EQUATIONS
DOI: 10.37394/232021.2021.1.1

Designing a Generalized PIO for Polytropic Discrete Time Systems

ADDISON RIOS-BOLIVAR
Universidad de Los Andes
Departamento de Sistemas de Control
Núcleo La Hechicera , Mérida
VENEZUELA

FRANCKLIN RIVAS
Universidad de Los Andes
Lab. de Sistemas Inteligentes, Mérida 5101
VENEZUELA
Pontificia Universidad Católica del Ecuador-sede Ibarra
Ibarra, Provincia Imbabura
ECUADOR

Abstract: Considering the Linear Parameter Varying (LPV) systems of discrete time, in this paper an approach for the synthesis of robust Proporcional+Integral Observers (PIO) is presented. From LPV systems characterized with polytopical uncertainties, the method of design is based on considering a dynamics extended of the typical PIO, in order to transform the design of the matrices of the dynamics of the observer, as a design of the gain of Static Output Feedback (SOF) of a problem of robust control. Under these conditions and from the norms $\mathscr{H}_2/\mathscr{H}_\infty$ described as Linear Matrix Inequalities (LMI), the criteria to obtain the gain in the SOF problem are established; taking into account performance indices in $\mathscr{H}_2$ and $\mathscr{H}_\infty$, under the presence of uncertainties and disturbances.

Key–Words: Discrete-time systems. LPV Systems. Proportional+Integral Observers (PIO). Linear Matrix Inequalities (LMI). $\mathcal{H}_2$-$\mathcal{H}_\infty$ Norms

1 Introduction

The Linear Parameter Variable (LPV) systems are referred as those linear dynamical systems whose representations in state space depends on exogenous nonstationary parameters [16]. The LPV systems are a generalization of LTV systems, establishing an intermediate model between linear and nonlinear dynamics, which can become a representative model for the control of nonlinear processes, allowing the use of all machinery linear control systems linear to the particular case of nonlinear processes [7, 2].

From a practical point of view, LPV system has at least two interesting interpretations [3]: 1) It can be seen as a LTI system with parametric uncertainty. 2) It can be seen as an LTV model, or a model resulting from the linearization of a nonlinear system (SNL) along the trajectories of the parameter α. This last statement is important since the LPV system represents a description of one intermediate system, which enables to design controllers sor observers for nonlinear systems, in a systematic way such as for linear systems. Also, if the nonlinear model is formulated as a linear system parameterized, where parameterization is dependent states, allows a description LPV represent a nonlinear system not locally, taking advantage of the consequences of a global stabilization [2].

On the other hand, the causal observation is the problem of finding estimates for the current values of a set of signals from the present and past values of another set of signals, where both sets of signals are interconnected by the action a dynamic system. The latter is called *observer*, and the procedure is known as an estimate or reconstruction of states. An important feature is that the estimate is asymptotically exact, ie, to converge the actual value of the observed signals as time goes to infinity (asymptotic observability). In the case of LPV systems, this idea is followed: designing a dynamic system that allows the asymptotic estimation of states, under the presence of parametric variations.

In principle, the observer design problem for LPV systems involves an analysis of the observability of such systems. Thus, in [1] the notion of invariant subspaces for LPV systems is presented, introducing the concept of invariant subspaces of parameter variations, which is very important for the design of state observers, characterizing a geometric condition for the observability.

In the same vein, in [17] some characterizations are presented, and necessary and sufficient conditions for the existence of observers for linear functions and variant finite dimensional systems are given. The results are evaluated for affine parameter variant systems and bilinear control systems.

For the design of observers in LPV discrete-time systems, in [6] a synthesis method based on interpolation is discussed. Estimate the stability of the error is evaluated by the existence of a Lyapunov function dependent, affine way, regarding parameters and a rate of asymptotic decay defined. In [10] observation state is used for the synthesis of state feedback controllers in discrete LPV systems. The observer is Luenberger raised type, regardless unknown entries.

On the other hand, in [4] a PIO for estimating state and unknown inputs in discrete systems without uncertainty applies. For the same class of systems, in [8] a PIO for estimating state and unknown inputs and outputs are presented.

In this paper, a method for designing observers for LPV discrete-time systems based on the Proportional+Integral (PI) observers, considering polytopic type parametric variations in the process and sensors, is presented. The stability of the observer system is analyzed by Lyapunov stability of LPV discrete time polytopic systems. The dynamic observer is constructed by a control formulation by static output feedback (SOF), considering performance indices in $\mathscr{H}_2/\mathscr{H}_\infty$; and also, the possibility of reconstructing unknown inputs.

Notation. $\Re$ is the set of real numbers. For a matrix A, A^T denote its transpose. $\mathrm{tr}\,(A)$ defines the trace of the matrix A. $\mathrm{diag}(A, B)$ is a diagonal matrix with entries A and B on its diagonal. In symmetric matrices partitions $\star$ denotes each of the symmetrical blocks. $\mathbb{R}_n$ defines the identity matrix of dimension n.

2 Theoretical Framework

In order to progress in the description of the key tools that will support the results to present, consider the discrete LTI system

$$\begin{aligned} x_{k+1} &= Ax_k + B\omega_k \\ z_k &= Cx_k + D\omega_k \end{aligned} \quad (1)$$

where $x \in \Re^n$ are the states, $\omega_k \in \Re^m$ are exogenous inputs (noise, disturbance); and $z_k \in \Re^q$ are regulated outputs. The matrices A, B, C and D be of appropriate dimensions.

From Lyapunov stability, it is very well known that (1) system is asymptotically stable if and only if there exists a matrix $P = P^T \in \Re^n \times n$, satisfying the following matrix inequalities:

$$P > 0, \qquad P - A^TPA > 0 \quad (2)$$

This condition can be described as follows:

Lemma 1 (Quadratic stability) *Be the system* (1). *If $P = P^T > 0$, the matrix $G \in \Re^{n\times n}$ and $Q = P^{-1}$, then the following conditions of asymptotic stability* (1) *are equivalents:*

i) $P > 0$ *and* $P - A^TPA > 0$.

ii) $P > 0$ *and*

$$\begin{bmatrix} P & A^TP \\ PA & P \end{bmatrix} > 0, \quad (3)$$

iii) $Q > 0$ *and*

$$\begin{bmatrix} Q & AQ \\ QA^T & Q \end{bmatrix} > 0, \quad (4)$$

iv) There exist G, such that $G + G^T > 0$ *and*

$$\begin{bmatrix} G + G^T - P & GA \\ G^TA^T & P \end{bmatrix} > 0, \quad (5)$$

Proof: See [14] □

Consider that is desired to place the poles in a particular stable region, such as in a stable region of radius r and center $(\sigma, 0)$, with $|\sigma| < 1$. In this case, the stability condition corresponds [5, 13]:

$$P = P^T > 0, \quad r^2P - (A - \sigma\mathbb{I}_n)^TP(A - \sigma\mathbb{I}_n) > 0 \quad (6)$$

It follows by considering the dynamic matrix is given by $\frac{A-\sigma\mathbb{I}_n}{r}$.

Lemma 2 (Quadratic stability and pole placement) *Consider the system* (1). *If there exist $P = P^T > 0$, the matrix $G \in \Re^{n\times n}$ and $Q = P^{-1}$,then the following conditions of asymptotic stability for* (1) *are equivalents*

i) $P > 0$ *and* $r^2P - (A - \sigma\mathbb{I}_n)^TP(A - \sigma\mathbb{I}_n) > 0$.

ii) $P > 0$ *and*

$$\begin{bmatrix} rP & A^TP - \sigma P \\ PA - \sigma P & rP \end{bmatrix} > 0, \quad (7)$$

iii) $Q > 0$ *and*

$$\begin{bmatrix} rQ & AQ - \sigma Q \\ QA^T - \sigma Q & rQ \end{bmatrix} > 0, \quad (8)$$

iv) There exist G, such that $G + G^T > 0$ *and*

$$\begin{bmatrix} r(G + G^T - P) & GA - \sigma G \\ G^TA^T - \sigma G^T & rP \end{bmatrix} > 0, \quad (9)$$

Proof: The procedure for the proof of Lema 1 is followed; whereas the dynamic matrix is $\frac{A-\sigma\mathbb{I}_n}{r}$. □

In the same vein, other important results to be taken into account, corresponds to the extended LMIs characterizations of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms for discrete time systems [19, 14]. These standard and extended characterizations of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms as LMIs can be combined with pole placement in particular regions, whereas the dynamic matrix is $\frac{A-\sigma\mathbb{I}_\ltimes}{r}$, which helps to improve the transient response of the dynamics of the state estimator.

2.1 Robust stability and performance in discrete LPV systems

Consider LPV discrete-time system given by

$$\begin{aligned} x_{k+1} &= A(\alpha(k))x_k + B(\alpha(k))\omega_k \\ z_k &= C(\alpha(k))x_k + D(\alpha(k))\omega_k \end{aligned} \quad (10)$$

The system (10) can be characterized as a polytope, defining

$$\mathcal{P} := \begin{pmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{pmatrix} \in \Omega. \quad (11)$$

where Ω is a polytopic set:

$$\Omega := \left\{ \mathcal{P} : \mathcal{P} = \sum_{i=1}^{l} \alpha_i \mathcal{P}_i; \quad \alpha_i \geq 0; \quad \sum_{i=1}^{l} \alpha_i = 1 \right\}; \quad (12)$$

such that any admissible matrix $\mathcal{P}$ of the system can be written as a convex combination of l vertices matrices given, so that

$$\mathcal{P}_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \quad (13)$$

where A_i, B_i, C_i, D_i, for $i = 1, \ldots, l$, are the vertices of the polytope, and they are known matrices. Thus, this system can be characterized by the convex hull of Ω considering the vertices of the polytope, ie

$$\mathcal{C}_o\,\Omega = \left\{ \begin{pmatrix} A_1 & B_1 \\ C_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_l & B_l \\ C_l & 0 \end{pmatrix} \right\} \quad (14)$$

for $\alpha_i \geq 0,\ \ i = 1, \ldots, l,\ \ \sum_{i=1}^{l} \alpha_i = 1$.

The stability of polytopic system [9] is given by

$$P_i = P_i^T > 0, \quad P_i - A_i^T P_i A_i > 0, \quad i = 1, \ldots, l \quad (15)$$

Following the results of Lemma 1, the robust stability condition for polytopic system (10) can be summarized as follows:

Lemma 3 (Robust stability) *Consider the system* (10). *If there exist $P_i = P_i^T > 0$, the matrix $G \in \Re^{n\times n}$, $i = 1, \ldots, l$, then the following conditions for robust stability are equivalents*

i) $P_i = P_i^T > 0, \quad P_i - A_i^T P_i A_i > 0,$

ii) There exist G, such that $G + G^T > 0$ and

$$\begin{bmatrix} G + G^T - P_i & GA_i \\ G^T A_i^T & P_i \end{bmatrix} > 0, \quad (16)$$

Proof: The procedure is similar to the proof of Lemma 1 and the considerations presented in [9]. □

Similarly, the robust performance of the polytopic system (10) can be analyzed from the following:

Lemma 4 ($\mathscr{H}_2$ Robust performance) *Consider the system* (10). *For $P_i = P_i^T > 0$, the following statements are equivalent*

i) System (10) *is robustly stable and $\left\| C_i(z\mathbb{I} - A_i)^{-1} B_i + D_i \right\|_2 < \mu$; for $i = 1, \ldots, l$*

ii) There exist $P_i = P_i^T \in \Re^{n\times n}$, $W = W^T \in \Re^{q\times q}$ y $G \in \Re^{n\times n}$, such that: $\mathrm{tr}(W) < \mu^2$ *and*

$$\begin{bmatrix} G + G^T - P_i & GA_i & GB_i \\ A_i^T G^T & P_i & 0 \\ B_i^T G^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad (17)$$

$$\begin{bmatrix} W & C_i & D_i \\ C_i^T & P_i & 0 \\ D_i^T & 0 & \mathbb{I} \end{bmatrix} > 0, \quad (18)$$

Proof: See [14] and [11]. □

Lemma 5 ($\mathscr{H}_\infty$ Robust performance) *Consider the system* (10). *If $P_i = P_i^T > 0 \in \Re^{n\times n}$, $i = 1, \ldots, l$; and the matrix $G \in \Re^{n\times n}$, the following statements are equivalent:*

i) System (10) *is robustly estable and $\left\| C_i(z\mathbb{I} - A_i)^{-1} B_i + D_i \right\|_\infty < \gamma$.*

ii) There exist $P_i = P_i^T > 0$ y G such that

$$\begin{bmatrix} G + G^T - P_i & GA_i & GB_i & 0 \\ A_i^T G^T & P_i & 0 & C_i^T \\ B_i^T G^T & 0 & \gamma\mathbb{I} & D_i^T \\ 0 & C_i & D_i & \gamma\mathbb{I} \end{bmatrix} > 0. \quad (19)$$

Proof: See [14] and [11]. □

Importantly, the robust performance in $\mathscr{H}_2$-$\mathscr{H}_\infty$ for polytopic system (10) can be combined with robust pole placement, considering that the dynamic matrix corresponds to $\frac{A_i - \sigma\mathbb{R}_n}{r}$.

3 Main results

In this section, the design procedures of generalized PIO for LPV discrete-time systems are explained.

3.1 A PIO for LPV discrete time systems

A PIO is characterized by the incorporation of an additional integral term estimate of the output estimation error in order to design the observer, which may offer certain degrees of freedom.

In [18] a generalization for PIO design is presented, considering an explicit parametric solution of Sylvester matrix equations for the observer gain. In the following proposal, the design of the generalized PIO gain is obtained by solving a control problem by SOF, considering the closed loop stability and following characterization of the quadratic stability as a feasibility problem LMIs. Accordingly, consider LPV system

$$\begin{aligned} x_{k+1} &= A(\alpha)x_k + Bu_k; \quad x(0) = x_0 \\ y_k &= C(\alpha)x_k \end{aligned} \tag{20}$$

taking the same considerations for the model (10), except that no uncertainty for for the operation of the actuators, since the control matrix B is assumed known and constant. It is recognized that the pair $(C(\alpha), A(\alpha))$ is observable, for all α.

From the PIO model given in [12], consider the following generalized version

$$\begin{aligned} \hat{x}_{k+1} &= F\hat{x}_k + K_I\vartheta_k + K_P\left(y_k - \hat{y}_k\right) + Bu_k, \\ \vartheta_{k+1} &= L\vartheta_k + H\left(y_k - \hat{y}_k\right) \\ \hat{y}_k &= J\hat{x}_k \end{aligned} \tag{21}$$

where the matrices F, L, H, J, K_P (proportional gain) and K_I(integral gain) are of appropiate dimensions, which are defined as the observer matrices to be determined, provided that

$$\lim_{k\to\infty}\begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} = 0 \tag{22}$$

being $e_k = x_k - \hat{x}_k$ the estimation error. The variable ϑ is related to the "weighted" integral of the output estimation error. The matrix L is a fading effect coefficient for regulating the transient response of the observer. The matrix H is a coefficient of additional integral effect, which improves the stability margin. If $L = 0$, a classical PIO is obtained. If $L \neq 0$, it can be interpreted as a generalized PIO, since the dynamics of the observer is enriched, reaffirming their behavior as PIO, which is important in applications of fault diagnosis based on observers. Also, if steady state $\vartheta_k = 0$, it implies that $y_k - J\hat{x}_k = 0$.

Definition 6 *Dynamic system*(21) *is said to be a generalized PIO of full order for the system* (20)*, if only if, the matrices F, L, H, J, K_P and K_I are such that the expression* (22) *is satisfied.*

Accordingly, the following theorem can be stated:

Theorem 7 *Dynamic system* (21) *is said to be a generalized PIO of full order for the system* (20)*, if only if, the matrices F, L, H, J, K_P and K_I are such that*

1. *$F = A(\alpha)$ and $J = C(\alpha)$.*
2. *The matrix $\mathfrak{A}(\alpha)$:*

$$\mathfrak{A}(\alpha) = \begin{pmatrix} F - K_PC(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix} \tag{23}$$

is stable in the sense of Lyapunov.

Proof: From the dynamic error

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} = \left(A(\alpha) - K_PC(\alpha)\right)x_k - \\ &\quad \left(F - K_PJ\right)\hat{x}_k - K_I\vartheta_k \\ \vartheta_{k+1} &= L\vartheta_k + H\left(C(\alpha)x_k - J\hat{x}_k\right) \end{aligned}$$

then, if $F = A(\alpha)$ y $J = C(\alpha)$,

$$\begin{aligned} \begin{pmatrix} e_{k+1} \\ \vartheta_{k+1} \end{pmatrix} &= \begin{pmatrix} F - K_PC(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix}\begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} \\ &= \mathfrak{A}(\alpha)\begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} \end{aligned} \tag{24}$$

The dynamic system (24) must be quadratically stable, so that the condition condition (22) is satisfied, which implies that the matrix $\mathfrak{A}(\alpha)$ be quadratically stable. □

The matrix $\mathfrak{A}(\alpha)$ can be expressed as

$$\begin{aligned} \mathfrak{A}(\alpha) &= \begin{pmatrix} A(\alpha) & 0 \\ 0 & 0 \end{pmatrix} + \\ &\quad \begin{pmatrix} -K_P & -K_I \\ H & L \end{pmatrix}\begin{pmatrix} C(\alpha) & 0 \\ 0 & \mathbb{I}_q \end{pmatrix} \end{aligned} \tag{25}$$

Thus, consider the matrices

$$\begin{aligned} \mathcal{A}_o(\alpha) &= \begin{pmatrix} A(\alpha) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}_o = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_q \end{pmatrix}, \\ \mathcal{C}_o(\alpha) &= \begin{pmatrix} C(\alpha) & 0 \\ 0 & \mathbb{I}_q \end{pmatrix}; \end{aligned} \tag{26}$$

and the matrix

$$\mathbb{K} = \begin{pmatrix} -K_P & -K_I \\ H & L \end{pmatrix}. \tag{27}$$

The matrices given in (26), defining the dynamic system

$$\begin{aligned} z_{k+1} &= \mathcal{A}_o(\alpha)z_k + \mathcal{B}_ov_k \\ \eta_k &= \mathcal{C}_o(\alpha)z_k \end{aligned} \tag{28}$$

donde $z_k = \begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix}$. The system (28) is characterized by the polytope

$$\mathcal{P}_i = \begin{pmatrix} \mathcal{A}_{o_i} & \mathcal{B}_o \\ \mathcal{C}_{o_i} & 0 \end{pmatrix} \qquad i = 1, \dots, l \tag{29}$$

It is easily verifiable that the pair $(\mathcal{A}_o(\alpha), \mathcal{B}_o)$ is controllable. In addition, the system satisfies the conditions for the design of a gain to the problem of SOF [15], given that $\mathcal{B}_o = \mathbb{I}_{n+q}$, the necessary condition is that the pair $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ be observable, which corresponds that the pair $(C(\alpha), A(\alpha))$ be observable. Therefore, the matrix $\mathbb{K}$ corresponding to the feedback gain to the control problem by SOF for the system (28).

Lemma 8 *Consider the system* (28)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable (equivalently, the pair* $(C(\alpha), A(\alpha))$ *observable). Then, the system admits a SOF control, given by* $v(t) = \mathbb{K}\eta(t)$*, such that closed loop dynamics is asymptotically stable.*

Proof: In effect, if $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ is observable and the structure of the matrix $\mathcal{B}_o$, the control $v(t) = \mathbb{K}\eta(t)$ stabilizes the closed loop dynamic matrix, given by

$$\mathcal{A}_o(\alpha) + \mathcal{B}_o\mathbb{K}\mathcal{C}_o(\alpha) = \mathfrak{A}(\alpha)$$

□

Consequently, the stabilization problem of the dynamic matrix $\mathfrak{A}(\alpha)$ for the generalized PIO corresponds to design the gain $\mathbb{K}$ into the problem of stabilization by SOF of the system (28). This has the advantage that the design is obtained by direct solution of a problem of stabilizing LPV control systems, which has been extensively studied for those uncertain systems.

3.2 Designing a generalized PIO by SOF

The main result of this work is described by the following theorem:

Theorem 9 *Let be system* (28)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. There exist a gain* $\mathbb{K}$*, for the control by SOF, provided that the matrix* $\mathfrak{A}(\alpha)$ *be stable, if only if, there exist* $P_i = P_i^T > 0$*, for* $i = 1, \dots, l$*; the matrix* G *and the matrix* Y *such that the following LMI is satisfied*

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & P_i \end{bmatrix} > 0, \tag{30}$$

where the gain $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \tag{31}$$

Proof: According to the robust stability established in Lemma 3, $\mathfrak{A}(\alpha)$ will be stable if exist $P_i = P_i^T > 0$ and G are such that

$$\begin{bmatrix} G + G^T - P_i & G\mathfrak{A}(\alpha) \\ \star & P_i \end{bmatrix} > 0$$

By substitution, the matrix inequalities appear, which are linearized by changes of variables $G\mathcal{B}_o = \mathcal{B}_o M$ and $Y = M\mathbb{K}$.

Therefore

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & P_i \end{bmatrix} > 0$$

resulting the LMI defined by (30) and the expression (31) that allows to obtain $\mathbb{K}$. □

Calculating the gain $\mathbb{K}$, the matrices G, H, K_P y K_I of the generalized PIO are obtained. If the location of poles is relevant, then the following result can be applied:

Lemma 10 *Consider the system* (28)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. There exist a gain* $\mathbb{K}$ *for control by SOF, provided that the matrix* $\mathfrak{A}(\alpha)$ *has its poles in the circular region of radius* r *and center* $(\sigma, 0)$*, if only if, there exist* $P_i = P_i^T > 0$ *for* $i = 1, \dots, l$*, the matrix* G*, and the matrix* Y *such that the following LMI is satisfied*

$$\begin{bmatrix} r(G + G^T - P_i) & G\mathcal{A}_{o_i} - \sigma G + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & rP_i \end{bmatrix} > 0, \tag{32}$$

where the gain $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \tag{33}$$

Proof: Proof follows from Theorem 9. □

Given that $F = A(\alpha)$ and $J = C(\alpha)$, for purposes of practical implementation of the generalized PIO, as most observers to LPV systems, consider the matrix $F = A_0$ and $J = C_0$, where A_0 and C_0 are the central matrices of the respective polytopes, which allows to preserver the robustness condition in the design of the generalized PIO.

3.3 Designing a generalized PIO with $\mathscr{H}_2 - \mathscr{H}_\infty$ performance

Consider LPV perturbed discrete time system

$$\begin{aligned} x_{k+1} &= A(\alpha)x_k + B_1(\alpha)\omega_k + Bu_k \\ y_k &= C(\alpha)x_k + D(\alpha)\omega_k, \end{aligned} \tag{34}$$

where $\omega_k \in \Re^r$ are unknown perturbations. For all parameter α, it is assumed that $(C(\alpha), A(\alpha))$ is observable. In principle, the design of the observer must correspond to minimizing effects of disturbances in the state estimation.

Again, it is considered that the uncertain matrices $A(\alpha), B_1(\alpha), C(\alpha), D(\alpha)$ belong to a convex polytopic set, $\forall\ \alpha_i \geq 0$, $\sum_{i=1}^{l} \alpha_i = 1$, defined by

$$\Omega = \left\{ \sum_{i=1}^{l} \alpha_i \Big(A^{(i)}, B_1^{(i)}, C^{(i)}, D^{(i)} \Big) \right\}. \tag{35}$$

For the generalized PIO given by (21), if $F = A(\alpha)$ y $J = C(\alpha)$, then

$$\begin{pmatrix} e_{k+1} \\ \vartheta_{k+1} \end{pmatrix} = \begin{pmatrix} F - K_P C(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix} \begin{pmatrix} e_k \\ \vartheta_k \end{pmatrix} + \begin{pmatrix} B_1(\alpha) - K_P D(\alpha) \\ HD(\alpha) \end{pmatrix} \omega_k \tag{36}$$

Considering the matrices defined by (26), the gain matrix given by (27), and the following matrices

$$B_{1_o}(\alpha) = \begin{pmatrix} B_1(\alpha) \\ 0 \end{pmatrix}, \quad D_o(\alpha) = \begin{pmatrix} D(\alpha) \\ 0 \end{pmatrix}, \tag{37}$$

the following dynamic system is derived

$$\begin{aligned} z_{k+1} &= \mathcal{A}_o(\alpha) z_k + \mathcal{B}_{1_o}(\alpha)\omega_k + \mathcal{B}_o v_k, \\ \eta_k &= \mathcal{C}_o(\alpha) z_k + D_o(\alpha)\omega_k, \end{aligned} \tag{38}$$

which admits a SOF control $v_k = \mathbb{K}\eta_k$, such that the closed loop dynamics (36), with the output η_k, satisfying a $\mathscr{H}_2 - \mathscr{H}_\infty$ performance index. The gain $\mathbb{K}$, that defines the generalized PIO, is obtained by solving of a robust optimal control in $\mathscr{H}_2 - \mathscr{H}_\infty$. Consequently, the closed-loop dynamic is

$$\begin{aligned} z_{k+1} &= \left(\mathcal{A}_o(\alpha) + \mathcal{B}_o \mathbb{K} \mathcal{C}_o(\alpha)\right) z_k + \\ &\quad \left(\mathcal{B}_{1_o}(\alpha) + \mathcal{B}_o \mathbb{K} \mathcal{D}_o(\alpha)\right) \omega_k \\ \eta_k &= \mathcal{C}_o(\alpha) z_k + D_o(\alpha)\omega_k \end{aligned} \tag{39}$$

Such as has been stated, the observer design should consider the minimizing of the effects of the disturbance ω_k in the state estimation. This performance criterion can be imposed by minimizing of the $\mathscr{H}_2$ norm or $\mathscr{H}_infty$ norm for the transfer function of the system (39).

3.3.1 Design in $\mathscr{H}_2$

Let be $T_{\omega\eta}(\mathbf{z})$ the transfer function of the disturbance ω to the regulated output η for the system (39). The design problem of a generalized PIO with robust performance in $\mathscr{H}_2$ corresponds to the following:

Theorem 11 *Consider the system* (34) *on the polytope* (35)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (21)*, that is determined by the gain* $\mathbb{K}$ *solving the SOF control problem for the system* (38)*, guaranteeing a suboptimal performance in* $\mathscr{H}_2$ *for* (39)*, ie,* $\left\| T_{\omega\eta}(\mathbf{z}) \right\|_2 < \mu$*, from the following optimization problem:*

$$\min_{\substack{P_i, Y, W, G \\ i = 1, \dots, l.}} \ \mathrm{tr}(W), \qquad \textit{such that}$$

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} & G\mathcal{B}_{1_{oi}} + \mathcal{B}_o Y \mathcal{D}_{o_i} \\ \star & P_i & 0 \\ \star & \star & \mathbb{I} \end{bmatrix} > 0,$$

$$\begin{bmatrix} W & \mathcal{C}_{o_i} & \mathcal{D}_{o_i} \\ \star & P_i & 0 \\ \star & \star & \mathbb{I} \end{bmatrix} > 0, \tag{40}$$

where, $P_i = P_i^T > 0$*,* $i = 1, \dots, l$ *and the matrices* G *(con* $G + G^T > 0$*),* W*,* Y *have appropriate dimensions. Thus, the gain* $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1} Y, \quad \textit{where} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \tag{41}$$

Proof: Assuming that there is a feasible solution to the formulated optimization problem, in accord to Lemma 4, the inequality (17) is linearized by changing variable $Y = \mathcal{B}_o M$, where $G\mathcal{B}_o = \mathcal{B}_o M$; previously the respective substitutions are made, in order to obtain the LMIs (40). □

This formulation reduces the conservatism, which occurs when a fixed Lyapunov matrix $P^T = P$ is used. Thus, it is possible to obtain a Lyapunov function for each vertex of the polytope without forcing a single Lyapunov matrix for the LPV estimation system.

The immediate result that is derived, corresponding to the location of poles in a particular region, together with the minimization of the $\mathscr{H}_2$ norm.

Lemma 12 *Consider the system* (34) *on the polytope* (35)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (21)*, provided that the closed loop dynamic matrix* $\mathfrak{A}(\alpha)$ *has its poles in the circular region of radius* r *and center* $(\sigma, 0)$*, it is determined by the gain* $\mathbb{K}$ *solving the SOF control problem for the system* (38)*, guaranteeing a suboptimal performance in* $\mathscr{H}_2$ *for* (39)*, ie,* $\left\| T_{\omega\eta}(\mathbf{z}) \right\|_2 < \mu$*, from the following optimization problem:*

$$\min_{\substack{P_i, Y, W, G \\ i = 1, \dots, l.}} \ \mathrm{tr}(W), \qquad \textit{such that}$$

$$\begin{bmatrix} r(G+G^T-P_i) & \Upsilon_o & r\left(G\mathcal{B}_{1_{oi}}+\mathcal{B}_oY\mathcal{D}_{o_i}\right) \\ \star & rP_i & 0 \\ \star & \star & r\mathbb{I} \end{bmatrix} > 0,$$

$$\begin{bmatrix} W & \mathcal{C}_{o_i} & \mathcal{D}_{o_i} \\ \star & P_i & 0 \\ \star & \star & \mathbb{I} \end{bmatrix} > 0, \qquad (42)$$

where $\Upsilon_o = G\mathcal{A}_{o_i} - \sigma G + \mathcal{B}_oY\mathcal{C}_{o_i}$, $P_i = P_i^T > 0$, $i = 1,\ldots,l$ *and the matrices* G, W, Y *have appropriate dimensions. Thus, the gain* $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{where} \quad M = \mathcal{B}_o^{-1}G\mathcal{B}_o \qquad (43)$$

Proof: Proof follows from Theorem 11. □

3.3.2 Design in $\mathscr{H}_\infty$

Theorem 13 *Consider system* (34) *on the polytope* (35)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (21) *is determined by the gain* $\mathbb{K}$*, solving the SOF control problem for the system* (38)*, guaranteeing a suboptimal performance in* $\mathscr{H}_\infty$ *for* (39)*, ie,* $\left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty < \gamma$*, from the following optimization problem:*

$$\min_{\substack{P_i, G, Y,\\ i=1,\ldots,l.}} \left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty, \qquad \textit{such that}$$

$$\begin{bmatrix} G+G^T-P_i & G\mathcal{A}_{o_i}+\mathcal{B}_oY\mathcal{C}_{o_i} & G\mathcal{B}_{1_{oi}}+\mathcal{B}_oY\mathcal{D}_{o_i} & 0 \\ \star & P_i & 0 & \mathcal{C}_{o_i}^T \\ \star & \star & \gamma\mathbb{I} & \mathcal{D}_{o_i}^T \\ \star & \star & \star & \gamma\mathbb{I} \end{bmatrix} > 0 \qquad (44)$$

where $P_i = P_i^T > 0$, $i = 1,\ldots,l$ *and the matrices* G, Y *have appropriate dimensions. Thus, the gain* $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1}G\mathcal{B}_o \qquad (45)$$

Proof: Similarly, assuming that there is a feasible solution to the formulated optimization problem, by the change of variable $Y = \mathcal{B}_oM$, with $Q^T\mathcal{B}_o = \mathcal{B}_oM$, to Item *iii)* of the Lemma 5, the respective substitutions are made, in order to obtain the LMI given by (44). □

For $\mathscr{H}_\infty$ robust performance and location of poles in a particular circular region, then:

Lemma 14 *Consider system* (34) *on the polytope* (35)*, with* $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *observable. A generalized PIO given by* (21)*, provided that the closed loop dynamic matrix* $\mathfrak{A}(\alpha)$ *has its poles in the circular region of radius* r *and center* $(\sigma, 0)$*, it is determined by the gain* $\mathbb{K}$ *solving the SOF control problem for the system* (38)*, guaranteeing a suboptimal performance in* $\mathscr{H}_\infty$ *for* (39)*, ie,* $\left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty < \gamma$*, from the following optimization problem:*

$$\min_{\substack{P_i, Y, G\\ i=1,\ldots,l.}} \left\|T_{\omega\eta}(\mathbf{z})\right\|_\infty, \qquad \textit{such that}$$

$$\begin{bmatrix} r(G+G^T-P_i) & \Gamma_o & r\left(G\mathcal{B}_{1_{oi}}+\mathcal{B}_oY\mathcal{D}_{o_i}\right) & 0 \\ \star & rP_i & 0 & r\mathcal{C}_{o_i}^T \\ \star & \star & r\gamma\mathbb{I} & r\mathcal{D}_{o_i}^T \\ \star & \star & \star & r\gamma\mathbb{I} \end{bmatrix} > 0 \qquad (46)$$

where $\Gamma_o = G\mathcal{A}_{o_i} - \sigma G + \mathcal{B}_oY\mathcal{C}_{o_i}$, $P_i = P_i^T > 0$, $i = 1,\ldots,l$ *and the matrices* G, Y *have appropriate dimensions. Thus, the gain* $\mathbb{K}$ *is obtained from*

$$\mathbb{K} = M^{-1}Y, \quad \textit{with} \quad M = \mathcal{B}_o^{-1}G\mathcal{B}_o \qquad (47)$$

Proof: The procedure for the proof of Theorem 13 is followed. □

The results in $\mathscr{H}_2$ and $\mathscr{H}\infty$ can be combined to establish mixed performance indices. Importantly, once the gain obtained $\mathbb{K}$ proceed to determine the matrices of the generalized PIO. Similarly, some conditions transient performance in the dynamic of the observer can be imposed, which represents a method of tuning of the PIO design parameters. Thus, a LMIs characterizing the pole location in a particular circular region can be specified, which ensures faster for estimating, important when controllers are designed from the estimated states.

3.4 Generalized PIO design by perturbation reconstruction

Section 3.3 has been dedicated to the design of generalized PIO, imposing performance indices in $\mathscr{H}_2/\mathscr{H}_\infty$ in order to minimize the effects of disturbance signals in the estimation error e_k and in the signal ϑ_k that, as mentioned, is related to the "weighted" integral of the estimation error of the output. At this point, the idea is to consider that ϑ_k represents an approximation of the disturbance ω_k, that is, the signal $\vartheta_k = \hat{\omega}_K$ is defined, then the design of the PIO must ensure that

$$\lim_{k\to\infty} \begin{pmatrix} e_k \\ \xi_k \end{pmatrix} = 0 \qquad (48)$$

where $\xi_k = \omega_k - \hat{\omega}_k$. This is the more relevant design specification for the synthesis of UIO. Under certain conditions, this design specification can be satisfied by a PIO. In effect, consider the discrete polytopic system (34), with $B_1(\alpha) = B_1$ a constant matrix and $D(\alpha) = 0$, that means the unknown input is in system dynamic. The unknown input may represent actuator or process failures, which which are to rebuild. There, $(C(\alpha), A(\alpha))$ is assumed observable. A generalized PIO (21) guarantees the condition (48) if some criteria for the choice of design matrices are satisfied, as it is shown below. Under the new features of the dynamics of (34) and if $\vartheta_k = \hat{\omega}_k$, then:

$$\begin{pmatrix} e_{k+1} \\ \hat{\omega}_{k+1} \end{pmatrix} = \begin{pmatrix} A(\alpha) - K_P C(\alpha) & -K_I \\ HC(\alpha) & L \end{pmatrix} \begin{pmatrix} e_k \\ \hat{\omega}_k \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \omega_k \quad (49)$$

In order to satisface (48), ξ_k dynamically should be evaluated, thus

$$\begin{aligned} \xi_{k+1} &= \omega_{k+1} - \hat{\omega}_{k+1} = \omega_{k+1} - \hat{\omega}_{k+1} + \xi_k - \xi_k \\ &= \xi_k - LHC(\alpha)e_k + L\hat{\omega} - \hat{\omega} + \omega_{k+1} - \omega_k \end{aligned} \quad (50)$$

Consider $\varepsilon = \omega_{k+1} - \omega_k$. The first constraint imposed is that ω should be a smooth signal in the sense that $\varepsilon \approx 0$. Combining the dynamic equations of closed loop for errors e and ξ, then

$$\begin{aligned} e_{k+1} &= (A(\alpha) - K_P C(\alpha))e_k - K_I \hat{\omega}_k + B_1 \omega_k \\ \xi_{k+1} &= -LHC(\alpha)e_k + \xi_k + L\hat{\omega}_k - \hat{\omega}_k + \varepsilon \end{aligned} \quad (51)$$

From (51), if $K_I = B_1$ and $L = \mathbb{I}$, then the closed loop dynamics will be

$$\begin{pmatrix} e_{k+1} \\ \xi_{k+1} \end{pmatrix} \begin{pmatrix} A(\alpha) - K_P C(\alpha) & B_1 \\ -HC(\alpha) & \mathbb{I} \end{pmatrix} \begin{pmatrix} e_k \\ \xi_k \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}, \quad (52)$$

which must be asymptotically stable in order to satisfy the condition (48), through the appropriate choice of design matrices K_P and H.

Following the design procedure, the synthesis of K_P and H of the PIO must be transformed, once established the conditions for the matrices K_I and L, in a control problem by SOF. Thus, in this case, the following matrices are considered

$$\begin{aligned} \mathcal{A}_o(\alpha) &= \begin{pmatrix} A(\alpha) & B_1 \\ 0 & \mathbb{I}_r \end{pmatrix}, \quad \mathcal{B}_o = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_r \end{pmatrix}, \\ \mathcal{C}_o(\alpha) &= \begin{pmatrix} C(\alpha) & 0 \end{pmatrix}; \end{aligned} \quad (53)$$

and the matrix

$$\mathbb{K} = \begin{pmatrix} -K_P \\ -H \end{pmatrix}. \quad (54)$$

The matrices in (53) defining a dynamic system given by (28), where $z_k = \begin{pmatrix} e_k \\ \xi_k \end{pmatrix}$. In addition, matrices defined in (53) characterize a polytope given by (29). If $\mathfrak{A}(\alpha)$ is the closed loop dynamic matrix for (52), the Lemma 8 can be applied, as

$$\mathfrak{A}(\alpha) = \mathcal{A}_o(\alpha) + \mathcal{B}_o \mathbb{K} \mathcal{C}_o(\alpha).$$

Theorem 15 *Consider system* (51)*, defining the matrices $\mathcal{A}_o(\alpha)$, $\mathcal{B}_o$, $\mathcal{C}_o(\alpha)$. If*

1. $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ *is observable, (necessary condition).*
2. $\varepsilon \approx 0$.
3. $K_I = B_1$ *and* $L = \mathbb{I}_r$*, for the PIO given by* (21).

There exist a gain $\mathbb{K}$ for SOF control, provided that the matrix $\mathfrak{A}(\alpha)$ be stable: $\lim_{k\to\infty} \begin{pmatrix} e_k \\ \xi_k \end{pmatrix} = 0$, if only if, there exist $P_i = P_i^T > 0$, with $i = 1, \dots, l$; the matrix G and the matrix Y such that following LMI is satisfied

$$\begin{bmatrix} G + G^T - P_i & G\mathcal{A}_{o_i} + \mathcal{B}_o Y \mathcal{C}_{o_i} \\ \star & P_i \end{bmatrix} > 0, \quad (55)$$

where $\mathbb{K}$ is obtained from

$$\mathbb{K} = M^{-1}Y, \quad \text{with} \quad M = \mathcal{B}_o^{-1} G \mathcal{B}_o \quad (56)$$

Proof: See the proof of Theorem 9. □

Remark 16 *The necessary condition $(\mathcal{C}_o(\alpha), \mathcal{A}_o(\alpha))$ be observable not only depends on the matrices of the original system, but also to impose L and K_I, such as these design matrices are defined in* (53)*, establishing difference from the procedure presented in Section 3.1 and Section 3.3. Thus, this design technique proves to be very restrictive.*

From $\mathbb{K}$ the design matrices K_P and H are obtained. For LPV discrete time systems, this procedure is a generalization of the results presented in [4]. Moreover, the Lemma 10 can be applied to ensure that the closed loop poles are located in a particular stable region. In order to show the effectiveness of the technique, some numerical examples and simulations have been made, details of which are not shown for reasons of space.

4 Conclusions

A technique for designing of generalized Proportional+Integral observers for linear discrete-time systems with polytopic uncertainties has been presented. The generalization is to incorporate, in the dynamics of PI observer, a design matrix representing a fading effect coefficient, which allows adjusting the transient response of the observer. Thereafter, the method involves extending the dynamics of PIO, this allows the synthesis of design matrices of the observer, through its transformation into a control problem by static output feedback for LPV systems. Then, considering the extended LMIs characterizations of $\mathscr{H}_2/\mathscr{H}_\infty$ norms and the location of poles in a particular region, the design technique is generalized to LPV systems with disturbances, imposing performance indices in $\mathscr{H}_2/\mathscr{H}_\infty$, specifications for the synthesis of the gain in the control problem by SOF, which results in the matrices of the IOP. The synthesis can be extended to consider multi-objective object specifications. Similarly, exploiting the ability of unknown input reconstruction PI observer, it has design ensuring PIO approximate reconstruction of the disturbance, under appropriate technical considerations. This last method is very restrictive, limiting the selection of parameters PIO, which can result in marginal stability or infeasibility of the solution. Through numerical examples to show the effectiveness of the technique.

Acknowledgements: The research was supported by the CDCHTA-ULA under grant project **I-1302-12-02-B**, and support given by the Secretaría de Educación Superior, Ciencia, Tecnología e Innovación of Ecuador and Prometeo Program.

References:

[1] Gary J. Balas, Jozsef Bokor, and Zoltán Szabó. Invariant subspaces for lpv systems and their applications. *IEEE Trans. Automat. Control*, 48:2065–2069, 2003.

[2] Corentin Briat. *Robust control and observation of LPV time-delay systems*. PhD thesis, Institut National Polytechnique de Grenoble-INPG, 2008.

[3] F. Bruzelius, S. Pettersson, and C. Breitholtz. Linear parameter-varying descriptions of nonlinear systems. In *Proc. American Control Conf.*, pages 1374–1379, Boston, Massachusetts, 2004. ACC.

[4] Jeang-Lin Chang. Applying discrete-time proportional integral observers for state and disturbance estimations. *Automatic Control, IEEE Transactions on*, 51(5):814–818, 2006.

[5] M. Chilali and P. Gahinet. $\mathcal{H}_\infty$ design with pole placement constraints: An LMI approach. *IEEE Tran. Aut. Control*, 41(3):358–367, 1996.

[6] Jamal Daafouz, Iula Bara, Frédéric Kratz, and José Ragot. State observers for discrete-time LPV systems: an interpolation based approach. In *39th IEEE Conference on Decision and Control, CDC 2000*, pages 4571–4572, 2000.

[7] Guang-Ren Duan and Hai-Hau Yu. *LMIs in Control Systems: Analysis, Design and Applications*. CRC Press, Center for Control Theory and Guidance Technology, China, 2013.

[8] Zhiwei Gao, Tim Breikin, and Hong Wang. Discrete-time proportional and integral observer and observer-based controller for systems with both unknown input and output disturbances. *Optimal Control Applications and Methods*, 29(3).

[9] Ĺubomír Grman, Danica RosinovÁ, Vojtech Veselỳ, and Alena KozÁ KovÁ. Robust stability conditions for polytopic systems. *International Journal of Systems Science*, 36(15):961–973, 2005.

[10] WP Maurice H Heemels, Jamal Daafouz, and Gilles Millerioux. Observer-based control of discrete-time LPV systems with uncertain parameters. *Automatic Control, IEEE Transactions on*, 55(9):2130–2135, 2010.

[11] G. Hilhorst, G. Pipeleers, R.C.L.F Oliveira, P.L.D. Peres, and J. Swevers. On extended LMI conditions for H_2/H_∞ control of discrete-time linear systems. In *Proc. 19th IFAC World Congress*, volume 1, pages 9307–9312, Cape Town, South Africa, 2014. IFAC.

[12] Tadeusz Kaczorek. *Pole-Zero Assignment Techniques and Some Their Applications*. Number 159 in III. IAC Istituto per le Applicazioni del Calcolo, 1978.

[13] Vinícius E Montagner, Valter JS Leite, and Pedro LD Peres. Discrete-time switched systems: Pole location and structural constrained control. In *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, volume 6, pages 6242–6247. IEEE, 2003.

[14] Goele Pipeleers, Bram Demeulenaere, Jan Swevers, and Lieven Vandenberghe. Extended LMI characterizations for stability and performance of linear systems. *Systems & Control Letters*, 58(7):510–518, July 2009.

[15] A. Ríos-Bolívar. *Control de Sistemas Lineales: Realimentando la salida*. Talleres Gráficos Universitarios, ULA, Mérida, 2014.

[16] Jeff S. Shamma. *Control of Linear Parameter Varying Systems with Applications*, chapter An Overview of LPV Systems, pages 3–26. Springer, 2012.

[17] Jochen Trumpf. Observers for linear time-varying systems. *Linear Algebra and its Applications*, 425:303–312, 2007.

[18] Ai-Guo Wu and Guang-Ren Duan. Generalized PI observer design for linear systems. *IMA J. of Math. Control & Information*, 25(2):239–250, 2008.

[19] Shengyuan Xu, James Lam, and Yun Zou. New versions of bounded real lemmas for continuous and discrete uncertain systems. *Circuits, Systems & Signal Processing*, 26(6):829–838, 2007.