Usage of the Fuzzy Laplace Transform Method for Solving One-Dimensional Fuzzy Integral Equations

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Abstract: - In this paper, we propose the solution of fuzzy Volterra and Fredholm integral equations with convolution type kernel using fuzzy Laplace transform method (FLTM) under Hukuhara differentiability. It is shown that FLTM is a simple and reliable approach for solving such equations analytically. Finally, the method is illustrated with few examples to show the ability of the proposed method.

Key-Words: - Fuzzy integral equation, Fuzzy Laplace transform method, Fuzzy convolution.

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1 Introduction

Modeling many problems of science, engineering, physics, and other disciplines-leads to linear and nonlinear Fredholm and Volterra integral equations of the second kind. These are usually difficult to solve analytically and in many cases the solution must be approximated. Therefore, in recent years several numerical approaches have been proposed [5, 6, 7, 20].

The basic idea and arithmetics of fuzzy sets were first introduced by Zadeh in [25]. The concept of fuzzy derivatives and fuzzy integration were studied in [10, 12] and then some generalization have been investigated in [9, 10]. The topic of fuzzy integral equations has been rapidly grown recent years.

Abbasbandy et. al [1] proposed a numerical algorithm for solving linear Fredholm fuzzy integral equations of the second kind by using parametric form of fuzzy number and converting a linear fuzzy Fredholm integral equation to two linear systems of integral equation of the second kind in crisp case. Babolian et. al [3] proposed another numerical procedure for solving fuzzy linear Fredholm integral of the second kind using Adomian method. Moreover, Friedman et. al [11] proposed an embedding method to solve fuzzy Volterra and Fredholm integral equations. However, there are several research papers about obtaining the numerical integration of fuzzy-valued functions and solving fuzzy Volterra and Fredholm integral equations [2, 4, 8, 13, 14, 15, 16, 17, 18, 22, 24].

The fuzzy Laplace transform method solves FDEs and corresponding fuzzy initial and boundary value problems [2]. In this way fuzzy Laplace transforms reduce the problem of solving a FDE to an algebraic problem [19]. This switching from operations of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics, and for engineers, the fuzzy Laplace transform method is practically the most important operational method.

Recently, Allahviranloo and Barkhordari in [2] proposed fuzzy Laplace transforms for solving first order fuzzy differential equations under generalized H-differentiability. By such benefits, we develop fuzzy Laplace transform method to solve fuzzy convolution Volterra integral equation (FCVIE) of the second kind. So, the original FCVIE is converted to two crisp convolution integral equations in order to determine the lower and upper function of solution, using fuzzy convolution operator.

The paper is organized as follows. In section 2, some basic definitions which will be used later in the paper are provided. In section 3, the fuzzy Laplace transform is studied. In section 4, the fuzzy convolution Volterra and Fredholm integral equations of the second kind with fuzzy convolution kernel is studied. Then, the fuzzy Laplace transforms are applied to solve such special fuzzy integral equation in section 5. Illustrative examples are also considered to show the ability of the proposed method in section 6, and the conclusion is drawn in section 7.

2 Preliminaries

In this section, we will recall some basics definitions and theorems needed throughout the paper such as fuzzy number, fuzzy-valued function and the derivative of the fuzzy-valued functions [9, 10, 12, 23, 25].

We denote the set of all real numbers by $\mathbb{R}$.

A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0, 1]$ with the following properties:

- (a) u is upper semi-continuous,
- (b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$,
- (c) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

- (d) $suppu = \{x \in \mathbb{R}|u(x) > 0\}$ is the support of the u, and its closure $cl(suppu)$ is compact. Let E be the set of all fuzzy numbers on $\mathbb{R}$. The $\alpha - level$set of a fuzzy number $u \in E, 0 \leq \alpha \leq 1$, denoted by $[u]_\alpha$, is defined as

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R}|u(x) \geq \alpha\} & if\ \ 0 < \alpha \leq 1 \\ cl(suppu) & if\ \ \alpha = 0. \end{cases}$$

It is clear that the $\alpha - level$ set of a fuzzy number is a closed and bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$, where $\underline{u}(\alpha)$ denotes the left-hand endpoint of $[u]_\alpha$ and $\overline{u}(\alpha)$ denotes the right-hand endpoint of $[u]_\alpha$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\tilde{y}$ defined by

$$\tilde{y}(t) = \begin{cases} 1 & if\ t = y \\ 0 & if\ t \neq y. \end{cases}$$

An equivalent parametric definition is also given in [12] as:

Definition 1. *A fuzzy number u in parametric form is a pair* $(\underline{u}, \overline{u})$ *of functions* $\underline{u}(\alpha)$, $\overline{u}(\alpha)$, $0 \leq \alpha \leq 1$, *which satisfy the following requirements:*

- 1. $\underline{u}(\alpha)$ *is a bounded non-decreasing left continuous function in* $(0, 1]$, *and right continuous at 0,*
- 2. $\overline{u}(\alpha)$*, is a bounded non-increasing left continuous function in* $(0, 1]$, *and right continuous at* 0*,*
- 3. $\underline{u}(\alpha) \leq \overline{u}(\alpha),\ 0 \leq \alpha \leq 1$.

A crisp number α *is simply represented by* $\underline{u}(\alpha) = \overline{u}(\alpha) = \alpha,\ 0 \leq \alpha \leq 1$. *We recall that for* $a < b < c$ *which* $a, b, c \in \mathbb{R}$, *the triangular fuzzy number* $u = (a, b, c)$ *determined by* a, b, c *is given such that represented by* $\underline{u}(\alpha) = a + (b - a)\alpha$ *and* $\overline{u}(\alpha) = c - (c - b)\alpha$ *are the endpoints of the alpha-level sets, for all* $\alpha \in [0, 1]$. *The Hausdorff distance between fuzzy numbers given by* $d : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}_+ \cup \{0\}$.

$$d(u, v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}$$

where $u = (\underline{u}(\alpha), \overline{u}(\alpha)), v = (\underline{v}(\alpha), \overline{v}(\alpha)) \subset \mathbb{R}$ is utilized in [10]. Then, it is easy to see that d is a metric in E and has the following properties

- (i) $d(u + w, v + w) = d(u, v),\ \ \forall u, v, \omega \in \mathbb{E}$,
- (ii) $d(ku, kv) = |k|d(u, v),\ \ \forall k \in \mathbb{R}, u, v \in \mathbb{E}$,
- (iii) $d(u + v, w + e) \leq d(u, w) + d(v, e),\ \ \forall u, v, w, e \in \mathbb{E}$,
- (iv) $(d, \mathbb{E})$, is a complete metric space

Definition 2. *[8], Let* $f : \mathbb{R} \longrightarrow \mathbb{E}$ *be a fuzzy valued function. If for arbitrary fixed* $t_0 \in \mathbb{R}$ *and* $\epsilon > 0,\ a\ \delta > 0$ *such that*

$$|t - t_0| < \delta \Longrightarrow d(f(t), f(t_0)) < \epsilon.$$

f is said to be continuous

Theorem 1. *[12] Let* $f(x)$ *be a fuzzy-valued function on* $[a, \infty)$ *and it is represented by* $(\underline{f}(x, \alpha), \overline{f}(x, \alpha))$. *For any fixed* $r \in [0, 1]$, *assume* $\underline{f}(x, \alpha)$ *and* $overline f(x, \alpha)$ *are Riemann integrable on* $[a, b]$ *for every* $b \geq a$, *and assume there are two positive* $\underline{M}(\alpha)$ *and* $\overline{M}(\alpha))$ *such that* $\int_a^b |\underline{f}(x, \alpha)|dx \leq \underline{M}(\alpha)$ *and* $\int_a^b |\overline{f}(x, \alpha)|dx \leq \overline{M}(\alpha)$ *for every* $b \geq a$. *Then* $f(x)$ *is improper fuzzy Riemann integrable on* $[a, \infty)$ *and the improper fuzzy Riemann integral is a fuzzy number. Further more, we have:*

$$\int_a^\infty f(x)dx = \Big(\int_a^\infty \underline{f}(x, \alpha)dx, \int_a^\infty \overline{f}(x, \alpha)|dx\Big).$$

Proposition 1. *[12] If each of* $f(x)$ *and* $g(x)$ *is fuzzy-valued function and fuzzy Riemman integrable on* $I = [a, \infty)$ *then* $f(x) + g(x)$ *is fuzzy Riemman integrable on* I. *Moreover, we have*

$$\int_1 (f(x) + g(x))dx = \int_1 f(x)dx + \int_1 g(x)dx.$$

3 Fuzzy Laplace transforms

Suppose that f is a fuzzy-valued function and s is a real parameter. We define the fuzzy Laplace transform of f as following:

Definition 3. *The fuzzy Laplace transform of fuzzy-valued function* $f(t)$ *is defined as following*

$$\begin{aligned} F(s) &= L(f(t)) = \int_0^\infty e^{-st} f(t)dt \qquad (1) \\ &= \lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st} f(t)dt, \end{aligned}$$

whenever the limits exist.
The $L(f)$ *be also used to denote the fuzzy Laplace transform of fuzzy-valued function* $f(t)$, *and the integral is the fuzzy Riemann improper integral. The symbol* $\mathbf{L}$ *is the fuzzy Laplace transformation, which acts on fuzzy-valued function* $f = f(t)$ *and generates a new fuzzy-valued function,* $F(s) = \mathbf{L}(f(t))$. *Consider fuzzy-valued function* f, *then the lower and upper fuzzy Laplace transform of this function are denoted, based on the lower and upper of fuzzy-valued function* f *and* $0 \leq \alpha \leq 1$ *as following:*

$$F(s; \alpha) = \mathbf{L}(f(t; \alpha)) = [\mathbf{L}(\underline{f}(t; \alpha)), \mathbf{L}(\overline{f}(t; \alpha)),$$

where

$$\mathbf{L}(\underline{f}(t;\alpha)) = \int_0^\infty e^{-st}\underline{f}(t;\alpha))dt = \lim_{\tau\longrightarrow\infty}\int_0^\tau e^{-st}\underline{f}(t;\alpha))dt,$$

$$\mathbf{L}(\overline{f}(t;\alpha)) = \int_0^\infty e^{-st}\overline{f}(t;\alpha))dt = \lim_{\tau\longrightarrow\infty}\int_0^\tau e^{-st}\overline{f}(t;\alpha))dt.$$

4 Fuzzy Volterra and Fredholm integral equations

In this section, we investigate the solution of fuzzy convolution Fredholm and Volterra equations of the second kind [21].

4.1 Fuzzy Fredholm integral equations

The fuzzy convolution Fredholm integral equation of the second kind is defined as

$$\tilde{x}(t) = \tilde{f}(t) + \int_0^T \tilde{k}(s-t)\tilde{x}(s)ds, t \in [0,T], \quad (2)$$

where $\tilde{x}(t) = \tilde{x}(t;\alpha) = [\underline{x}(t;\alpha), \overline{x}(t;\alpha)]$, $\tilde{f}(t) = \tilde{f}(t;\alpha) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$, and $\tilde{k}(s-t)$ is an arbitrary given fuzzy-valued convolution kernel function and $\tilde{f}$ is a continuous fuzzy-valued function. The solution of Eq. (2) can be obtained by solving the following system integral equations:

$$\underline{x}(t;\alpha) = \underline{f}(t;\alpha) + \int_0^T \underline{k}(s-t;\alpha)\underline{x}(s;\alpha)ds, \quad (3)$$

$$\overline{x}(t;\alpha) = \overline{f}(t;\alpha) + \int_0^T \overline{k}(s-t;\alpha)\overline{x}(s;\alpha)ds. \quad (4)$$

We adopt fuzzy Laplace transform to solve the given problem such that by taking fuzzy Laplace transform on both sides of Eqs. (3)-(4) and using fuzzy convolution, we get solution of Eq. (2) directly.

4.2 Fuzzy Volterra integral equations

The fuzzy convolution Volterra integral equation of the second kind is defined a

$$\tilde{x}(t) = \tilde{f}(t) + \int_0^t \tilde{k}(s-t)\tilde{x}(s)ds, t \in [0,T], \quad (5)$$

where $\tilde{x}(t) = \tilde{x}(t;\alpha) = [\underline{x}(t;\alpha), \overline{x}(t;\alpha)]$, $\tilde{f}(t) = \tilde{f}(t;\alpha) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$, and $\tilde{k}(s-t)$ is an arbitrary given fuzzy-valued convolution kernel function and $\tilde{f}$ is a continuous fuzzy-valued function. The solution of Eq. (5) can be obtained by solving the following system integral equations:

$$\underline{x}(t;\alpha) = \underline{f}(t;\alpha) + \int_0^t \underline{k}(s-t;\alpha)\underline{x}(s;\alpha)ds, \quad (6)$$

$$\overline{x}(t;\alpha) = \overline{f}(t;\alpha) + \int_0^t \overline{k}(s-t;\alpha)\overline{x}(s;\alpha)ds. \quad (7)$$

We adopt fuzzy Laplace transform to solve the given problem such that by taking fuzzy Laplace transform on both sides of Eqs. (6)-(7) and using fuzzy convolution, we get solution of Eq. (5) directly. So, the concept of fuzzy convolution must be introduced.

4.3 Fuzzy convolution

The convolution of two fuzzy-valued functions f and g defined for $t > 0$ by

$$(f * g)(t) = \int_0^t f(\tau).g(t-\tau)d\tau, \quad (8)$$

which of course exists if f and g are, say, piece-wise continuous. Substituting $u = t - \tau$ give

$$(f * g)(t) = \int_0^t f(\tau).g(t-u)du = (g * f)(t) \quad (9)$$

that is, the fuzzy convolution is commutative. Other basic properties of the fuzzy convolution are as follows:

(i) $\quad c(f * g) = cf * g = f * cg, \quad c$ is constant

(ii) $\quad f*(g*h) = (f*g)*h$ (associative property)

Property (i) is routine to verify. As for (ii)

$$\begin{aligned} & [f*(g*h)](t) \\ = & \int_0^t f(\tau)(g*h)(t-\tau)d\tau \\ = & \int_0^t f(\tau)\Big(\int_0^{t-\tau} g(x)h(t-\tau-x)dx\Big)d\tau \\ = & \int_0^t \Big(\int_0^{t-\tau} f(\tau)g(u-\tau)h(t-\tau)dx\Big)d\tau \\ = & [(f*g)*h](t), \end{aligned}$$

while having reverse the order of integration. One of the very significant properties possessed by the fuzzy convolution in connection with the fuzzy Laplace transform is that the fuzzy Laplace transform of the convolution of two fuzzy-valued functions is the product of their fuzzy Laplace transform.

Theorem 2. *(Convolution Theorem), If* f *and* g *are piecewise continuous fuzzy-valued functions on* $[0,\infty]$ *and of exponential order* p, *then*

$$\mathbf{L}\{(f*g)(t)\} = \mathbf{L}\{(f(t)\}.\mathbf{L}\{(g(t)\} = F(s).G(s). \quad (10)$$

Proof. Let us start with the produce

$$
\begin{aligned}
& \mathbf{L}\{(f(t)\}.\mathbf{L}\{(g(t)\} \\
= & \Big(\int_0^\infty e^{-s\tau} f(\tau)d\tau\Big).\Big(\int_0^\infty e^{-su} g(u)du\Big) \\
= & \int_0^\infty \Big(\int_0^\infty e^{-s(\tau+u)} f(\tau)g(u)du\Big)d\tau,
\end{aligned}
$$

substituting $t=\tau+u,$ and noting that τ is fixed in the interior integrals, so that $du=dt$, we have

$$
\begin{aligned}
& \mathbf{L}\{(f(t)\}.\mathbf{L}\{(g(t)\} \qquad (11) \\
= & \int_0^\infty \Big(\int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dt\Big)d\tau.
\end{aligned}
$$

If we define $g(t)=\widetilde{0}$ for $t<0$, then $g(t-\tau)=\widetilde{0}$ for $t<\tau$ and we can write (11) as

$$
\begin{aligned}
& \mathbf{L}\{(f(t)\}.\mathbf{L}\{(g(t)\} \\
= & \int_0^\infty \Big(\int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dt\Big)d\tau
\end{aligned}
$$

Due to the hypotheses on $f, g,$ the fuzzy Laplace integrals of f, g converge absolutely and hen

$$
\int_0^\infty \int_0^\infty |e^{-st)} f(\tau)g(t-\tau)|d\tau,
$$

converges. This fact allows us to reverse the order of integration, so that

$$
\begin{aligned}
& \mathbf{L}\{(f(t)\}.\mathbf{L}\{(g(t)\} \\
= & \int_0^\infty \int_0^\infty e^{-st} f(\tau)g(t-\tau)d\tau dt \\
= & \int_0^\infty \Big(\int_0^t e^{-st} f(\tau)g(t-\tau)d\tau\Big)dt \\
= & \int_0^\infty e^{-st}\Big(\int_0^t f(\tau)g(t-\tau)d\tau\Big)dt \\
= & \mathbf{L}\{(f*g)(t)\}.
\end{aligned}
$$

Please notice that in the fuzzy case, we should investigate more accurately than the deterministic case. So, mentioned calculation is assumed valid under suitable conditions. □

5 Fuzzy Laplace transform method for FIE

Here, we shall obtain the solution of fuzzy convolution Fredholm integral equation using fuzzy Laplace transform. Indeed, our method is constructed on applying fuzzy convolution. Consider the original Eq.(2), then by taking fuzzy Laplace transform on both sides of it we get the following:

$$
\mathbf{L}\{(\tilde{x}(t)\} = \mathbf{L}\{(\tilde{f}(t)\} + \mathbf{L}\Big\{\int_0^T \tilde{k}(s-t)\tilde{x}(s)ds\Big\},
$$

then, we get by using fuzzy convolution and definition of fuzzy Laplace transform:

$$
\mathbf{L}\{\underline{x}(t;\alpha)\} = \mathbf{L}\{\underline{f}(t;\alpha)\} + \underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}},
$$

and

$$
\mathbf{L}\{\overline{x}(t;\alpha)\} = \mathbf{L}\{\overline{f}(t;\alpha)\} + \overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}}.
$$

Now, we should discuss $\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}}$ and $\mathbf{L}\underline{k(t;\alpha)}\ \mathbf{L}\underline{x(t;\alpha)}$.

To this end, we have the following cases:

Case 1- if $k(t;\alpha)$ and $x(t;\alpha)$ are positive, then we get

$$
\begin{aligned}
\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \underline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}}, \\
\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \overline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}}.
\end{aligned}
$$

Case 2- if $k(t;\alpha)$ is positive and $x(t;\alpha)$ is negative, then we get

$$
\begin{aligned}
\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \overline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}}, \\
\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \underline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}}.
\end{aligned}
$$

Case 3- if $k(t;\alpha)$ is negative and $x(t;\alpha)$ is positive, then we get

$$
\begin{aligned}
\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \underline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}}, \\
\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \overline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}}.
\end{aligned}
$$

Case 4- if $k(t;\alpha)$ and $x(t;\alpha)$ are negative, then we get

$$
\begin{aligned}
\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \overline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}}, \\
\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} &= \underline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}}.
\end{aligned}
$$

Notice that, it is assumed that zero does not exist in support. We obtain explicit formula for Case 1 and the others are similar. Indeed, we can write case 1 in a compact form

$$
\mathbf{L}\{\underline{x}(t;\alpha)\} = \frac{\mathbf{L}\{\underline{f}(t;\alpha)\}}{1-\mathbf{L}\{\underline{k}(t;\alpha)\}}
$$

and

$$
\mathbf{L}\{\overline{x}(t;\alpha)\} = \frac{\mathbf{L}\{\overline{f}(t;\alpha)\}}{1-\mathbf{L}\{\overline{k}(t;\alpha)\}}.
$$

Finally, using the inverse of fuzzy Laplace transform we get the solution:

$$
\underline{x}(t;\alpha) = \mathbf{L}^{-1}\Big(\frac{\mathbf{L}\{\underline{f}(t;\alpha)\}}{1-\mathbf{L}\{\underline{k}(t;\alpha)\}}\Big)
$$

and

$$
\overline{x}(t;\alpha) = \mathbf{L}^{-1}\Big(\frac{\mathbf{L}\{\overline{f}(t;\alpha)\}}{1-\mathbf{L}\{\overline{k}(t;\alpha)\}}\Big)
$$

for all $0 \leq \alpha \leq 1$, provided that a fuzzy valued function is define.

5.1 Fuzzy Laplace transform method for VIE

Here, we shall obtain the solution of fuzzy convolution Fredholm and Volterra integral equations using fuzzy Laplace transform. Indeed, our method is constructed on applying fuzzy convolution. Consider the original Eq.(2) and Eq.(5), then by taking fuzzy Laplace transform on both sides of it we get the following:

$$\mathbf{L}\{(\tilde{x}(t)\} = \mathbf{L}\{(\tilde{f}(t)\} + \mathbf{L}\Big\{\int_0^T \tilde{k}(s-t)\tilde{x}(s)ds\Big\},$$

and

$$\mathbf{L}\{(\tilde{x}(t)\} = \mathbf{L}\{(\tilde{f}(t)\} + \mathbf{L}\Big\{\int_0^t \tilde{k}(s-t)\tilde{x}(s)ds\Big\},$$

then, we get by using fuzzy convolution and definition of fuzzy Laplace transform:

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \mathbf{L}\{\underline{f}(t;\alpha)\} + \underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}},$$

and

$$\mathbf{L}\{\overline{x}(t;\alpha)\} = \mathbf{L}\{\overline{f}(t;\alpha)\} + \overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}}.$$

Now, we should discuss $\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}}$ and $\mathbf{L}\underline{k(t;\alpha)}\ \mathbf{L}\overline{x(t;\alpha)}$.

To this end, we have the following cases:
Case 1- if $k(t;\alpha)$ and $x(t;\alpha)$ are positive, then we get

$$\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \underline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}},$$
$$\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \overline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}}.$$

Case 2- if $k(t;\alpha)$ is positive and $x(t;\alpha)$ is negative, then we get

$$\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \overline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}},$$
$$\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \underline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}}.$$

Case 3- if $k(t;\alpha)$ is negative and $x(t;\alpha)$ is positive, then we get

$$\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \underline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}},$$
$$\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \overline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}}.$$

Case 4- if $k(t;\alpha)$ and $x(t;\alpha)$ are negative, then we get

$$\underline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \overline{\mathbf{L}\{k(t;\alpha)\}}\ \overline{\mathbf{L}\{x(t;\alpha)\}},$$
$$\overline{\mathbf{L}\{k(t;\alpha)\}\mathbf{L}\{x(t;\alpha)\}} = \underline{\mathbf{L}\{k(t;\alpha)\}}\ \underline{\mathbf{L}\{x(t;\alpha)\}}.$$

Notice that, it is assumed that zero does not exist in support. We obtain explicit formula for Case 1 and the others are similar. Indeed, we can write case 1 in a compact form

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \frac{\mathbf{L}\{\underline{f}(t;\alpha)\}}{1-\mathbf{L}\{\underline{k}(t;\alpha)\}}$$

and

$$\mathbf{L}\{\overline{x}(t;\alpha)\} = \frac{\mathbf{L}\{\overline{f}(t;\alpha)\}}{1-\mathbf{L}\{\overline{k}(t;\alpha)\}}.$$

Finally, using the inverse of fuzzy Laplace transform we get the solution:

$$\underline{x}(t;\alpha) = \mathbf{L}^{-1}\Big(\frac{\mathbf{L}\{\underline{f}(t;\alpha)\}}{1-\mathbf{L}\{\underline{k}(t;\alpha)\}}\Big)$$

and

$$\overline{x}(t;\alpha) = \mathbf{L}^{-1}\Big(\frac{\mathbf{L}\{\overline{f}(t;\alpha)\}}{1-\mathbf{L}\{\overline{k}(t;\alpha)\}}\Big)$$

for all $0 \le \alpha \le 1$, provided that a fuzzy valued function is define.

6 Examples

In this section, we give some examples to obtain the solution of fuzzy convolution Volterra and Fredholm integral equations of the second kind.

Example 1. Consider the following fuzzy Volterra integral equation

$$\tilde{x}(t) = (\alpha, 2-\alpha).e^t + \int_0^t sin(t-\tau).\tilde{x}(\tau)d\tau.$$

We apply the fuzzy Laplace transform to both sides of the equation, so tha

$$\mathbf{L}\{\tilde{x}(t)\} = \mathbf{L}\{(\alpha,2-\alpha).e^t\} + \mathbf{L}\{sin(t)\}.\mathbf{L}\{\tilde{x}(t)\}$$

i.e

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \mathbf{L}\{(\alpha).e^t\} + \mathbf{L}\{sin(t)\}.\mathbf{L}\{\underline{x}(t;\alpha)\},$$

$$\mathbf{L}\{\overline{x}(t;\alpha)\} = \mathbf{L}\{(2-\alpha).e^t\} + \mathbf{L}\{sin(t)\}.\mathbf{L}\{\overline{x}(t;\alpha)\},$$

Hence . we get

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = (\alpha)\Big(\frac{2}{s-1} - \frac{1}{s^2} - \frac{1}{s}\Big),\ 0 \le \alpha \le 1$$

$$\mathbf{L}\{\overline{x}(t;r)\} = (2-\alpha)\Big(\frac{2}{s-1} - \frac{1}{s^2} - \frac{1}{s}\Big),\ 0 \le \alpha \le 1$$

By taking the inverse of fuzzy Laplace transform on both sides of above relations, we have the following

$$\begin{aligned}\underline{x}(t;\alpha) &= (\alpha)(2e^t - t - 1),\\ \overline{x}(t;\alpha) &= (2-\alpha)(2e^t - t - 1).\end{aligned}$$

Example 2. Consider the following fuzzy Fredholm integral equation

$$\tilde{x}(t) = \frac{1}{2}(\alpha+1), \frac{1}{2}(3-\alpha)).t + \int_0^2 \frac{1}{4}(t-\tau).\tilde{x}(\tau)d\tau.$$

We apply the fuzzy Laplace transform to both sides of the equation, so that

$$\mathbf{L}\{\tilde{x}(t)\} = \frac{1}{2}\mathbf{L}\{((\alpha+1),(3-\alpha)).t\} + \frac{1}{4}\mathbf{L}\{t\}.\mathbf{L}\{\tilde{x}(t)\}$$

i.e

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \frac{1}{2}\mathbf{L}\{(\alpha+1).t\} + \frac{1}{4}\mathbf{L}\{t\}.\mathbf{L}\{\underline{x}(t;\alpha)\},$$

$$\mathbf{L}\{\overline{x}(t;\alpha)\} = \frac{1}{2}\mathbf{L}\{(3-\alpha).t\} + \frac{1}{4}\mathbf{L}\{t\}.\mathbf{L}\{\overline{x}(t;\alpha)\}.$$

Hence we get

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \frac{2(\alpha+1)}{4s^2-1},$$

$$\mathbf{L}\{\overline{x}(t;r)\} = \frac{2(3-\alpha)}{4s^2-1}.$$

By taking the inverse of fuzzy Laplace transform on both sides of above relations, we have the following

$$\underline{x}(t;\alpha) = (\alpha+1).\sinh(\frac{t}{2}),$$
$$\overline{x}(t;\alpha) = (3-\alpha).\sinh(\frac{t}{2}).$$

Example 3. Consider the following fuzzy Volterra integral equation

$$\tilde{x}(t) = (1+\alpha, 3-\alpha).\cosh t + \int_0^t e^{t-\tau}.\tilde{x}(\tau)d\tau.$$

Similarly, by taking fuzzy Laplace transform on both sides of equation, we get the

$$\begin{aligned}\mathbf{L}\{\tilde{x}(t)\} &= \mathbf{L}\{(1+\alpha,3-\alpha).\cosh(t)\} \\ &\quad +\mathbf{L}\{e^t\}.\mathbf{L}\{\tilde{x}(t)\}\end{aligned}$$

i.e

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \mathbf{L}\{(1+\alpha).\cosh t\} + \mathbf{L}\{e^t\}.\mathbf{L}\{\underline{x}(t;\alpha)\},$$

$$\mathbf{L}\{\overline{x}(t;\alpha)\} = \mathbf{L}\{(3-\alpha).\cosh t\} + \mathbf{L}\{e^t\}.\mathbf{L}\{\overline{x}(t;\alpha)\},$$

i.e

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = \frac{(1+\alpha)s}{(s+1)(s-2)},$$

$$\mathbf{L}\{\overline{x}(t;\alpha)\} = \frac{(3-\alpha)s}{(s+1)(s-2)},\ 0 \le \alpha \le 1.$$

Hence, we get

$$\mathbf{L}\{\overline{x}(t;r)\} = (1+\alpha)\Big(\frac{2}{3(s-2)} + \frac{1}{3(s+1)}\Big),$$

$$\mathbf{L}\{\underline{x}(t;\alpha)\} = (3-\alpha)\Big(\frac{2}{3(s-2)} + \frac{1}{3(s+1)}\Big).$$

Finally, by applying the inverse of fuzzy Laplace transform, we get the following:

$$\underline{x}(t;\alpha) = (1+\alpha)(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-t}),\ 0 \le \alpha \le 1$$

$$\overline{x}(t;\alpha) = (3-\alpha)(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-t}),\ 0 \le \alpha \le 1.$$

7 Conclusion

In the present paper, the fuzzy Laplace transform method was applied to approximate the solution of fuzzy Volterra and Fredholm integral equations. We transformed our problem to a system of algebraic equations so that by solving this system we obtained the solution of this kind of equations by considering the type of differentiability. Finally, the solution obtained using the suggested method shows that this approach can solve the problem effectively.

An interesting extension of our study would be to discuss method with neural networks and finite-time stability for the Volterra and Fredholm integral equations. This topic will be the subject of a forthcoming paper.

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