Taylor and Maclaurin Series Representations of the Nield-Kuznetsov Function of the First Kind

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Abstract: - Taylor and Maclaurin series and polynomial approximations of the Standard Nield-Kuznetsov function of the first kind are obtained in this work. Convergence and error criteria are developed. The obtained series represent alternatives to the existing asymptotic and ascending series approximations of this integral function, and are expected to provide an efficient method of computation that is valid for all values of the argument.

Key-Words: - Taylor series, Standard Nield-Kuznetsov function.

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1 Introduction

A series of recent articles discussed the importance of Airy's ordinary differential equation (ODE) in fundamental research in the fields of circuits, systems and signal processing (*c.f.* [1-3] and the references therein). Applications of Airy's ODE to the study of Schrodinger and Tricomi equations have also been emphasized, in addition to its importance in the analysis of Stark equation and the study of Stark effect, [1-4]. Recent research in the area of Airy's ODE reflects the fundamental importance of seeking solutions to the inhomogeneous version, and the need for representations and efficient computations of the integral functions that arise in the processes of obtaining its general and particular solutions, [5-8].

The above needs give ris to the current work whose main objective is to develop Taylor series representation, and Taylor polynomial approximation, to the Standard Nield-Kuznetsov function of the first kind, $Ni(x)$, [5,9]. This function arises in the general solution to Airy's, [10], inhomogeneous ordinary differential equation of the form, [11]:

$$\frac{d^2y}{dx^2} - xy = R \tag{1}$$

where R is any constant.

If $R = 0$, general solution of (1) is given by

$$y = c_i Ai(x) + c_2 Bi(x) \tag{2}$$

where c_1, c_2 are arbitrary constants and $Ai(x)$ and $Bi(x)$ are the two linearly independent functions known as Airy's homogeneous functions of the first and second kind, respectively. These functions are defined by the following integrals, [8]:

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right)dt \tag{3}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{t^3}{3}\right) + \exp\left(xt - \frac{t^3}{3}\right)dt \tag{4}$$

The Wronskian of $Ai(x)$ and $Bi(x)$ is non-zero, as given by, [13]:

$$W\big(Ai(x), Bi(x)\big) = Ai(x)Bi'(x) - Bi(x)Ai'(x) \quad (5)$$

If $R = \frac{1}{\pi}$ or $-\frac{1}{\pi}$, general solutions to (1) are given, respectively, by

$$y = c_1 Ai(x) + c_2 Bi(x) + Hi(x) \quad (6)$$

$$y = c_1 Ai(x) + c_2 Bi(x) + Gi(x) \quad (7)$$

where the functions $Gi(x)$ and $Hi(x)$ are the Scorer functions, [12], given by

$$Gi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{t^3}{3}\right)dt \quad (8)$$

$$Hi(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{t^3}{3}\right)dt \quad (9)$$

If $R \neq \mp\frac{1}{\pi}$, obtaining a general solution to (1) in terms of the Scorer functions requires non-trivial changes of variables, [8]. A practical need to solve (1) for values of $R \neq \mp\frac{1}{\pi}$ arose in the analysis of the transition layer by Nield and Kuznetsov, [9], who found it convenient and necessary to introduce an integral function, $Ni(x)$, defined by

$$Ni(x) = Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt. \quad (10)$$

Hamdan and Kamel, [5], showed that $Ni(x)$ possesses the integral representation

$$Ni(x) = \frac{2}{3\pi}\int_0^\infty \sin\left(xt + \frac{t^3}{3}\right)dt - \frac{1}{3\pi}\int_0^\infty \exp\left(xt - \frac{t^3}{3}\right)dt \quad (11)$$

and provided the following general solution to (1) in terms of $Ni(x)$, which they called the Standard Nield-Kuznetsov Function of the First Kind:

$$y = c_1 Ai(x) + c_2 Bi(x) - \pi R Ni(x) \quad (12)$$

Several properties of $Ni(x)$ were explored by Hamdan and Kamel, [5], and further features continue to arise in the literature alongside its non-trivial computations which invariably rely heavily on the infinite series representations. In particular, the following series representations for $Ni(x)$ have been developed and used in its computations, [5,14,15]:

1.1 Asymptotic Series Approximation:
Based on asymptotic series representations of Airy's functions, $Ai(x)$ and $Bi(x)$, Hamdan and Kamel, [5], obtained the following asymptotic series for $Ni(x)$:

$$Ni(x) = \frac{1}{2\pi x^2} - \frac{\exp(\mu)}{3\sqrt{\pi}x^{\frac{1}{4}}} \quad (13)$$

where $= \frac{2}{3}x^{3/2}$. If x is large, (13) can be approximated by the following, [9]:

$$Ni(x) = -\frac{\exp(\mu)}{3\sqrt{\pi}x^{\frac{1}{4}}} \quad (14)$$

1.2 Ascending Series Representation:
Airy's particles, a_1 and a_2, are defined as, [8, 13]:

$$a_1 = A_i(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)} \quad (15)$$

$$a_2 = -A'_i(0) = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)} \quad (16)$$

wherein $\Gamma(.)$ is the Gamma function. The Airy's particles and the following two series

$$F_1(x) = \sum_{k=0}^\infty \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \quad (17)$$

$$F_2(x) = \sum_{k=0}^\infty \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \quad (18)$$

which employ the *Pochhammer symbol*, or *shifted factorial*

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)(b+2)\dots(b+k-1); \quad (b)_0 = 1 \quad (19)$$

were employed by Hamdan and Kamel, [5], to obtain the following expression of $Ni(x)$:

$$Ni(x) = 2\sqrt{3}a_1 a_2\{F_2 F'_1 - F'_2\} \quad (20)$$

or, equivalently

$$Ni(x) = 2\sqrt{3}a_1 a_2 \begin{bmatrix} \left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!}\right\} * \\ \left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!}\right\} \\ -\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}\right\} * \\ \left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}\right\} \end{bmatrix} \quad (21)$$

Using Cauchy product, (21) can be written as, [14]:

$$Ni(x) = 2\sqrt{3}a_1 a_2 * \sum_{k=0}^{\infty} 3^k x^{3k+2} \left\{\sum_{l=0}^{k} \left(\frac{\left(\frac{1}{3}\right)_l \left(\frac{2}{3}\right)_{k-l} \\ -3k+6l-1}{(3l+1)!\,(3(k-l)+2)!}\right)\right\} \quad (22)$$

Series representations (21) and (22) are the ascending series representations of $Ni(x)$. They can be used in the efficient computations of $Ni(x)$ for small enough values of x, [14].

Both asymptotic and ascending series underscore the importance of investigating properties and representations of functions associated with the solutions of Airy's homogeneous and inhomogeneous equation. In addition to providing invaluable insight into the behaviour of solutions to Airy's equation and the expanded applications in mathematical physics, the relationships these arising functions have with other functions of mathematical physics serves not only to enrich, but to potentially expand and deepen mathematical knowledge (*cf.* [16-18] and the references therein). This motivates the current work whose scope is to examine representations of $Ni(x)$ using Taylor and Maclaurin series, and its approximations using Taylor and Maclaurin polynomials.

2 Taylor Series Expansion of $Ni(x)$

The function $Ni(x)$ is a smooth function with an n^{th} derivative expressible in terms of Airy's polynomials, [3]. It can therefore be expanded in a Taylor series, about $x = x_0$, of the form:

$$Ni(x) = \sum_{k=0}^{\infty} C_k (x-x_0)^k = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \cdots + C_n(x-x_0)^n + \cdots \quad (23)$$

where

$$C_k = \frac{Ni^{(k)}(x_0)}{k!} \quad (24)$$

and $Ni^{(k)}(x_0)$ denotes the k^{th} derivative of $Ni(x)$ evaluated at $x = x_0$.

If $x_0 = 0$ then Taylor series becomes Maclaurin series, namely:

$$Ni(x) = \sum_{k=0}^{\infty} C_k x^k \quad (25)$$

where

$$C_k = \frac{Ni^{(k)}(0)}{k!} \quad (26)$$

2.1 Derivatives of $Ni(x)$

In a recent article, Hamdan *et.al.*, [3], obtained expressions for the higher derivatives of $Ni(x)$. The first ten derivatives can be obtained from equation(10) by direct differentiation, and are tabulated below.

Table 1. The first ten derivatives of $Ni(x)$

$$Ni^{(1)}(x) = N'i(x) = Ai'(x)\int_0^x Bi(t)dt - Bi'(x)\int_0^t Ai(t)dt$$

$$Ni^{(2)}(x) = N''i(x) = xNi(x) - \frac{1}{\pi}$$

$$Ni^{(3)}(x) = Ni(x) + xN'i(x)$$

$$Ni^{(4)}(x) = x^2 Ni(x) + 2N'i(x) - \frac{x}{\pi}$$

$$Ni^{(5)}(x) = 4xNi(x) + x^2 N'i(x) - \frac{3}{\pi}$$

$$Ni^{(6)}(x) = (x^3+4)Ni(x) + 6xN'i(x) - \frac{x^2}{\pi}$$

$$Ni^{(7)}(x) = 9x^2 Ni(x) + (x^3 + 10)N'i(x) - \frac{8x}{\pi}$$

$$Ni^{(8)}(x) = (x^4 + 28x)Ni(x) + 12x^2 N'i(x) - \frac{(x^3 + 18)}{\pi}$$

$$Ni^{(9)}(x) = (16x^3 + 28)Ni(x) + (x^4 + 52x)N'i(x) - \frac{15x^2}{\pi}$$

$$Ni^{(10)}(x) = (x^5 + 100x^2)Ni(x) + (20x^3 + 80)N'i(x) - \frac{(x^4 + 82x)}{\pi}$$

The second and higher derivatives of $Ni(x)$ can be expressed in terms of $Ni(x)$, $Ni'(x)$ and $W\big(Ai(x), Bi(x)\big)(=\frac{1}{\pi})$ with coefficients that are polynomials in x. The k^{th} derivative of $Ni(x)$, for $k \geq 2$, can then be expressed as

$$Ni^{(k)}(x) = P_k(x)Ni(x) + Q_k(x)N'i(x) - \frac{R_k(x)}{\pi} \tag{27}$$

For derivative orders $k = 2$ to 10, **Table 2** lists $P_k(x), Q_k(x),$ and $R_k(x)$:

Table 2. Polynomial Coefficients of $Ni(x)$, $Ni'(x)$ and $W\big(Ai(x), Bi(x)\big)$

k	$P_k(x)$	$Q_k(x)$	$R_k(x)$
2	x	0	1
3	1	x	0
4	x^2	2	x
5	$4x$	x^2	3
6	$x^3 + 4$	$6x$	x^2
7	$9x^2$	$x^3 + 10$	$8x$
8	$x^4 + 28x$	$12x^2$	$x^3 + 18$
9	$16x^3 + 28$	$x^4 + 52x$	$15x^2$
10	$x^5 + 100x^2$	$20x^3 + 80$	$x^4 + 82x$

More generally, the degrees of these polynomials may be determined for arbitrary k, and are provided in the following **Table 3** in terms of the floor function.

Table 3. Degrees of Coefficient Polynomials

Polynomial	Degree
$P_k(x)$	$3\left\lfloor\frac{k-2}{2}\right\rfloor - k + 3,\ k \geq 2$
$Q_k(x)$	$3\left\lfloor\frac{k-3}{2}\right\rfloor - k + 4,\ k \geq 3$
$R_k(x)$	$3\left\lfloor\frac{k-4}{2}\right\rfloor - k + 5,\ k \geq 5$

The k+1st derivative of $Ni(x)$ takes the form:

$$Ni^{(k+1)}(x) = P_{k+1}(x)Ni(x) + Q_{k+1}(x)Ni'(x) - \frac{R_{k+1}(x)}{\pi}. \tag{28}$$

In order to be able to compute the k+1st derivative with the knowledge of the k^{th} derivative, Hamdan *et.al.*, [3], established the following relationships between the polynomial coefficients in (27) and (28):

$$P_{k+1}(x) = P'_k(x) + xQ_k(x) \tag{29}$$

$$Q_{k+1}(x) = Q'_k(x) + P_k(x) \tag{30}$$

$$R_{k+1}(x) = R'_k(x) + Q_k(x) \tag{31}$$

Once the k^{th} derivative is known, the polynomial coefficients from (27) can be used in (29)-(31) to compute the polynomial coefficients in (28).

Now, using (27) in (24), the following coefficients are obtained in the Taylor series expansion of $Ni(x)$:

$$C_k = \frac{Ni^{(k)}(x_0)}{k!} = \{P_k(x_0)Ni(x_0) + Q_k(x)N'i(x_0) - \frac{R_k(x_0)}{\pi}\}/k! \quad (32)$$

and (23) can be written as:

$$Ni(x) = Ni(x_0) + Ni'(x_0)(x - x_0) + Ni(x_0)\sum_{k=2}^{\infty}\frac{P_k(x_0)(x-x_0)^k}{k!} + N'i(x_0)\sum_{k=2}^{\infty}\frac{Q_k(x_0)(x-x_0)^k}{k!} - \frac{1}{\pi}\sum_{k=2}^{\infty}\frac{R_k(x_0)(x-x_0)^k}{k!} \quad (33)$$

In order to find $P_k(x)$, $Q_k(x)$ and $R_k(x)$ in terms of previously calculated polynomials, (29)-(31) are used in the form:

$$P_k(x) = P'_{k-1}(x) + xQ_{k-1}(x) \quad (34)$$

$$Q_k(x) = Q'_{k-1}(x) + P_{k-1}(x) \quad (35)$$

$$R_k(x) = R'_{k-1}(x) + Q_{k-1}(x) \quad (36)$$

Using (34)-(36) in (32), the following coefficients are obtained:

$$C_k = \frac{\{[P'_{k-1}(x_0) + x_0Q_{k-1}(x_0)]Ni(x_0) +\}}{k!} + \frac{\{[Q'_{k-1}(x_0) + P_{k-1}(x_0)]N'i(x_0)\}}{k!} - \frac{\left\{\frac{[R'_{k-1}(x_0)+Q_{k-1}(x_0)]}{\pi}\right\}}{k!} \quad (37)$$

Equation (45)(33) can then by replaced by the following final form of Taylor series expansion of $Ni(x)$ about $x = x_0$:

$$Ni(x) = Ni(x_0) + Ni'(x_0)(x - x_0) + \frac{1}{2}Ni''(x_0)(x - x_0)^2 + Ni(x_0)\sum_{k=3}^{\infty}\frac{\{P'_{k-1}(x_0)+x_0Q_{k-1}(x_0)\}(x-x_0)^k}{k!} + N'i(x_0)\sum_{k=3}^{\infty}\frac{\{Q'_{k-1}(x_0)+P_{k-1}(x_0)\}(x-x_0)^k}{k!} - \frac{1}{\pi}\sum_{k=3}^{\infty}\frac{\{R'_{k-1}(x_0)+Q_{k-1}(x_0)\}(x-x_0)^k}{k!} \quad (38)$$

If $x_0 = 0$ then $Ni(0) = Ni'(0) = 0$ and

$$Ni^{(k)}(0) = -\frac{R_k(0)}{\pi} \quad (39)$$

$$Ni^{(k+1)}(0) = -\frac{R_{k+1}(0)}{\pi} = -\frac{[Q_k(0)+R'_k(0)]}{\pi} \quad (40)$$

Equation (38) becomes the following Maclaurin series expansion of $Ni(x)$:

$$Ni(x) = -\frac{1}{2\pi}x^2 - \frac{1}{\pi}\sum_{k=3}^{\infty}\frac{\{R'_{k-1}(0)+Q_{k-1}(0)\}x^k}{k!} \quad (41)$$

3 Convergence of Taylor Series of $Ni(x)$

This series converges for values of x satisfying $|x - x_0| < r$, where r is the radius of convergence defined by

$$\frac{1}{r} = \lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right| = \lim_{k\to\infty}\left|\frac{Ni^{(k+1)}(x_0)}{(k+1)Ni^{(k)}(x_0)}\right| = \lim_{k\to\infty}\left|\frac{L}{(k+1)}\right| = 0 \quad (42)$$

where $L = \frac{Ni^{(k+1)}(x_0)}{Ni^{(k)}(x_0)}$ is finite since the maximum degrees of the polynomials involved in $Ni^{(k)}(x_0)$ and $Ni^{(k+1)}(x_0)$ are comparable. Hence, the radius of convergence r is infinite and series (38) converges for all x. The same is true for Maclaurin series (41).

This furnishes the following Theorem on convergence.

Theorem 1.
The Taylor series expansion (38) of $Ni(x)$ about x_0 converges for all values of x.

4 Values of Polynomials and Derivatives of $Ni(x)$ at $x = 0$

Values of polynomials $P_k(x), Q_k(x),$ and $R_k(x)$ and derivatives of $Ni(x)$ at $x = 0$ are shown in the following **Table 4** for $k = 2$ to 15. At the outset, it is noted that for any given derivative, only one of the polynomials is non-zero at $x = 0$. Furthermore, $Ni^{(k)}(0)$ is non-zero whenever $R_k(0)$ is non-zero, and the following recursive relations can easily be established:

$$R_{k+3}(0) = (k+1)R_k(0);\ k \geq 2 \tag{43}$$

$$P_{k+3}(0) = (k+1)P_k(0);\ k \geq 3 \tag{44}$$

$$Q_{k+3}(0) = (k+1)Q_k(0);\ k \geq 4 \tag{45}$$

$$Ni^{(k)}(0) = -\frac{R_k(0)}{\pi}$$

$$= (k-2)Ni^{(k-3)}(0);\ k \geq 3 \tag{46}$$

$$Ni^{(k+1)}(0) = -\frac{R_{k+1}(0)}{\pi}$$
$$= (k-1)Ni^{(k-2)}(0);\ k \geq 2 \tag{47}$$

Table 4. Values of Coefficient Polynomials and Derivatives of $Ni(x)$ at $x = 0$

k	$P_k(0)$	$Q_k(0)$	$R_k(0)$	$Ni^{(k)}(0)$
2	*0*	0	1	$-1/\pi$
3	*1*	0	0	0
4	0	2	0	0
5	0	0	3	$-3/\pi$
6	4	0	0	0
7	0	10	0	0
8	0	0	18	$-18/\pi$
9	28	0	0	0
10	0	80	0	0
11	0	0	162	$-162/\pi$
12	280	0	0	0
13	0	880	0	0
14	0	0	1944	$-1944/\pi$
15	3640	0	0	0

It may also be convenient to note the closed formulae for the respective values as represented in the following table. As indicated, the values depend on the congruence of k modulo 3, and employ the *triple factorial notation,* $\mathrm{n!!!} = \mathrm{n(n-3)(n-6)(n-9)\cdots(n-a)};\ 3 \geq \mathrm{(n-a)} > 0$.

Table 5. Values of Coefficient Polynomials and $Ni^{(k)}(x)$ at $x = 0$ as related to congruence of k modulo 3.

k	$P_k(0)$	$Q_k(0)$	$R_k(0)$	$Ni^{(k)}(0)$
$k = 3m$	$(3m-2)!!!$	0	0	0
$k = 3m+1$	*0*	$(3m-1)!!!$	0	0
$k = 3m+2$	0	2	$(3m)!!!$	$-\frac{(3m)!!!}{\pi}$

5 Taylor Polynomial Approximation to $Ni(x)$

If the Taylor series of $Ni(x)$ is terminated after *n+1* terms, then a Taylor polynomial, $T_n(x)$, of degree *n* results. This polynomial approximates the function $Ni(x)$ *near* $x = x_0$, namely

$$Ni(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{Ni^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= Ni(x_0) + Ni'(x_0)(x-x_0) + Ni''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + Ni^{(n)}(x_0)\frac{(x-x_0)^n}{n!} \quad (48)$$

Equation (48) takes the following form in terms of Airy's polynomials:

$$Ni(x) \approx T_n(x) = Ni(x_0) + Ni'(x_0)(x-x_0) + \frac{1}{2}Ni''(x_0)(x-x_0)^2$$

$$+Ni(x_0)\sum_{k=3}^{n} \frac{\{P'_{k-1}(x_0) + x_0 Q_{k-1}(x_0)\}(x-x_0)^k}{k!}$$

$$+N'i(x_0)\sum_{k=3}^{n} \frac{\{Q'_{k-1}(x_0) + P_{k-1}(x_0)\}(x-x_0)^k}{k!}$$
$$-\frac{1}{\pi}\sum_{k=3}^{n} \frac{\{R'_{k-1}(x_0) + Q_{k-1}(x_0)\}(x-x_0)^k}{k!} \quad (49)$$

If $x_0 = 0$ then the Taylor polynomial becomes *Maclaurin polynomial*, $M_n(x)$:

$$Ni(x) \approx M_n(x) = -\frac{1}{2\pi}x^2 - \frac{1}{\pi}\sum_{k=3}^{n} \frac{\{R'_{k-1}(0) + Q_{k-1}(0)\}\, x^k}{k!} \quad (50)$$

$$= -\frac{1}{2\pi}x^2 - \frac{1}{\pi}\sum_{k=3}^{n} \frac{R_k(0)x^k}{k!} \quad (51)$$

where $R_k(0)$ can be generated using (43).

As an example, the 14[th] degree Maclaurin polynomial approximation of $Ni(x)$ takes the form

$$M_{14}(x) = -\frac{1}{\pi}\cdot\left(\begin{array}{c}\frac{1}{2!}x^2 + \frac{3}{5!}x^5 + \frac{18}{8!}x^8 + \\ \frac{162}{11!}x^{11} + \frac{1944}{14!}x^{14}\end{array}\right) \quad (52)$$

6 Remainder and Error Terms

When approximating $Ni(x)$ by an n^{th} degree Taylor polynomial, $T_n(x)$, an error term, $E_n(x) = Ni(x) - T_n(x)$, is introduced. Explicitly, $E_n(x)$ is given by:

$$E_n(x) = \sum_{k=0}^{\infty} \frac{Ni^{(k)}(x_0)}{k!}(x-x_0)^k - \sum_{k=0}^{n} \frac{Ni^{(k)}(x_0)}{k!}(x-x_0)^k$$
$$= \sum_{k=n+1}^{\infty} \frac{Ni^{(k)}(x_0)}{k!}(x-x_0)^k \quad (53)$$

On an arbitrary interval $[a, b]$ around x_0, continuity of $Ni(x)$ and each of it's derivatives deems that $Ni^{(n+1)}(x)$ is bounded, say $|Ni^{(n+1)}(x)| \le M$. As such, *Taylor's inequality* provides

$$|E_n(x)| \le M\frac{|x-\tau|^{n+1}}{(n+1)!} \quad (54)$$

For all $\tau \in [a, b]$. Consequently,

$$0 \le |E_n(x)| \le M\frac{|x-\tau|^{n+1}}{(n+1)!} \le M\cdot\frac{(b-a)^{n+1}}{(n+1)!} \quad (55)$$

Taking limits in (55) shows

$$\lim_{n\to\infty} E_n = 0 \quad (56)$$

In other words, $Ni(x)$ is equal to it's Taylor Series (everywhere).

7 Tangent Line Approximation

If *n=1* then Taylor polynomial approximation to $Ni(x)$ becomes:

$$Ni(x) \approx T_1(x) = Ni(x_0) + Ni'(x_0)(x-x_0) \quad (57)$$

In the first derivative of $Ni(x)$, Airy's polynomials are not involved. Therefore, using (10), the following expressions for $Ni(x_0)$ and $Ni'(x_0)$ are obtained, respectively

$$Ni(x_0) = Ai(x_0)\int_0^{x_0} Bi(t)dt - Bi(x_0)\int_0^{x_0} Ai(t)dt \quad (58)$$

$$Ni'(x_0) = Ai'(x_0)\int_0^{x_0} Bi(t)dt - Bi'(x_0)\int_0^{x_0} Ai(t)dt \quad (59)$$

Using (59) and (60) in (58) yields

$$Ni(x) \approx T_1(x)$$

$$= [Ai(x_0) + (x - x_0)Ai'(x_0)]\int_0^{x_0} Bi(t)dt - [Bi(x_0) + (x - x_0)Bi'(x_0)]\int_0^{x_0} Ai(t)dt \quad (60)$$

Equation (61) is the tangent line approximation to $Ni(x)$ near $x = x_0$. It is written here in terms of Airy's functions and integrals.

Equation (58) also gives an approximation to the slope of the tangent line, $Ni'(x_0)$, in terms of the slope of the secant line, namely

$$Ni'(x_0) \approx \frac{Ni(x) - Ni(x_0)}{(x - x_0)} \quad (61)$$

If $x_0 = 0$, then the right-hand-side of (59) is zero and

$$Ni'(0) = 0. \quad (62)$$

8 Sample Results

In using 10 terms of series ascending series (21), Alzahrani et.al. obtained the following values for $Ni(x)$ when 10 decimal places are retained:

$Ni(1) = -0.1672560919$

$Ni(0.1) = -0.0015911629$

In using asymptotic series (13), which is valid for $x \gg 1$, the following approximation is obtained:

$Ni(1) = -0.2071421427$

It is believed that the computed value of $Ni(1)$ is more accurate when the ascending series is used.

By comparison, in using Maclaurin polynomial, (52), of various degrees, the following approximations are obtained for $Ni(1)$ (**Table 6**) and $Ni(0.1)$ (**Table 7**) while retaining 10 decimal places. All Maclaurin polynomials used give an excellent agreement with the values computed using ascending series (21).

Table 6. Computed Values of $Ni(1)$ Using Maclaurin Polynomials

$\boldsymbol{Ni(x) \approx M_n(x)}$	$\boldsymbol{Ni(1) \approx M_n(1)}$
$M_2(x) = -\frac{1}{\pi}\frac{x^2}{2!}$	-0.1591549430
$M_5(x) = -\frac{1}{\pi}(\frac{x^2}{2!} + \frac{3x^5}{5!})$	-0.1671126902
$M_8(x) = -\frac{1}{\pi}(\frac{x^2}{2!} + \frac{3x^5}{5!} + \frac{18x^8}{8!})$	-0.1672547928
$M_{11}(x) = -\frac{1}{\pi}(\frac{x^2}{2!} + \frac{3x^5}{5!} + \frac{18x^8}{8!} + \frac{162x^{11}}{11!})$	-0.1672560818
$M_{14}(x) = -\frac{1}{\pi}\cdot\left(\frac{x^2}{2!} + \frac{3x^5}{5!} + \frac{18x^8}{8!} + \frac{162x^{11}}{11!} + \frac{1944x^{14}}{14!}\right)$	-0.1672568251

Table 7. Computed Values of $Ni(0.1)$ Using Maclaurin Polynomials

$\boldsymbol{Ni(x) \approx M_n(x)}$	$\boldsymbol{Ni(0.1) \approx M_n(0.1)}$
$M_2(x) = -\frac{1}{\pi}\frac{x^2}{2!}$	-0.0015915494
$M_5(x) = -\frac{1}{\pi}(\frac{x^2}{2!} + \frac{3x^5}{5!})$	-0.001591628977
$M_8(x) = -\frac{1}{\pi}(\frac{x^2}{2!} + \frac{3x^5}{5!} + \frac{18x^8}{8!})$	-0.0015916289784
$M_{11}(x) = -\frac{1}{\pi}(\frac{x^2}{2!} + \frac{3x^5}{5!} + \frac{18x^8}{8!} + \frac{162x^{11}}{11!})$	-0.0015916289784
$M_{14}(x) = -\frac{1}{\pi} \cdot \left(\frac{x^2}{2!} + \frac{3x^5}{5!} + \frac{18x^8}{8!} + \frac{162x^{11}}{11!} + \frac{1944x^{14}}{14!}\right)$	-0.0015916289784

9 Conclusion

In this work, Taylor and Maclaurin series expansions of the Standard Nield-Kuznetsov function of the first kind, $Ni(x)$, were obtained in order to provide further insight into the behavior of this integral function. Convergence criteria were also investigated in order to show that Taylor series representation of $Ni(x)$ converges for all x. Errors incurred in representing this function by Taylor and Maclaurin polynomials were quantified and tangent line approximation was obtained. Results obtained in computing $Ni(x)$ using Maclaurin polynomial agree well with results obtained using ascending series representation for small values of x.

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Contribution of individual authors

Both authors reviewed the literature, formulated the problem, provided independent analysis, and jointly wrote and revised the manuscript.

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