An Algorithm with the Even-odd Splitting of the Wavelet Transform of Non-Hermitian Splines of the Seventh Degree, II

BORIS M. SHUMILOV
Department of Applied Mathematics
Tomsk State University of Architecture and Building
2 Solyanaya, Tomsk, 634003
RUSSIA

Abstract: - In this study, the zeroing property of the first two moments is used to construct an algorithm for splitting spline wavelets of the seventh degree. The presentation is based on the system of basic spline wavelets of the seventh degree, constructed in the previous article, which implements the conditions of orthogonality to all polynomials of any degree. Then, using homogeneous Dirichlet boundary conditions, the system is adapted to orthogonality to all polynomials up to the first degree on a finite interval. Implicit finite relationships are obtained between the spline coefficients in the original scale, on the one hand, and the spline coefficients and wavelet coefficients in the nested scale, on the other hand. After eliminating the even rows of the system, the transformation matrix has seven diagonals instead of five, as in the previous case studied. The resulting system has been modified to ensure strict diagonal dominance and, hence, computational stability, in contrast to the five-diagonal case.

Key-Words: - B-splines, wavelets, implicit decomposition relations, sweep method, data processing

1 Introduction

Haar wavelets and Daubechies wavelets were the first compactly supported orthonormal wavelets [1-3]. Compactness means that there are explicit finite formulas for the discrete wavelet decomposition. The locality property of wavelets gives them an advantage over Walsh and sine-cosine functions commonly used in digital signal processing [4, 5]. The unique hierarchical property of wavelets allow the development of a basis in which the data representation can be expressed with a small number of non-zero coefficients. This property makes wavelets attractive for data compression, including video and audio information. The wavelet transform can be considered as one of the methods of primary signal processing to increase the efficiency of its compression. In this case, direct compression is carried out by classical methods only for significant coefficients of the wavelet decomposition of the signal, and its reconstruction from these coefficients is performed at the stage of restoration (decompression) [6]. As multimedia becomes more and more popular, the conflict between massive data and limited storage devices is ever-increasing; thus, a more convenient, efficient and high-quality transmission and storage technology is required, and fast wavelet analysis is what people want to use for a high-performance compression technology [7-10].

The generalization of orthonormal wavelets were constructed by Cohen et al. in the form of biorthogonal wavelets [11, 12]. But the disadvantage of these wavelets is that the expansion coefficients are calculated by the formulas of local averaging, that is, when processing data, information is lost at the edges of the image [13]. It is for this reason that it is usually advised to use interpolating cubic spline wavelets [14] instead of orthogonal wavelets. Wavelet transforms based on interpolation splines have their drawbacks. First, in the problem of processing measurement information, interpolating spline wavelets [15] are calculated by solving interpolation problems on sequences of nested grids, so the expansion coefficients are equal to the values of the function at some nodes of a dense grid, which is a kind of data decimation. Secondly, from the point of view of data noise reduction, the function values are not filtered at all, although the best root-mean-square approximation of the derivative is provided [16].

Meanwhile, in the work of the author [17], non-orthogonal wavelets of the third degree with the first six zero moments, i.e., orthogonal to all polynomials of the fifth degree, were considered; the existence of finite implicit decomposition relations was proved and an efficient even-odd splitting algorithm based on them for wavelet analysis was substantiated. The importance of the new algorithm in wavelet theory lies in its stability and ease of implementation, since at each resolution step a matrix with strict diagonal dominance is solved.

The first attempt to find out whether there is an algorithm for splitting non-orthogonal wavelets of the seventh degree was made in the work of the author [18], the method of five-diagonal splitting was studied, and the absence of strict diagonal dominance in the resulting system was established. In this article, we justify the stability of the solution for the seven-diagonal splitting method; presents the results of numerical experiments on the approximation of a discretely given function. Section 2 discusses the properties of splines of degree m of smoothness C^{m-1} on a uniform infinite grid of knots and of non-orthogonal spline wavelets with $n+1$ vanishing moments. In subsection 2.1, similar properties of splines of the seventh degree are discussed in the case of a finite segment, with corresponding changes near the boundaries. Subsection 2.2 discusses the use of matrix notation for the wavelet transform of hierarchical spline bases, and section 3 proposes the basic idea of preprocessing the system of wavelet transform equations. As a result, in Theorem 1, the case of seventh-degree spline wavelets with two zero moments is completely resolved and in subsection 3.1 the wavelet decomposition algorithm on a finite interval is implemented.

2 Construction of spline wavelets with vanishing moments [18]

To construct wavelets, we need a set of approximating spaces $\ldots V_{L-1} \subset V_L \subset V_{L+1} \ldots$ such that each basis function in V_L can be represented as a linear combination of basis functions in V_{L+1}. In particular, this property is possessed by splines, which are smooth functions glued together from pieces of polynomials of degree m on a nested sequence of grids. The essence of the wavelet transform is formulated as follows: it allows one to decompose a given function V_{L+1} into a rough approximate representation V_L and the locally refined details $W_L = V_{L+1} - V_L$. This procedure can be applied recursively to V_L. Hence, the original function can be represented as a hierarchy of rough versions of $V_L, V_{L-1}, \ldots$ and refinements $W_L, W_{L-1}, \ldots$. Such a recursive process is called direct wavelet transformation (decomposition or analysis) [1, p. 46]. Conversely, the function V_{L+1} can be reconstructed from the most compact representation (reconstruction). Moreover, the values of the coefficients of the wavelet decomposition can be used to judge the significance of the corresponding details of the refinement. Insignificant details can be removed to compress the information. In the case when splines are suitable as a basis for the space W_L, the main thing here is to find fast one-to-one formulas for the direct and inverse wavelet transforms for it.

As the space V_L we will use the space of splines of degree m of smoothness C^{m-1} on the uniform grid of knots $\Delta^L : x_{i+1} = x_i + 1/2^L$, which continues indefinitely in both directions for all i. It is well known that the basis in this space is generated by functions $\varphi_m(v-i)\forall i$, where $v = 2^L x$, formed by a function of the form [1, p. 89]:

$$\varphi_m(t) = \frac{1}{m!}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}(t-j)_+^m,$$

where $t_+^m = (\max\{t,0\})^m$.

It is known that they satisfy the calibration relation [1, p. 91]:

$$\varphi_m(t) = 2^{-m}\sum_{k=0}^{m+1}\binom{m+1}{k}\varphi_m(2t-k), \qquad (1)$$

and they have the following supports,

$$\operatorname{supp}\varphi_m = [0, m+1].$$

As a result, any spline on the mesh Δ^L can be represented as

$$s^L(x) = \sum_{-\infty}^{\infty} c_i^L \varphi_m(2^L x - i). \qquad (2)$$

We need to solve, for example, the cardinal interpolation problem:

$$s^L(x_i) = f(x_i),\ -\infty < i < \infty,$$

to determine the coefficients $c_i^L \forall i$.

Let the grid Δ^{L-1} be obtained from the grid Δ^L by removing every second node. Then the corresponding base functions $\varphi_m(v/2-i)\forall i$ have supports twice as wide, and the space V_{L-1} is embedded in V_L. The complement of V_{L-1} to V_L is defined as wavelet space: $W_{L-1} = V_L - V_{L-1}$ [1].

Non-orthogonal wavelets [19] orthogonal to all polynomials of degree n, are defined as

$$w_{m,n}(t) = 2^{-m}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\varphi_m(2t-k). \quad (3)$$

It can be evaluated that they have t $n+1$ zero moments

$$\int_{-\infty}^{\infty} x^k w_{m,n}(x)dx = 0,\ k = 0,1,\ldots,n,$$

and, accordingly, they have the following supports,

$$\operatorname{supp} w_{m,n} = [0, \frac{m+n}{2} + 1].$$

2.1 The case of a finite segment

Recall that for the case of interpolation by splines on a finite interval $[0, 2^L]$, the most productive approach to constructing basis functions is to set multiple nodes at the ends of the interval, which corresponds to zeroing of the approximating spline and some of its derivatives at the ends of the interval [2]. Then the left seventh degree basic functions have the view forms [18]. They have the following supports,

$$\operatorname{supp}\varphi_{b1} = [0, 7],\ \operatorname{supp}\varphi_{b2} = [0, 6],$$

and they satisfy the calibration relations

$$\begin{aligned}\varphi_{b1}(t) = {} & \frac{1}{64}\varphi_{b1}(2t) + \frac{19}{134}\varphi_7(2t) + \\ & + \frac{59}{160}\varphi_7(2t-1) + \frac{327}{640}\varphi_7(2t-2) + \\ & + \frac{41}{96}\varphi_7(2t-3) + \frac{167}{768}\varphi_7(2t-4) + \\ & + \frac{1}{16}\varphi_7(2t-5) + \frac{1}{128}\varphi_7(2t-6), \\ \varphi_{b2}(t) = {} & \frac{1}{32}\varphi_{b2}(2t) + \frac{147}{640}\varphi_{b1}(2t) + \frac{12299}{25600}\varphi_7(2t) + \\ & + \frac{371}{800}\varphi_7(2t-1) + \frac{399}{1600}\varphi_7(2t-2) + \\ & + \frac{7}{96}\varphi_7(2t-3) + \frac{7}{768}\varphi_7(2t-4).\end{aligned}$$

As to boundary seventh-degree basic wavelets, then because of the future need for splitting the decomposition matrix we can use the method of constructing wavelets that are orthogonal to all first-degree polynomials and include only even basic splines [18],

$$\begin{aligned}w_{b1}(t) &= \frac{7}{48}\varphi_{b2}(2t) - \frac{15}{64}\varphi_{b1}(2t) + \frac{49}{512}\varphi_7(2t), \\ w_{b2}(t) &= \frac{1}{16}\varphi_{b2}(2t) - \frac{11}{128}\varphi_7(2t) + \frac{5}{128}\varphi_7(2t-2).\end{aligned}$$

They have the following supports

$$\operatorname{supp} w_{b1} = [0, 5],\ \operatorname{supp} w_{b2} = [0, 4],$$

and, accordingly, they have two zero moments

$$\int_0^5 x^k w_{b1}(x)dx = \int_0^4 x^k w_{b2}(x)dx = 0,$$

for $k = 0, 1$.

The basic functions at the right end of the segment mirror the functions $\varphi_{b1,2}(t), w_{b1,2}(t)$. So for any grid Δ^L, $L \geq 3$, a seventh-degree spline can be represented as

$$\begin{aligned}s^L(v) = {} & c_{-2}^L\varphi_{b2}(v) + c_{-1}^L\varphi_{b1}(v) + \quad (4)\\ & + \sum_{i=0}^{2^L-8} c_i^L\varphi_7(v-i) + c_{2^L-7}^L\varphi_{b1}(2^L - v) + \\ & + c_{2^L-6}^L\varphi_{b2}(2^L - v), 0 \leq v \leq 2^L.\end{aligned}$$

To difine the coefficients $c_i^L \forall i$ we can solve, for example, the interpolation problem:

$$s^L(i) \;=\; f(i),\ i = 2, 3, \ldots, 2^L - 2,$$

while zero boundary conditions are satisfied automatically

$$\left(s^L\right)^{(r)}(v) = 0,\ r = 0, 1, \ldots, 4,\ v = 0, 2^L.$$

The graphs of basis spline functions and wavelets of the 7th degree, orthogonal to all polynomials of the 1st degree, were shown in [18].

2.2 Construction of the defining system of wavelet transform equations

Let us write the basic spline functions in a single-line matrix form,

$$\begin{aligned}\varphi^L(v) = [&\varphi_{b2}(v), \varphi_{b1}(v), \varphi_7(v), \ldots \\ &\ldots, \varphi_7(v - 2^L + 8), \varphi_{b1}(2^L - v), \varphi_{b2}(2^L - v)\Big].\end{aligned}$$

Introduce the notation

$$\mathbf{c}^L = \left[c_{-2}^L, c_{-1}^L, c_0^L, \ldots, c_{2^L-7}^L, c_{2^L-6}^L\right]^T$$

for a vector consisting of spline coefficients. Then we can write formula (2) in vector form

$$s^L(x) = \varphi^L(x)\mathbf{c}^L.$$

In the same way, we can write basic wavelet functions at the decomposition level $L-1$ in the form of a single row matrix as

$$\begin{aligned}\psi^{L-1}(v) = [&w_{b2}(v), w_{b1}(v), w_{7,1}(v), w_{7,1}(v-2), \ldots \\ &\ldots, w_{7,1}(v - 2^L + 10), w_{b1}(2^L - v), w_{b2}(2^L - v)\Big].\end{aligned}$$

We denote the corresponding wavelet approximation coefficients by $d_i^{L-1}, -2 \leq i \leq 2^{L-1} - 3$, and introduce the column vector

$$\mathbf{d}^{L-1} = \left[d_{-2}^{L-1}, d_{-1}^{L-1}, \ldots, d_{2^{L-1}-3}^{L-1}\right]^T.$$

By definition the spaces V_{L-1} and W_{L-1} are subspaces of V_L. So the functions $\varphi^{L-1}(x)$ and $\psi^{L-1}(x)$ can be represented as linear combinations of the functions $\varphi^L(x)$:

$$\begin{aligned}\varphi^{L-1}(x) &= \varphi^L(x)P^L, \\ \psi^{L-1}(x) &= \varphi^L(x)Q^L.\end{aligned}$$

Here the columns of the matrix P^L are built from the relation coefficients (1), and the elements of the columns of the matrix Q^L consist of the relation coefficients (3) with the corresponding changes near the boundaries.

Therefore, there is a chain of equalities:

$$\begin{aligned}\varphi^L(x)\mathbf{c}^L = &\ \varphi^{L-1}(x)\mathbf{c}^{L-1} + \psi^{L-1}(x)\mathbf{d}^{L-1} = \\ = &\ \varphi^L(x)P^L\mathbf{c}^{L-1} + \varphi^L(x)Q^L\mathbf{d}^{L-1}.\end{aligned}$$

Then the coefficients $\mathbf{c}^L$ can be obtained from the known coefficients $\mathbf{c}^{L-1}$ and $\mathbf{d}^{L-1}$ as follows

$$\mathbf{c}^L = P^L\mathbf{c}^{L-1} + Q^L\mathbf{d}^{L-1}. \tag{5}$$

Equality (5) can be rewritten in the form of block matrices,

$$\mathbf{c}^L = \left[P^L \mid Q^L\right]\left[\frac{\mathbf{c}^{L-1}}{\mathbf{d}^{L-1}}\right]. \tag{6}$$

Formula (6) is nothing more than a recovery algorithm [1, p. 248], for the implementation of which, due to the tape matrices P^L and Q^L the moving average scheme is successfully used. For example, for the case $m = 7,\ n = 1$, the matrix $\left[P^L \mid Q^L\right]$ takes the following form:

$$\left[P^L \mid Q^L\right] = \frac{1}{128}\cdot$$

$$\cdot\left[\begin{array}{cccc|cccc}
 & \vdots & & & & & & \\
\ddots & 0 & & & & & & \\
\ddots & 1 & \vdots & & & & & \\
\ddots & 8 & 0 & & & & & \\
\ddots & 28 & 1 & \ddots & & \vdots & & \\
\ddots & 56 & 8 & \ddots & \ddots & 0 & & \\
\ddots & 70 & 28 & \ddots & \ddots & 1 & \vdots & \\
\ddots & 56 & 56 & \ddots & \ddots & -2 & 0 & \ddots \\
\ddots & 28 & 70 & \ddots & \ddots & 1 & 1 & \ddots \\
\ddots & 8 & 56 & \ddots & \ddots & 0 & -2 & \ddots \\
\ddots & 1 & 28 & \ddots & & \vdots & 1 & \ddots \\
\ddots & 0 & 8 & \ddots & & & 0 & \ddots \\
 & \vdots & 1 & \ddots & & & \vdots & \\
 & & 0 & \ddots & & & & \\
 & & \vdots & & & & &
\end{array}\right].$$

Unfortunately, for the reverse process of calculating from the coefficients $\mathbf{c}^L$ the coarser version $\mathbf{c}^{L-1}$ and the refining coefficients $\mathbf{d}^{L-1}$, following the decomposition algorithm [1, p. 247]

$$\left[\frac{\mathbf{c}^{L-1}}{\mathbf{d}^{L-1}}\right] = \left[\frac{A^L}{B^L}\right]\mathbf{c}^L,$$

we obtain, that the rows of matrices A^L and B^L are completely filled numerical sequences, and their truncation leads to errors.

3 The new algorithm with splitting

Now we will use the method of even-odd splitting to the resulting system solving [17]. We choose for this purpose some preconditioning matrix R^L to receive an easy invertible matrix

$$G^L = \left[P^L \mid Q^L\right] R^L$$

by the conditions:
a) a matrix G^L is a tape matrix with the minimum possible number of nonzero diagonals;
b) R^L is a tape matrix, with the minimum possible number of elements.

By zeroing the elements spaced by an odd number of steps from the main diagonal of the matrix G^L, it is possible to use an efficient sweep method to solve the computational scheme; and additionally zeroing the elements outside the main diagonal of the matrix G^L, one can look for the possibility of splitting the system into even and odd rows. And also place as many zeros as possible in the upper and lower parts of each column of the matrix R^L to ensure its compactness.

Then, assuming that the matrix G^L is nonsingular, we multiply the left and right sides of equality (5) by the matrix $R^L G^{L^{-1}}$ to receive the equalities

$$\begin{gathered} R^L G^{L^{-1}}\mathbf{c}^L = \\ = R^L\left(\left[P^L \mid Q^L\right] R^L\right)^{-1}\left[P^L \mid Q^L\right]\left[\frac{\mathbf{c}^{L-1}}{\mathbf{d}^{L-1}}\right] = \\ = R^L R^{L^{-1}}\left[\frac{A^L}{B^L}\right]\left[P^L \mid Q^L\right]\left[\frac{\mathbf{c}^{L-1}}{\mathbf{d}^{L-1}}\right] = \\ = \left[\frac{\mathbf{c}^{L-1}}{\mathbf{d}^{L-1}}\right]. \end{gathered}$$

Thus, instead of directly solving a system of a form (6), we can solve the system

$$G^L\mathbf{h}^L = \mathbf{c}^L \tag{7}$$

with respect to some values of $\mathbf{h}^L$ and then just calculate the values of $\mathbf{c}^{L-1}$ and $\mathbf{d}^{L-1}$ using the linear transformation

$$\left[\frac{\mathbf{c}^{L-1}}{\mathbf{d}^{L-1}}\right] = R^L\mathbf{h}^L. \tag{8}$$

For the matrix, $\left[P^L \mid Q^L\right]$, the nine-diagonal matrix

$$\left[P^L \mid Q^L\right]' = \frac{1}{128}\cdot$$

$$\cdot\begin{bmatrix}
\ddots & \vdots & \vdots & & & & & & \\
\ddots & 0 & 0 & \vdots & & & & & \\
\ddots & 0 & 1 & 0 & \vdots & & & & \\
\ddots & 0 & 8 & 0 & 0 & \vdots & & & \\
\ddots & 1 & 28 & 0 & 1 & 0 & \ddots & & \\
\ddots & -2 & 56 & 0 & 8 & 0 & 0 & \cdots & \\
\ddots & 1 & 70 & 1 & 28 & 0 & 1 & 0 & \cdots \\
\ddots & 0 & 56 & -2 & 56 & 0 & 8 & 0 & \cdots \\
\ddots & 0 & 28 & 1 & 70 & 1 & 28 & 0 & \ddots \\
\cdots & 0 & 8 & 0 & 56 & -2 & 56 & 0 & \ddots \\
\cdots & 0 & 1 & 0 & 28 & 1 & 70 & 1 & \ddots \\
 & \ddots & 0 & 0 & 8 & 0 & 56 & -2 & \ddots \\
 & & \vdots & 0 & 1 & 0 & 28 & 1 & \ddots \\
 & & & \vdots & 0 & 0 & 8 & 0 & \ddots \\
 & & & & \vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}$$

is obtained by permuting the columns of the matrix $\left[P^L \mid Q^L\right]$ so that the columns of the matrices P^L and Q^L alternate. In practice, such a permutation is accompanied by a change in the order of the unknowns in the system (6) and is often done to give the system a tape-like form to facilitate the numerical solution of the system [2]. Let the matrix corresponding to the indicated permutation of columns is denoted by T. Then the representation is true [20]

$$\left[P^L \mid Q^L\right]' = \left[P^L \mid Q^L\right] T. \tag{9}$$

From the representation (9) we find

$$\left[P^L \mid Q^L\right]^{-1} \cdot G^L = T \cdot \left[P^L \mid Q^L\right]'^{-1} \cdot G^L. \tag{10}$$

Thus, the problem of finding the matrices R^L and G^L is reduced to finding a solution of the system of matrix equalities

$$\left[P^L \mid Q^L\right]'_j R^{L\prime}_j = G^L_j,\ \forall j. \tag{11}$$

Here, the subscripts in the notation of the matrices indicate which elements of the columns of the matrix $R^{L\prime}$ are calculated by the corresponding system (11). Specifically, according to the assumed stepped structure of the matrices $R^{L\prime}$ and G^L and provided that the equations corresponding to zero rows of the matrix G^L are removed from the system, the inner part of the system (11) splits into blocks with matrices of the following form:

$$\left[P^L \mid Q^L\right]'_{j,j+1,\ldots,j+6} = \frac{1}{128}\cdot\begin{bmatrix}
1 & 28 & 0 & 1 & & & & & & & \\
-2 & 56 & 0 & 8 & & & & & & & \\
1 & 70 & 1 & 28 & 0 & 1 & & & & & \\
0 & 56 & -2 & 56 & 0 & 8 & & & & & \\
0 & 28 & 1 & 70 & 1 & 28 & 0 & 1 & & & \\
0 & 8 & 0 & 56 & -2 & 56 & 0 & 8 & & & \\
 & 1 & 0 & 28 & 1 & 70 & 1 & 28 & 0 & 1 & \\
 & 0 & 0 & 8 & 0 & 56 & -2 & 56 & 0 & 8 & \\
 & & 0 & 1 & 0 & 28 & 1 & 70 & 1 & 28 & \\
 & & & & 0 & 8 & 0 & 56 & -2 & 56 & \\
 & & & & & 1 & 0 & 28 & 1 & 70 & 1 \\
 & & & & & & 0 & 8 & 0 & 56 & -2 \\
 & & & & & & & 1 & 0 & 28 & 1
\end{bmatrix}.$$

This system is solvable and underdetermined. Therefore, we can choose non-trivial solutions that interest us from the point of view of diagonal domination, namely:

1) $$\begin{gathered} r_0 = r_{10} = -4;\ r_1 = r_9 = 0; \\ r_2 = r_8 = 20;\ r_3 = r_7 = -1; \\ r_4 = r_6 = 304;\ r_5 = 12; \\ g_0 = g_{12} = -5; \\ g_1 = g_2 = g_3 = g_5 = g_7 = \\ = g_9 = g_{10} = g_{11} = 0; \\ g_4 = g_8 = 589;\ g_6 = 1392; \end{gathered}$$

2) $$\begin{gathered} r_0 = r_1 = r_2 = r_3 = r_5 = r_6 = \\ = r_7 = r_8 = r_9 = r_{10} = 0;\ r_4 = 1; \\ g_0 = g_1 = g_2 = g_3 = g_7 = g_8 = \\ = g_9 = g_{10} = g_{11} = g_{12} = 0; \\ g_4 = g_6 = 1;\ g_5 = -2; \end{gathered}$$

$$
\begin{aligned}
3)\quad & r_0 = r_1 = r_2 = r_3 = r_4 = r_5 = \\
& = r_7 = r_8 = r_9 = r_{10} = 0;\; r_6 = 1; \\
& g_0 = g_1 = g_2 = g_3 = g_4 = g_5 = \\
& = g_9 = g_{10} = g_{11} = g_{12} = 0; \\
& g_6 = g_8 = 1;\; g_7 = -2.
\end{aligned}
$$

As a result, the matrix G^L acquires a tape structure with seven nonzero diagonals of the form

$$
\begin{bmatrix}
\ddots & \ddots & \ddots & & & & \\
\ddots & 0 & 0 & 0 & -5 & & \\
\ddots & 0 & 0 & 0 & 0 & \ddots & \\
\ddots & 1 & 589 & 0 & 0 & \ddots & \\
\ddots & -2 & 0 & 0 & 0 & \ddots & \\
\ddots & 1 & 1392 & 1 & 589 & 0 & \ddots \\
\ddots & 0 & 0 & -2 & 0 & 0 & \ddots \\
\ddots & 0 & 589 & 1 & 1392 & 1 & \ddots \\
\ddots & 0 & 0 & 0 & 0 & -2 & \ddots \\
\ddots & 0 & 0 & 0 & 589 & 1 & \ddots \\
 & \ddots & 0 & 0 & 0 & 0 & \ddots \\
 & & -5 & 0 & 0 & 0 & \ddots \\
 & & & \ddots & 0 & 0 & \ddots \\
 & & & & \ddots & \ddots & \ddots
\end{bmatrix},
$$

while the matrix $R^{L'}$ turns out to have the following form:

$$
\begin{bmatrix}
\ddots & \ddots & \ddots & & & & \\
\ddots & 0 & 0 & \ddots & & & \\
\ddots & 0 & 20 & 0 & -4 & \ddots & \\
\ddots & 0 & -1 & 0 & 0 & \ddots & \\
\ddots & 1 & 304 & 0 & 20 & 0 & \ddots \\
\ddots & 0 & 12 & 0 & -1 & 0 & \ddots \\
\ddots & 0 & 304 & 1 & 304 & 0 & \ddots \\
\ddots & 0 & -1 & 0 & 12 & 0 & \ddots \\
\ddots & 0 & 20 & 0 & 304 & 1 & \ddots \\
 & \ddots & 0 & 0 & -1 & 0 & \ddots \\
 & \ddots & -4 & 0 & 20 & 0 & \ddots \\
 & & & \ddots & 0 & 0 & \ddots \\
 & & & & \ddots & \ddots & \ddots
\end{bmatrix}. \quad (12)
$$

To prepare the system (6) for the odd-even splitting near the boundaries we need to solve the system (11) for indices $j = -2, -1, \ldots, 5$ with the following matrix

$$
\left[P^L \mid Q^L\right]'_{-2,-1,\ldots,5} = \frac{1}{128}\cdot
$$

$$
\cdot\begin{bmatrix}
8 & \frac{56}{3} & 4 & 0 & & & & \\
0 & -30 & \frac{147}{5} & 0 & 2 & & & \\
-11 & \frac{49}{4} & \frac{12299}{200} & 1 & \frac{1216}{67} & 0 & 1 & \\
0 & 0 & \frac{1484}{25} & -2 & \frac{236}{5} & 0 & 8 & \\
5 & 0 & \frac{798}{25} & 1 & \frac{327}{5} & 1 & 28 & 0 \\
0 & 0 & \frac{28}{3} & 0 & \frac{164}{3} & -2 & 56 & 0 \\
 & 0 & \frac{7}{6} & 0 & \frac{167}{6} & 1 & 70 & 1 \\
 & & 0 & 0 & 8 & 0 & 56 & -2 \\
 & & & 0 & 1 & 0 & 28 & 1
\end{bmatrix},
$$

provided that the equations corresponding to zero rows of the matrix G^L are removed from the system. This system is solvable and underdetermined. Therefore, we can choose any non-trivial solutions that interest us from the point of view of diagonal domination, for example:

$$
\begin{aligned}
1)\quad & r_{-2} = 1; \\
& r_{-1} = r_0 = \ldots = r_9 = 0; \\
& g_{-1} = g_1 = g_3 = g_4 = \ldots = g_{10} = 0; \\
& g_{-2} = 8;\; g_0 = -11;\; g_2 = 5; \\
2)\quad & r_{-2} = r_0 = r_1 = \ldots = r_9 = 0; \\
& r_{-1} = 1; \\
& g_{-2} = \tfrac{56}{3};\; g_{-1} = -30;\; g_0 = \tfrac{49}{4}; \\
& g_1 = g_2 = \ldots = g_{10} = 0; \\
3)\quad & r_{-2} = -\tfrac{418}{25};\; r_{-1} = \tfrac{147}{25}; \\
& r_0 = 6;\; r_1 = \tfrac{4452}{25};\; r_3 = 28; \\
& r_2 = r_4 = r_5 = \ldots = r_9 = 0; \\
& g_{-2} = g_{-1} = g_1 = g_3 = \\
& = g_5 = g_6 = \ldots = g_{10} = 0; \\
& g_0 = 803;\; g_2 = 314;\; g_4 = 35; \\
4)\quad & r_{-2} = r_{-1} = r_0 = \\
& = r_2 = r_3 = \ldots = r_9 = 0; \\
& r_1 = 1; \\
& g_{-2} = g_{-1} = g_3 = g_4 = \ldots = g_{10} = 0; \\
& g_0 = g_2 = 1;\; g_1 = -2; \\
5)\quad & r_{-2} = \tfrac{2811263}{3668250};\; r_{-1} = -\tfrac{6928323}{26900500}; \\
& r_0 = -\tfrac{533981}{1614030};\; r_1 = \tfrac{278031899}{20175375}; \\
& r_2 = 1; r_3 = \tfrac{62437363}{2421045};\; r_5 = 4; \\
& r_4 = r_6 = r_7 = \ldots = r_9 = 0; \\
& g_{-2} = g_{-1} = g_0 = g_1 = g_3 = \\
& = g_5 = g_7 = g_8 = g_9 = g_{10} = 0;
\end{aligned}
$$

$$g_2 = \tfrac{475695383}{4842090};$$
$$g_4 = \tfrac{110858263}{1936836};\ g_6 = 5;$$

6) $$r_{-2} = r_{-1} = r_0 = r_1 = = r_2 = r_4 = r_5 = \ldots = r_9 = 0;$$
$$r_3 = 1;$$
$$g_{-2} = g_{-1} = g_0 = g_1 = = g_5 = g_6 = \ldots = g_{10} = 0;$$
$$g_2 = g_4 = 1;\ g_3 = -2;$$

7) $$r_{-2} = r_{-1} = r_0 = r_2 = r_8 = 0;$$
$$r_1 = 4; r_3 = \tfrac{394}{15}; r_4 = 1; r_5 = \tfrac{238}{15};$$
$$r_6 = -\tfrac{13}{30}; r_7 = -\tfrac{122}{15}; r_9 = -\tfrac{26}{15};$$
$$g_{-2} = g_{-1} = g_1 = g_3 = = g_5 = g_7 = g_9 = 0;$$
$$g_0 = 5; g_2 = \tfrac{347}{6};\ g_4 = 100;$$
$$g_6 = \tfrac{27}{5}; g_8 = -21; g_{10} = -\tfrac{13}{6}.$$

So, the first seven columns of the matrix G^L are

$$\begin{bmatrix}
8 & \frac{56}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & -30 & 0 & 0 & 0 & 0 & 0 \\
-11 & \frac{49}{4} & 803 & 1 & 0 & 0 & 5 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 \\
5 & 0 & 314 & 1 & \frac{475695383}{4842090} & 1 & \frac{347}{6} \\
0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 35 & 0 & \frac{110858263}{1936836} & 1 & 100 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & \frac{27}{5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{13}{6} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.$$

From the structure of the matrix G^L it immediately follows, that the values of $\mathbf{h}^L$ at odd nodes are calculated from the explicit equations

$$\frac{c_i^L}{h_i} = \begin{cases} 30, & i = \ -1, 2^L-3; \\ 2, & i = \ 1, 3, \ldots, 2^L-5, \end{cases},$$

while

$$8h_i = r_i,\ i = -2, 2^L-2,$$

and for the rest values of $\mathbf{h}^L$ at even nodes a system of linear equations is solved:

$$\left.\begin{array}{r}
803h_0 + 5h_4 - 5h_6 \\
314h_0 + \frac{475695383}{4842090}h_2 + \frac{347}{6}h_4 - 5h_8 \\
35h_0 + \frac{110858263}{1936836}h_2 + 100h_4 + 589h_6 - 5h_{10} \\
5h_2 + \frac{27}{5}h_4 + 1392h_6 + 589h_8 - 5h_{12} \\
-21h_4 + 589h_6 + 1392h_8 + 589h_{10} - 5h_{14} \\
-\frac{13}{6}h_4 + 589h_8 + 1392h_{10} + 589h_{12} - 5h_{16} \\
i = 12, 14, \ldots, 2^L-16: \\
-5h_{i-6} + 589h_{i-2} + 1392h_i + 589h_{i+2} - 5h_{i+6} \\
i = 2^L-14: \\
-5h_{i-6} + 589h_{i-2} + 1392h_i + 589h_{i+2} - \frac{13}{6}h_{i+6} \\
i = 2^L-12: \\
-5h_{i-6} + 589h_{i-2} + 1392h_i + 589h_{i+2} - 21h_{i+4} \\
i = 2^L-10: \\
-5h_{i-6} + 589h_{i-2} + 1392h_i + \frac{27}{5}h_{i+2} + 5h_{i+4} \\
i = 2^L-8: \\
-5h_{i-6} + 589h_{i-2} + 100h_i + \frac{110858263}{1936836}h_{i+2} + 35h_{i+4} \\
i = 2^L-6: \\
-5h_{i-6} + \frac{347}{6}h_{i-2} + \frac{475695383}{4842090}h_i + 314h_{i+2} \\
i = 2^L-4: \\
-5h_{i-6} + 5h_{i-4} + 803h_i
\end{array}\right\} = r_i,\ i = 0, 2, \ldots, 2^L-4. \quad (13)$$

Here the right-hand sides of the equations (13) are calculated from the formulas

$$
\begin{aligned}
r_{-2} &= c_{-2}^L + \frac{56}{3} h_{-1}, \\
r_0 &= c_0^L + 11 h_{-2} + \frac{49}{4} h_{-1} + h_1, \\
r_2 &= c_2^L - 5 h_{-2} + h_1 + h_3, \\
r_i &= \begin{cases} c_i^L + h_{i-1} + h_{i+1}, \\ i = 4, 6, \ldots, 2^L - 8, \\ c_i^L + h_{i-1} + h_{i+1} - 5 h_{i+4}, \\ i = 2^L - 6, \\ c_i^L + h_{i-1} + \frac{49}{4} h_{i+1} + 11 h_{i+2}, \\ i = 2^L - 4, \\ c_i^L + \frac{56}{3} h_{i-1}, \\ i = 2^L - 2. \end{cases}
\end{aligned}
$$

Theorem 1 *The stability of the calculations is guaranteed.*

Proof. Note that the matrix of the system (13) has a strict diagonal dominance [21, p. 78] over the columns of the system. So far, meanwhile the system of equations has a unique solution because of linear independence of basis functions, the stability of the calculations by the sweep method is guaranteed.

Now, from the equality (10) it follows that to find the matrix $R^L = \left[P^L \mid Q^L \right]^{-1} \cdot G^L$, it is required to apply to the rows of the matrix (12) an inverse permutation, that is, a permutation in which the records of the image and the inverse image are interchanged. From this, we can obtain the required in (8) representation of the matrix R^L.

To recheck the above analytical estimate, one can simply multiply the matrices $\left[P^L \mid Q^L \right]^{-1}$ and G^L to obtain that the matrix R^L consists of two blocks: the first one for $2^{L-1} - 3$ basic spline functions V_{L-1}, the second one for 2^{L-1} basic wavelets W_{L-1}

$$
\left[\begin{array}{ccccccc|cc|c}
0 & 0 & 6 & 0 & -\frac{533981}{1614030} & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \ddots \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & -\frac{13}{30} & 0 & 12 & \ddots \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & -1 & \ddots \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & 0 & \ddots \\
 & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\hline
1 & 0 & -\frac{418}{25} & 0 & \frac{2811263}{3668250} & 0 & 0 & 0 & & \\
0 & 1 & \frac{147}{25} & 0 & -\frac{6928323}{26900500} & 0 & 0 & 0 & & \\
0 & 0 & \frac{4452}{25} & 1 & \frac{278031899}{20175375} & 0 & 4 & 0 & -4 & \ddots \\
0 & 0 & 28 & 0 & \frac{62437363}{2421045} & 1 & \frac{394}{15} & 0 & 20 & \ddots \\
0 & 0 & 0 & 0 & 4 & 0 & \frac{238}{15} & 1 & 304 & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{122}{15} & 0 & 304 & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{25}{15} & 0 & 20 & \ddots \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & -4 & \ddots \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & 0 & \ddots \\
 & & & & \vdots & \vdots & \ddots & \ddots & \ddots &
\end{array} \right].
$$

Here, the diagonal dots mean that the previous two columns are repeated the corresponding number of times, each time moving two positions to the right and moving one position down. The last eight columns of both blocks of the matrix R^L mirror the first eight columns; the empty positions of matrices are equal to zero.

As a result, the values of the spline coefficients on a thinned grid and the wavelet coefficients are calculated by the explicit formulas (8).

3.1 The algorithm of wavelet analysis consists of performing the following steps:

1. On a finite segment $[a, b]$ the values of the observation results are reset to zero at the ends by subtracting values of the ninth degree Hermitian polynomial [16]

$$\sum_{i=0}^{4}(b-a)^i\left[(-1)^i f^{(i)}(a)\eta_i(1-v)+ \right. \tag{14}$$

$$\left. +f^{(i)}(b)\eta_i(v)\right], v=\frac{x-a}{b-a},$$

where

$$\eta_i(t)=(1-t)^5\sum_{\beta=0}^{4-i}\frac{(4+\beta)!}{4!i!\beta!}t^{i+\beta},\ i=0,1,\ldots,4,$$

from the entire time series.

2. The wavelet decomposition algorithm for a given L is incorporated.
3. If $L>3$, then the value of L decreases by 1, and the algorithm goes to step 2.
4. Otherwise, at each level L of the decomposition, the rejection of insignificant wavelet coefficients is performed according to some criterion [2], and the spline coefficients are sequentially restored according to the moving average algorithm (4).
5. After wavelet analysis of the differences obtained at the first stage of the algorithm and reconstructing (of course, with some approximation) the spline coefficients for the densest mesh, the values of the polynomial (14) are added to values of the approximating spline of the seventh degree.

4 Precision check for polynomials

Let $x \in [0, 1]$. By setting $L = 4$ at the top resolution level, we get the grid step length $2^{-4} = 1/16$. To perform the wavelet transform, need to take zero values of the function and derivatives at the ends of the segment, while the values of the function at the nodes of the Δ^L grid will be used as initial data, 13 numbers in total.

Because the seventh-degree polynomial with ten zero boundary conditions does not exist we will bound considering the tenth-degree polynomial $f(x) = x^5(1-x)^5$ as a test function. We find at the last stage of the recursive wavelet decomposition algorithm, five values of the coefficients of the spline $s^3(x)$, $\mathbf{c}^3 = [1.293\cdot10^{-6}, 6.203\cdot10^{-6}, 9.755\cdot10^{-6}, 6.203\cdot10^{-6}, 1.293\cdot10^{-6}]^T$. In this case, the wavelet coefficients are equal to $\mathbf{d}^3 = [1.196\cdot10^{-8}, 1.281\cdot10^{-8}, 1.082\cdot10^{-7}, -1.284\cdot10^{-8}, -1.284\cdot10^{-8}, 1.082\cdot10^{-7}, 1.281\cdot10^{-8}, 1.196\cdot10^{-8}]^T$. To demonstrate the accuracy property of the approximation scheme for this example of polynomials, we will neglect all the wavelet coefficients, providing a compression factor of 13/5=2.6, and we will show here the graph of the difference between the approximation spline and polynomial (Fig. 1), which has the alternating character near the ends of the interval.

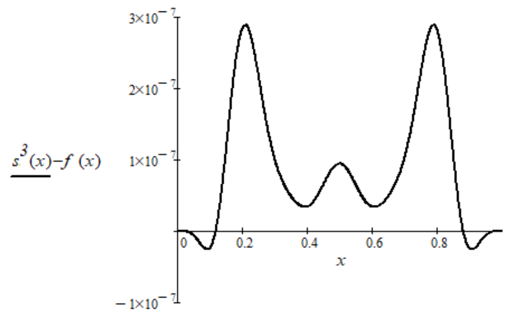

Figure 1: Graph of difference between the 7th-degree approximation spline and the tenth-degree polynomial

The compression coefficient in the considered example is small (only 8 wavelet coefficients are discarded). But the more will be the length of smoothness intervals, the higher will be the compression quality.

5 Conclusion

The article considers the further development of the author's procedure [22] of even-odd partitioning of the defining system of the Hermite wavelet expansion for the practically important case of approximation that does not require values of derivatives of functions, based on B-splines of the seventh degree.

The advantage of the new algorithm is the ease of implementation, since at each decomposition step a seven-diagonal matrix with strict diagonal dominance is solved.

The directions of our future research are to extend the proposed approach to splines of a higher degree and a greater number of zero moments, which can provide new opportunities for developing algorithms for performing wavelet decomposition and signal recomposition.

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