = \left[-L_{i3}(e^{-x}) - \frac{x\pi^2}{6}\right] - [L_{i3}(1)] = -\left[L_{i3}(e^{-x}) + \frac{x\pi^2}{6}\right] - \zeta(3) \quad (28)$$

and the following general solution is then obtained:

$$y = c_1 Ai(x) + c_2 Bi(x) - \left[L_{i3}(e^{-x}) + \frac{x\pi^2}{6}\right] - \zeta(3) \quad (29)$$

Graph of the particular solution is shown in **Fig. 2**, below.

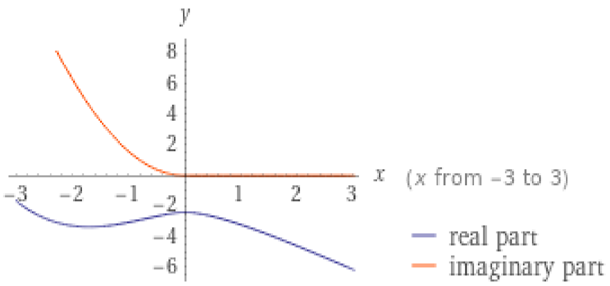

Fig. 2. Graph of y_p when $f(x) = E2(x)$

Case 3: $E_3(x) = \frac{x}{e^x - 1} - \log(1 - e^{-x})$

Integrating $E3(x)$ yields a convergent improper integral that can be used to obtain y_p, as follows.

$$\int_0^\tau E3(t)dt = \lim_{r\to 0^+} \int_r^\tau \left[\frac{t}{e^t - 1} - \log(1 - e^{-t})\right] dt = \tau log(1 - e^{-\tau}) - 2L_{i2}(e^{-\tau}) + \frac{\pi^3}{3} \quad (30)$$

$$y_p = \int_0^x \left\{\int_0^\tau E3(t)dt\right\} d\tau = \lim_{r\to 0^+} \int_r^x \left[\tau log(1 - e^{-\tau}) - 2L_{i2}(e^{-\tau}) + \frac{\pi^3}{3}\right] d\tau \quad (31)$$

Evaluating the limit, y_p takes the following final form:

$$y_p = -\left\{x[L_{i2}(e^{-x})] + 3L_{i3}(e^{-x}) + \frac{\pi^3}{3}x\right\} - 6\,\zeta(3) \quad (32)$$

and the following general solution is then obtained:

$$y = c_1 Ai(x) + c_2 Bi(x) - \left\{x[L_{i2}(e^{-x})] + 3L_{i3}(e^{-x}) + \frac{\pi^3}{3}x\right\} - 6\,\zeta(3) \quad (33)$$

Graph of the particular solution is shown in **Fig. 3**, below.

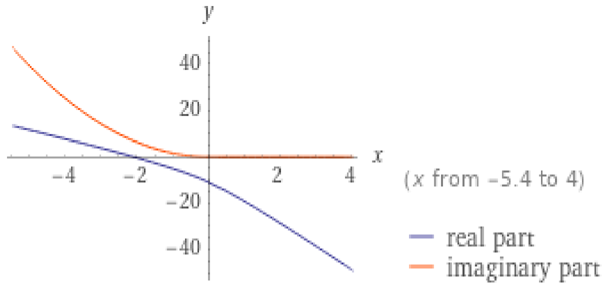

Fig. 3. Graph of y_p when $f(x) = E3(x)$

Case 4: $E_4(x) = \frac{x^2 e^x}{(e^x - 1)^2}$

Integrating $E3(x)$ yields a convergent improper integral that can be used to obtain y_p, as follows.

$$\int_0^\tau E4(t)dt = \lim_{r\to 0^+} \int_r^\tau \frac{t^2 e^t}{(e^t - 1)^2} dt = 2L_{i2}(e^\tau) + 2\,\tau\log(1 - e^\tau) + \frac{\tau^2 e^\tau}{1 - e^\tau} - \frac{\pi^2}{3} \quad (34)$$

$$y_p = \int_0^x \left\{\int_0^\tau E4(t)dt\right\} d\tau = \lim_{r\to 0^+} \int_r^x \left[2L_{i2}(e^\tau) + 2\,\tau\log(1 - e^\tau) + \frac{\tau^2 e^\tau}{1 - e^\tau} - \frac{\pi^2}{3}\right] d\tau \quad (35)$$

Evaluating the limit, y_p takes the following final form:

$$y_p = -x^2 log(1 - e^x) - 4xL_{i2}(e^x) + 6L_{i3}(e^x) - \frac{x\pi^2}{3} - 6\,\zeta(3) \quad (36)$$

and the following general solution is then obtained:

$$y = c_1 Ai(x) + c_2 Bi(x) - x^2 log(1 - e^x) - 4xL_{i2}(e^x) + 6L_{i3}(e^x) - \frac{x\pi^2}{3} - 6\,\zeta(3) \quad (37)$$

Graph of the particular solution is shown in **Fig. 4**, below.

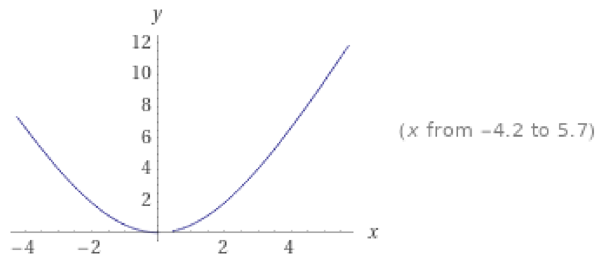

Fig. 4. Graph of y_p when $f(x) = E4(x)$

3 Asymptotic Series Representations

When $x \gg 1$, Airy's and the Nield-Kuznetsov functions have the following asymptotic series representations (*cf.* Hamdan and Kamel, [13], and the references therein) that can be used in the evaluation of the general solution of ODE (1):

$$Ai(x) \approx \frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}} \quad (38)$$

$$Bi(x) \approx \frac{\exp(\mu)}{\sqrt{\pi}x^{\frac{1}{4}}} \quad (39)$$

$$Ni(x) \approx -\frac{\exp(\mu)}{3\sqrt{\pi}x^{\frac{1}{4}}} \quad (40)$$

$$Ki(x) \approx \frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}} \int_0^x \left\{\frac{\exp(\varphi)}{\sqrt{\pi}t^{\frac{3}{4}}}\right\} Ei'(t)dt - \frac{\exp(\mu)}{3\sqrt{\pi}x^{\frac{1}{4}}} Ei(x) \quad (41)$$

wherein $\mu = \frac{2}{3}x^{\frac{3}{2}}$ and $\varphi = \frac{2}{3}x^{\frac{2}{3}}$.

While asymptotic series representations of $Ai(x)$, $Bi(x)$, and $Ni(x)$ are independent of the forcing function of Airy's ODE (1), that of $Ki(x)$ takes into account the forcing function. Equation (41) illustrates how Einstein's function, $Ei(x)$, fits into the asymptotic series representation of $Ki(x)$.

4 Relationships between Einstein's, Airy's and the Nield-Kuznetsov Functions

Equations (12), (13) and (17) establish the following theorem on relationships between Einstein functions, $Ei(x)$, Airy's functions, $Ai(x)$ and $Bi(x)$, and the Nield-Kuznetsov functions$Ni(x)$ and $Ki(x)$:

Theorem 1: *Airy's functions of the first and second kind, the Nield-Kuznetsov functions of the forst and second kind, and Einstein's function, are related by:*

$$Ki(x) = Ei(x)Ni(x) + \frac{1}{\pi}\int_0^x \left\{\int_0^\tau Ei(t)dt\right\} d\tau \quad (i)$$

$$Ki(x) = Ei(x)\left\{Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt\right\} + \frac{1}{\pi}\int_0^x \left\{\int_0^\tau Ei(t)dt\right\} \quad (ii)$$

Relationships involving the polylogarithm functions are developed using the obtained particular solutions, and take the following forms:

$$Ki(x) = E1(x)Ni(x) + \frac{x}{\pi}L_{i2}e^{-x} + \frac{2}{\pi}L_{i3}e^{-x} - \frac{\pi x}{6} - \frac{2}{\pi}\zeta(3) \quad (42)$$

$$Ki(x) = E2(x)Ni(x) - \left[\frac{1}{\pi}L_{i3}(e^{-x}) + \frac{x\pi}{6}\right] - \frac{1}{\pi}\zeta(3) \quad (43)$$

$$Ki(x) = E3(x)Ni(x) - \left\{\frac{x}{\pi}L_{i2}(e^{-x}) + \frac{3}{\pi}L_{i3}(e^{-x}) + \frac{\pi^2}{3}x\right\} - \frac{6}{\pi}\zeta(3) \quad (44)$$

$$Ki(x) = E4(x)Ni(x) - \frac{x^2}{\pi}log(1-e^x) - \frac{4x}{\pi}L_{i2}(e^x) + \frac{6}{\pi}L_{i3}(e^x) - \frac{x\pi}{3} - \frac{6}{\pi}\zeta(3) \quad (45)$$

5 Conclusion

In this work, particular solutions and general solutions to the inhomogeneous Airy's ordinary differential equation were obtained when the sources of inhomogeneity are Einstein's functions. The obtained solutions were expressed in terms of Airy's functions, the Nield-Kuznetsov functions and the polylogarithm functions. These solutions set the stage for the challenging tasks of computing solutions to initial and boundary value problems, and establish a connection between Airy's functions, the Nield-Kuznetsov functions and Einstein's functions.

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Contribution of individual authors

Both authors reviewed the literature, formulated the problem, provided independent analysis, and jointly wrote and revised the manuscript.

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