A Simplified Novel Link for A Simplified Stability Analysis of Finite Difference Time Domain Method

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Abstract: - Numerical stability and numerical dispersion analyses are critical subjects for Finite Difference Time Domain (FDTD) method. To perform these analyses, first of all, an equivalency of the FDTD numerical dispersion equation for Maxwell's equations and wave equation is proven in this study. Then, based on those calculations, a simplified version of a novel link is developed. Using this simplified version, a stability criterion and an amplification factor of the FDTD method are more easily extracted. Therefore, the FDTD stability analysis becomes simpler. The theoretical findings are validated by a numerical example of a late time simulation interval in the FDTD method. In particular, the effect of a hard FDTD source and a soft FDTD source on the growth (amplification) factor is also investigated.

Key-Words: - FDTD method, stability analysis, stability criterion, amplification factor, numerical dispersion.

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1 Introduction

Finite Difference Time Domain (FDTD) is a popular and effective numerical method for the solution of complex realistic electromagnetic problems, [1], [2]. Therefore, investigations on numerical analyses of the FDTD method are a valuable concern that can be formulated in two folds: numerical stability analysis and numerical dispersion analysis, [3]. To perform these two analyses, first of all, a numerical dispersion equation (NDE) of the FDTD method must be extracted. This can be performed in two different ways using Maxwell's equations (ME) or wave equation (WE). If the NDEs are different for ME and WE, the numerical analyses of any numerical methods in electromagnetics will inherently differ for ME and WE, [3]. This makes the numerical analyses more complex and tiresome. In this sense, one of the important examples of the numerical time domain methods is the Pseudo Spectral Time Domain (PSTD) method. The numerical analyses of the PSTD show that the PSTD method behaves differently in the case of ME and WE. This is due to the eigenvalues of WE PSTD having a second-order spatial differentiation matrix compared to ME PSTD having a first-order spatial differentiation matrix that is closer to the physical models. Therefore, WE PSTD is more robust to the numerical deficiencies rather than ME PSTD, [4], [5]. For this reason, in this paper, this concept is investigated especially for the FDTD method. First of all, an equivalency (unification) of the FDTD NDE for ME and WE are proven. Then, based on those calculations, a simplified version of a novel link approach is developed. Thus, a stability criterion and an amplification factor of the FDTD method are more easily extracted. This leads to a more simple numerical analysis of the FDTD method. Finally, the theoretical findings are validated by a numerical FDTD example at the late times.

The rest of this article is organized as follows. The equivalency of the FDTD NDE for ME and WE is proven in Section I. In Section II, the details for a complex-frequency approach and a novel link approach (classical) are revisited. In Section III, the details for the extraction of a simplified version of the novel link are given. In Section IV, the theoretical findings are validated by a numerical example of a late-time FDTD simulation result. In particular, the effect of a hard FDTD source and a soft FDTD source on the growth (amplification) factor is also investigated. In Section V, conclusions deducted from the theoretical findings and the numerical results are discussed.

2 Fundamentals

The numerical analysis of the FDTD method is intensively investigated by classical methods of

- The matrix eigenvalue method,

- The energy method,
- The von Neumann (Fourier) method.

The matrix eigenvalue and the energy method require cumbersome calculations. Therefore, the most common approach to analyze the stability of the FDTD method is von Neumann (or Fourier) method. It is based on a decomposition of the fields into a discrete complex spatial (position) function and a discrete real-time function. The first one is represented by an expansion of an exponential Fourier series. In the second one, the discrete-time function satisfies a quadratic equation. Then, the numerical stability results in a unity-or-less growth factor of the time function by evaluating the root locations of its quadratic equation. However, the rigorous (exact) stability criterion of the FDTD method cannot be found in this way, [1], [2]. Therefore, some alternative approaches (yet rigorous) are proposed. They are based on the numerical analysis of one of the extracted parameters in the FDTD NDE. In this sense, three different approaches are

- The complex-frequency approach based on a complex-valued ω,
- The novel link approach based on a complex-valued Δt,
- The simplified novel link approach is based on an equivalency of the NDE for ME and WE.

In this section, first of all, the complex-frequency approach and the novel link approach are aptly and briefly revisited. These two approaches have already known in the literature, [2], [3]. Then, the simplified novel link approach is presented in detail. The novelty of this paper is lying on the third approach.

In order to gain further physical insight and sake for simplicity, let us consider the one-dimensional (1D) FDTD NDE for WE as

$$\cos(\omega\Delta t) = 1 + \left(\frac{c^2\Delta t^2}{\Delta x^2}\right)(\cos(k_N\Delta x) - 1). \quad (1)$$

In a classical view, the numerical dispersion analysis of the FDTD method is based on the extraction of a complex-valued numerical wave number $k_N = k_N' + jk_N''$ from the 1D NDE as

$$k_N \frac{1}{\Delta x}\arccos\left[1 + \frac{1}{S^2}(\cos(\omega\Delta t) - 1)\right] \quad (2)$$

where $S = c\Delta t/\Delta x$, ω, and Δt are considered real-valued numbers. $\Delta x = \lambda/N$, λ is the wavelength at the given operating frequency f ($\omega = 2\pi f$) and N defines the grid resolution. In this step, an alternative compact form of k_N also can be formulated that

$$k_N = \frac{1}{\Delta x}\arccos\left(\frac{1}{S^2}\left(\cos\left(S\frac{2\pi}{N}\right) - 1\right) + 1\right). \quad (3)$$

Now, a transitional value for $N = N_{trans}$ at limiting cases of $\arccos(\cdot)$ is found as

$$N = N_{trans} = \frac{2\pi S}{\arccos(1 - 2S^2)} \quad (4)$$

where it is worth noting that the numerical dispersion analysis is based on the evaluations of N_{trans} only for its real value. The details for this analysis are given in [2]. Therefore, it is not repeated here.

For the FDTD stability analysis, the application of the methods based on the parameter extraction mentality is similar to the way the extraction of k_N. However, the extracted ω (or Δt) from the NDE is especially evaluated instead of k_N in the stability analysis. In the complex-frequency approach (the first way), k_N and ω are considered complex-valued numbers for the numerical dispersion and stability analyses, respectively. However, in the novel link approach (the second way), k_N and Δt are considered complex-valued numbers for the numerical dispersion and the stability analyses, respectively. This gives a chance to evaluate the NDE in different manners.

2.1 The Complex Frequency Approach

This approach is also known as a complex wavenumber method and is based on the extraction of the angular frequency ω from the NDE, [2]. Accordingly, $\omega = \omega' + j\omega''$ over the discrete wave $u_i^n = e^{-j(k_N i\Delta x - \omega n\Delta t)}$ is extracted from the NDE as

$$\omega = \frac{1}{\Delta t}\arccos\big(1 + S^2(\cos(k_N\Delta x) - 1)\big) \quad (5)$$

where S, k_N and Δt are considered real-valued numbers. The main concept of the complex-frequency approach in the sense of the input parameter (real-valued positive numbers of Δt and Δx) and the output parameter (the complex-valued number of ω) is shown in Fig.1. The complex-valued number nature of ω originates from the behavior of the function $\arccos(\cdot)$ its argument is given in (5).

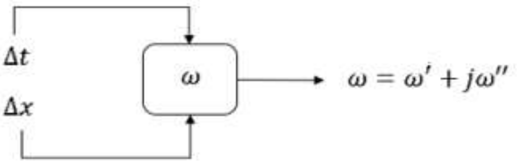

Fig. 1: The main concept of the complex-frequency approach.

Upon close scrutiny of the ω equation, ω' and ω'' are found as

$$\omega' = \frac{\pi}{\Delta t} \quad ; \quad \omega'' = \frac{1}{\Delta t} ln\left(-\eta - \sqrt{\eta^2 - 1}\right) \qquad (6)$$

where the stability and the instability correspond to the case of $\omega = \omega'$ and $\omega = \omega' + j\omega''$, respectively ($\eta = 1 - 2S^2$). Here, it is worth noting that ω has complex values in the unstable region.

Now, considering ω being a real-valued number ($\omega = \omega'$) corresponding to a real-valued number range of $\arccos(\cdot)$, the exact FDTD stability condition is formulated as

$$-1 \le 1 - 2S^2 \le 1 \quad \Rightarrow \quad 0 \le \underbrace{\frac{c\Delta t}{\Delta x}}_{S} \le 1 \qquad (7)$$

and considering ω being a complex-valued number ($\omega = \omega' + j\omega''$), the amplification factor (q_{amp}) corresponding to the FDTD instability ($S > 1$) is found to be

$$q_{amp} = \left(\frac{c\Delta t}{\Delta x} + \sqrt{\left(\frac{c\Delta t}{\Delta x}\right)^2 - 1}\right)^2 \qquad (8)$$

where specially for $S = 1$ ($\Delta x = c\Delta t$) known as a magic time step, $q_{amp} = 1$ leads to the stable algorithm.

Two main disadvantages of the complex frequency approach are

- ω must be thought of as a complex-valued number as $\omega = \omega' + j\omega''$. This causes a loss of physical insight for the operational (source) frequency and leads to controversial consideration in the time-domain solutions since $\omega = 2\pi f$ is a real-valued input parameter in reality as not a system (model) parameter in the FDTD method.
- It does not have the ability to perform the numerical dispersion analysis and the stability analysis simultaneously. The stability analysis and the dispersion analysis have to be independently performed in this way. Let us explain this situation in detail as k_N is assumed to be a complex-valued number while ω is assumed to be a real-valued number in the numerical dispersion analysis whereas ω is assumed to be a complex-valued number while k_N is assumed to be a real-valued number in the stability analysis. It means that two analyses must be performed, independently. There is no link between these two numerical analyses. All these disadvantages are resolved by proposing a novel link approach in [3].

2.2 The Novel Link Approach

This approach is based on the consideration of a complex-valued number of a discrete unit time step as $\Delta t = \Delta t' + j\Delta t''$ while keeping ω is a real-valued positive number not as before the complex-valued number of $\omega = \omega' + j\omega''$ in the complex-frequency approach for the stability analysis of the FDTD method, [2]. This is a more reasonable way since the complex-valued ω causes in loss of physical insight for the operational (source) frequency in the time domain. Moreover, ω is an input parameter (not a model parameter of the system), and its values are already known from the beginning that it must be a real-valued number rather than the complex-valued number due to its reality. The novel link approach resolves this conflict by accepting ω as the real-valued number. The main concept of the novel link approach as input (real-valued positive numbers of ω and Δx) and an output parameter (a complex-valued number of Δt) is shown in Fig.2.

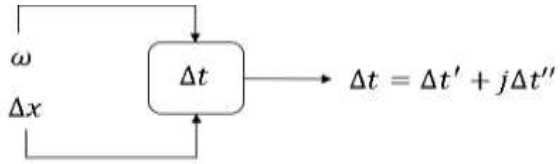

Fig. 2: The main concept of the novel link approach.

The novel link approach is based on the evaluation of N_{trans} that can be a real-valued number or a complex-valued number. This way is completely different from the previous technique since N_{trans} is never used in the complex frequency approach. In fact, N_{trans} is extracted from the numerical analysis that is based on the extraction of k_N. It means that a link is constructed between the numerical dispersion analysis and the stability analysis in the novel link approach. To summarize this method, let us revisit N_{trans} as

$$N_{trans} = \frac{2\pi S}{\arccos(1-2S^2)} \tag{9}$$

where considering N_{trans} being a real-valued number, the exact FDTD stability condition is

$$-1 \leq 1-2S^2 \leq 1 \quad \Rightarrow \quad 0 \leq \underbrace{\frac{c\Delta t}{\Delta x}}_{S} \leq 1 \tag{10}$$

where $\arccos(1-2S^2)$ has the real values in the $[-1,1]$ range of $\arccos(\cdot)$ function.

On the other hand, considering N_{trans} being a complex-valued number, Δt becomes

$$\Delta t = \Delta t' + j\Delta t'' = \frac{1}{N_T f} \tag{11}$$

where f is a real-valued number ($\omega = 2\pi f$) and N_T defines the time resolution, [3]. By analyzing a complex-valued number $N_T = N_{trans}/S$ in detail, $\Delta t'$ and $\Delta t''$can be calculated that

$$\Delta t' = \frac{1}{2f} \;\; ; \; \Delta t'' = \frac{1}{2\pi f} ln\left(-\eta - \sqrt{\eta^2-1}\right). \tag{12}$$

By using $\Delta t''$, the amplification factor corresponding to the FDTD instability ($S > 1$) is

$$\begin{aligned} q_{amp} &= \left(-\eta - \sqrt{\eta^2-1}\right)^n \\ &= \left(S + \sqrt{S^2-1}\right)^{2n} \end{aligned} \tag{13}$$

where let us set n to 1 for evaluation of the unit time step effect. Then, q_{amp} is obtained as

$$q_{amp} = \left(\frac{c\Delta t}{\Delta x} + \sqrt{\left(\frac{c\Delta t}{\Delta x}\right)^2 - 1}\right)^2. \tag{14}$$

This formula is the same as the previously published one of the complex frequency approaches, [2]. Two main advantages of the novel link approach are

- ω is considered a real-valued number whileΔt becomes the real- or the complex-valued number in the analyses. This is more meaningful for reality since Δt is a system (model) parameter and its behavior cannot be predicted and restricted from the beginning due to the fact that it is unknown, yet.
- The dispersion analysis and the stability analysis can be linked with the real-valued ω which gives a chance to unify the numerical analysis of the FDTD method. It is also worth noting that the novel link approach does not base on a simple extraction of Δt from the NDE. This is not simply possible since Δt is present two times in the transcendental form of the NDE. Therefore, a numerical root-finding technique must be used for the calculation of Δt. However, this is not a valuable step for the analytical calculations. Therefore, it cannot be extracted directly analytically as opposed to k_N and ω that they are present only one time in the NDE. The critical role of the novel link approach is shown in Fig.3. Accordingly, the numerical dispersion analysis and the stability analysis are unified in the novel link approach when they must be independently considered in the classical von Neumann (Fourier) method and the complex-frequency approach.

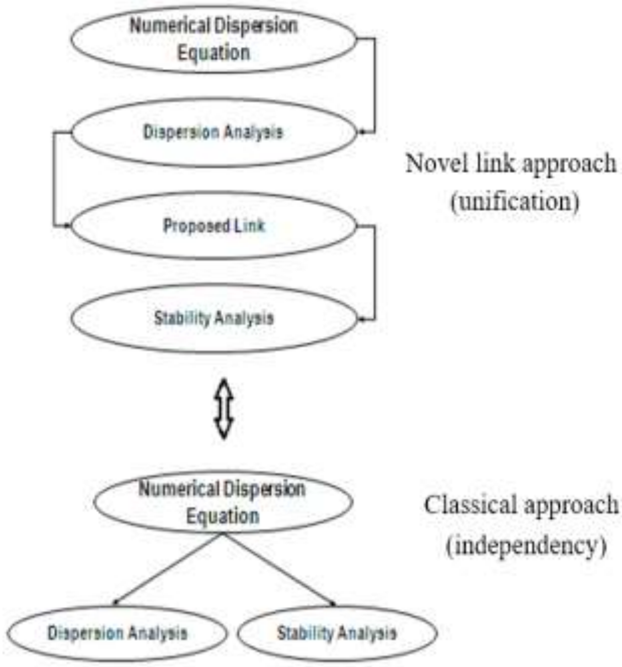

Fig. 3: The critical role of the novel link approach.

3 The Simplified Novel Link Approach

The simplification of the novel link is based on proving the equivalency of the NDEs for ME and WE. For this aim, more generally, let us consider the 3D NDE for Maxwell's equations

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\omega\frac{\Delta t}{2}\right) = \frac{1}{\Delta x^2}\sin^2\left(k_{Nx}\frac{\Delta x}{2}\right) + \frac{1}{\Delta y^2}\sin^2\left(k_{Ny}\frac{\Delta y}{2}\right) + \frac{1}{\Delta z^2}\sin^2\left(k_{Nz}\frac{\Delta z}{2}\right) \quad (15)$$

and also, the 3D FDTD NDE for the wave equation

$$\frac{\cos(\omega\Delta t)-1}{c^2\Delta t^2} = \frac{\cos(k_{Nx}\Delta x)-1}{\Delta x^2} + \frac{\cos(k_{Ny}\Delta y)-1}{\Delta y^2} + \frac{\cos(k_{Nz}\Delta z)-1}{\Delta z^2} \quad (16)$$

where, at first glance, it seems that they are completely different from each other. Now, considering a trigonometric relation of

$$\cos(2X) = 1-2\sin^2(X) \quad \Rightarrow \quad \sin^2(X) = \frac{1}{2}(1-\cos(2X)) \quad (17)$$

where $X = \omega\Delta t$ or $X = k_{Ni}\Delta x$ $(i = x, y, z)$ and applying the following relation for the NDE of ME

$$\sin^2\left(\omega\frac{\Delta t}{2}\right) = \frac{1}{2}(1-\cos(\omega\Delta t)) \quad (18)$$

$$\sin^2\left(k_{Ni}\frac{\Delta x}{2}\right) = \frac{1}{2}(1-\cos(k_{Ni}\Delta x)) \quad (19)$$

and substituting them into the ME NDE with a multiplication of minus two, it becomes

$$\frac{\cos(\omega\Delta t)-1}{c^2\Delta t^2} = \frac{\cos(k_{Nx}\Delta x)-1}{\Delta x^2} + \frac{\cos(k_{Ny}\Delta x)-1}{\Delta y^2} + \frac{\cos(k_{Nz}\Delta x)-1}{\Delta z^2} \quad (20)$$

where it is clear that this form extracted from ME is exactly equal (unified) to the NDE of WE without any approximation. They are exactly the same equations. This makes it possible to remove a complexity in which the same and unique numerical analysis is valid and enough for ME or WE. Extracting the equivalency of the 2D and 1D NDEs of ME and WE is a straightforward job. Therefore, it is not shown here. Using this idea, a novel link proposed in the previous section can be simplified as shown in the next section. For this aim, first, to gain further physical insight and to sake for simplicity, let us again consider the NDE for the 1D Maxwell's equations as ($k_N = k_{Nx}$)

$$\sin^2\left(\omega\frac{\Delta t}{2}\right) = S^2\sin^2\left(k_N\frac{\Delta x}{2}\right) \quad \Rightarrow \quad \sin\left(\omega\frac{\Delta t}{2}\right) = S\sin\left(k_N\frac{\Delta x}{2}\right) \quad (21)$$

where it is necessary to remember that this form of the ME NDE is equivalent to the WE NDE. However, the WE NDE (in the form of $\cos(\cdot)$) is used only for the stability analysis in the literature. Now, for the first time, the ME NDE proving its equality to the WE NDE is used for the stability analysis that enables us to find out the simplified novel link. The simplification of the novel link is based on using $\arcsin(\cdot)$ function instead of $\arccos(\cdot)$ function since the equality of the NDE between ME and WE gives this opportunity. For further progress, let us extract $k_N = k_{Nx}$ again from (2121) given

$$\sin\left(\omega\frac{\Delta t}{2}\right) = S\sin\left(k_N\frac{\Delta x}{2}\right) \quad \Rightarrow \quad k_N = k_{Nx} = \frac{2}{\Delta x}\arcsin\left(\frac{1}{S}\sin\left(\omega\frac{\Delta t}{2}\right)\right) \quad (22)$$

where after some mathematical steps, it becomes ($S = c\Delta t/\Delta x$)

$$k_N = k_{Nx} = \frac{2}{\Delta x}\arcsin\left(\frac{1}{S}\sin\left(S\frac{\pi}{N}\right)\right). \quad (23)$$

Now, evaluating the argument of this simplified version of k_N at the limit values, N_{trans} being a real-valued positive number

$$\frac{1}{S}\sin\left(S\frac{\pi}{N}\right) = +1 \quad \Rightarrow \quad N_{trans} = \frac{\pi S}{\arcsin(S)} \quad (24)$$

where the only $+1$ limit value is used in the numerical analysis since the -1 limit value leads to the negative Δt. The details are given in Appendix I. Here, this form of N_{trans} is inherently much simpler than the previous ones based on the WE NDE since the argument of $\arcsin(\cdot)$ is only S rather than η in the novel link approach.

Now, first of all, consider a region that $\arcsin(S)$ in the denominator of N_{trans} takes only the real-valued positive values as $0 \leq S \leq 1$ (S must be a positive valued number because $c, \Delta t,$ and Δx

cannot have negative values, individually). Importantly, this also corresponds to the *causality principle*. Then, the exact FDTD stability condition is extracted as follows

$$0 \le \underbrace{\frac{c\Delta t}{\Delta x}}_{S} \le 1 \quad \Rightarrow \quad \Delta t \le \frac{\Delta x}{c} \tag{25}$$

where S must be in the interval of [0,1] for the stable solution. In this way, the extraction is performed in an easier way than the previous two methods given in the earlier sections.

Second, let us extract the amplification factor of the FDTD method. Considering N_{trans} being a complex-valued number, it can be proven that, [7]

$$\begin{aligned}\arcsin(S) &= -j\ln\left(jS + \sqrt{1-S^2}\right)\\ &= \frac{\pi}{2} - j\ln\left(S + \sqrt{S^2-1}\right)\end{aligned} \tag{26}$$

then, N_{trans} becomes

$$\begin{aligned}N_{trans} &= \frac{\pi S}{\arcsin(S)}\\ &= \frac{\pi S}{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)}.\end{aligned} \tag{27}$$

Now, reconsidering S over the parameters of $\Delta t = 1/(N_T f)$ and $\Delta x = \lambda/N$ as [3]

$$S = c\frac{\Delta t}{\Delta x} = c\frac{1/N_T f}{\lambda/N} \;\Rightarrow\; N_T = \frac{N(=N_{trans})}{S} \tag{28}$$

where $N = N_{trans}$ shows the limit value of N. Then,

$$N_T = \frac{N_{trans}}{S} = \frac{\pi}{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)} \tag{29}$$

using this relation, the discrete unit time step Δt is formulated as

$$\begin{aligned}\Delta t = \Delta t' + j\Delta t'' &= \frac{1}{N_T f}\\ &= \frac{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)}{\pi f}\end{aligned} \tag{30}$$

where ω is assumed to be a real-valued number as an input parameter. Upon close scrutiny of the Δt equation, $\Delta t'$ and $\Delta t''$ are found to be

$$\Delta t' = \frac{1}{2f} \;\;;\;\; \Delta t'' = -\frac{1}{\pi f}\ln\left(S+\sqrt{S^2-1}\right). \tag{31}$$

where it is clearly figured out that the complex-valued number Δt is possible. Here, two $\Delta t'$ for the novel link and the simplified novel link approaches are already equal to each other. At first glance, it seems that $\Delta t''$ is different from the previous extraction of the novel link approach. However, their equivalency is proven in Appendix II.

Now, the amplification factor (q_{amp}) over Δt corresponding to the FDTD instability ($S > 1$) is extracted by a discrete plane-wave substitution as $e^{-j(k_N i\Delta x - \omega n\Delta t)}$

$$\begin{aligned}q_{amp} = e^{-\omega n\Delta t''} &= e^{2\ln\left(S+\sqrt{S^2-1}\right)}\\ &= \left(\frac{c\Delta t}{\Delta x} + \sqrt{\left(\frac{c\Delta t}{\Delta x}\right)^2 - 1}\right)^2\end{aligned} \tag{32}$$

where let us set n to 1 for evaluation of the unit time step effect. Here, it is clear that the same amplification factor is also extracted in an easier way without conversion of η to S as in the novel link approach. It is directly formulated as a function of S. Especially, the lower limit of the Nyquist criterion corresponding to $N = 2, (\Delta x = \lambda/2)$ is the worst numerical case of the FDTD method.

In fact, the stability and the instability occur in the case of $N < N_T$ and $N > N_T$, respectively. Particularly, the case $N = N_T$ leads to a numerically dispersionless solution, corresponding to $S = 1$ ($\Delta x = c\Delta t$) known as a magic time step. This also shows the validity of the simplified novel link analysis for ME and WE.

4 Numerical Example

A 1D problem in a lossless simple medium is numerically solved for validation of the theoretical findings over the amplification factor. The two FDTD solutions are obtained by developing two independent FDTD codes for WE and ME. The parameters of the problem are listed in Table 1.

In the two numerical experiments, the values of N and S are set to 2 ($\Delta x = \lambda/2$) and 1.0005, respectively. The FDTD update equations for the 1D ME and the 1D WE are well known and can be found in [1], [2]. Therefore, it is not repeated, here.

Table 1. The parameters of the problem

The FDTD unit space step ($\Delta x = \lambda/2$)	0.15 m
The FDTD unit time step (Δt)	0.5 ns
The number of unit cells (N_x)	33333
The problem space (L_x)	5000 m
The operational frequency (f)	1 GHz

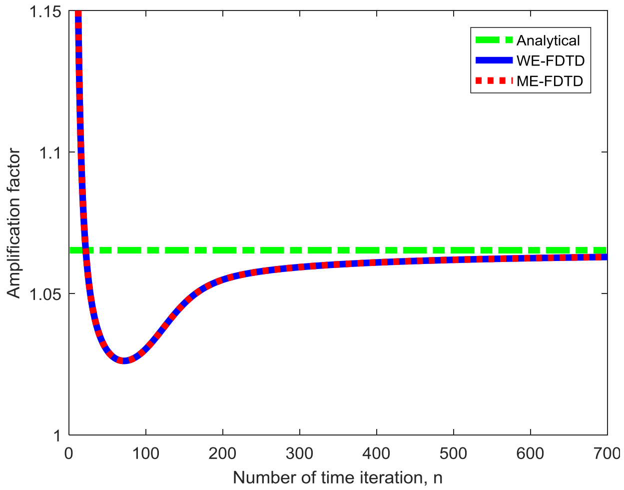

Fig. 4a: Comparison of the FDTD amplification factors between the analytical (formulated) solution, the ME FDTD solution, and the WE FDTD solution (the hard source case).

A hard (or a soft) point monochromatic source is located independently in the middle of the problem space. An observation point is positioned close to the source point. In particular, the effect of the hard FDTD source and the soft FDTD source on the amplification factor is also investigated. Accordingly, the comparison of the analytical (formulated) amplification factor and the independently FDTD calculated amplification factors for ME and WE are shown for the hard FDTD source and the soft FDTD source in Fig.4a and Fig.4b, respectively.

The numerical results show that

- the two FDTD solutions for ME and WE give the exactly same results as it is expected from the proven analytical results in the previous subsections.
- the numerically calculated amplification factors converge to the analytically (formulated) calculated ones. They are in good agreement with the steady-state regime. This is due to the fact that the obtained analytical solutions are valid at the steady state regime since the discrete plane wave representation is used in the analyses.
- the convergence speed of the hard source is slower than the convergence speed of the soft source.

Since it is shown in the previous chapter that the analytical formulations of the FDTD amplification factor for ME and WE are the same, one formula is enough for all the analytical calculations. On the other hand, its numerical calculations cannot be simply performed by a solution of the one FDTD equation. This is due to the fact that ME and WE have different orders and different codes. The main difference is that ME is the first-order equation while WE is the second-order equation.

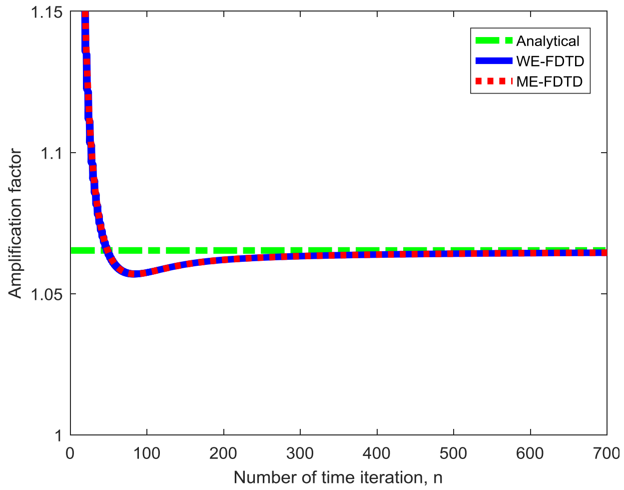

Fig. 4b: Comparison of the FDTD amplification factors between the analytical (formulated) solution, the ME FDTD solution, and the WE FDTD solution (the soft source case).

Here, an important example is worthy of mention that the order difference has a strong effect on the numerical behavior of the method such as the Pseudo Spectral Time Domain (PSTD) method, [5], [6].

5 Conclusion

In this study, a simplified version of the previously proposed novel link is formulated. This yields the simplified stability analysis of the FDTD method. The formulation is based on the equivalency of the ME FDTD and the WE FDTD numerical dispersion equations. Thus, the exact stability criterion and the amplification factor of the FDTD method are more easily extracted by proving their equivalency. The theoretical findings are validated by the two 1D numerical examples (independently for ME and WE) of the late time simulation interval in the FDTD method. In particular, the effect of the hard source and the soft source on the FDTD growth (amplification) factor is also investigated.

In the simplified version, the operational frequency is kept again as the real-valued number which is more physical, and logical and prevents the loss of physical insight. The unification concept of the dispersion analysis and the stability analysis is also kept in a simpler manner. Thus, all these calculations ensure a better and simple understanding of the numerical behavior of the FDTD method.

The simplified link concept may provide new openings for a better numerical understanding of the time domain methods such as the Finite Element Time Domain (FETD) method, PSTD method, and so on. In future works, this analysis can be extended to non-uniform FDTD meshes for more realistic media such as lossy and dispersive mediums.

6 Appendices

6.1 Appendix I

Considering the numerical wavenumber in the form

$$k_N = \frac{2}{\Delta x}\arcsin\left(\frac{1}{S}\sin\left(S\frac{\pi}{N}\right)\right) \tag{33}$$

where the argument of $\arcsin(\cdot)$ has two limit values as either -1 or $+1$. Let us analyze the -1 limit value by inserting it inside the $\arcsin(\cdot)$ as

$$\frac{1}{S}\sin\left(S\frac{\pi}{N}\right) = -1 \;\Rightarrow\; \sin\left(S\frac{\pi}{N}\right) = -S \;\Rightarrow\; S\frac{\pi}{N} = \arcsin(-S). \tag{34}$$

Using negative argument property of $\arcsin(\cdot)$

$$S\frac{\pi}{N} = \arcsin(-S) = -\arcsin(S) \;\Rightarrow\; N = N_{trans} = \frac{-\pi S}{\arcsin(S)}, \tag{35}$$

transforming N_{trans} to its complex equivalent as

$$N_{trans} = \frac{-\pi S}{\arcsin(S)} = \frac{-\pi S}{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)}. \tag{36}$$

Now, reconsidering the parameters of $\Delta t = 1/(N_T f)$ and $\Delta x = \lambda/N$

$$S = c\frac{\Delta t}{\Delta x} = c\frac{\frac{1}{N_T f}}{\frac{\lambda}{N}} \;\Rightarrow\; N_T = \frac{N_{trans}}{S} = \frac{-\pi}{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)} \tag{37}$$

and, using this relation, the discrete unit time step Δt is formulated as

$$\Delta t = \Delta t' + j\Delta t'' = \frac{1}{N_T f} = \frac{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)}{-\pi f}. \tag{38}$$

Equating Δt, $\Delta t'$ and $\Delta t''$can be calculated that

$$\Delta t' = -\frac{1}{2f} \;;\; \Delta t'' = \frac{1}{\pi f}\ln\left(S+\sqrt{S^2-1}\right). \tag{39}$$

Here, a detailed analysis is given for the -1 limit value in the argument of $\arcsin(\cdot)$. Since this case leads to the negative $\Delta t' = -1/2f$, there is no physical correspondence. From the beginning, it is declared that $\Delta t = \Delta t'$must be a real-valued positive number.

6.2 Appendix II

Let us reconsider N_{trans} and N_T from the novel link approach ($\eta = 1 - 2S^2$)

$$N_{trans} = \frac{2\pi S}{\arccos(\eta)} \;\Rightarrow\; N_T = \frac{N_{trans}}{S} = \frac{2\pi}{\pi + j\ln\left(-\eta - \sqrt{\eta^2-1}\right)}. \tag{40}$$

Then, Δt is found to be

$$\Delta t = \frac{1}{N_{trans} f} = \frac{\pi + j\ln\left(-\eta - \sqrt{\eta^2-1}\right)}{2\pi f} \tag{41}$$

where $\Delta t = \Delta t' + j\Delta t''$ is the complex-valued number of the real and the imaginary parts

$$\Delta t' = \frac{1}{2f} \;;\; \Delta t'' = \frac{\ln\left(-\eta - \sqrt{\eta^2-1}\right)}{2\pi f}. \tag{42}$$

Now, let us show N_{trans} and N_T from the simplified novel link approach

$$N_{trans} = \pi\frac{S}{\arcsin(S)} \;\Rightarrow\; N_{trans} = \frac{\pi S}{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)} \tag{43}$$

$$\Delta t = \frac{1}{N_T f} = \frac{\pi/2 - j\ln\left(S+\sqrt{S^2-1}\right)}{\pi f} \tag{44}$$

where again the real and the imaginary parts of the $\Delta t = \Delta t' + j\Delta t''$ are

$$\Delta t' = \frac{1}{2f} \quad ; \quad \Delta t'' = -\frac{\ln\left(S + \sqrt{S^2 - 1}\right)}{\pi f}. \quad (45)$$

Here, $\Delta t''$ seems to be different from the novel link. However, one can note that in the novel link's $\Delta t''$ does have η instead of S as an argument. By taking care of η, the equivalency of the two $\Delta t''$ can be demonstrated as

$$\Delta t'' = \frac{\ln\left(-\eta - \sqrt{\eta^2 - 1}\right)}{2\pi f} \quad \Rightarrow$$
$$\Delta t'' = \frac{\ln\left(2S^2 - 1 - \sqrt{4S^4 - 4S^2 + 1 - 1}\right)}{2\pi f} = \frac{\ln\left(2S^2 - 1 - 2S\sqrt{S^2 - 1}\right)}{2\pi f} \quad (46)$$

$$\Delta t'' = \frac{\ln\left(S^2 + (S^2 - 1) - 2S\sqrt{S^2 - 1}\right)}{2\pi f} = \frac{\ln\left(S^2 - 2S\sqrt{S^2 - 1} + (S^2 - 1)\right)}{2\pi f} \quad (47)$$

Here, let us define $a = S$ and $b = \sqrt{S^2 - 1}$ and rearrange the above equation

$$\Delta t'' = \frac{\ln(a^2 - 2ab + b^2)}{2\pi f}. \quad (48)$$

Then, it becomes

$$\frac{\ln(a - b)^2}{2\pi f} = \frac{2\ln(a - b)}{2\pi f} = \frac{\ln(a - b)}{\pi f} \quad (49)$$

$$\Delta t'' = \frac{\ln\left(S - \sqrt{S^2 - 1}\right)}{\pi f} = -\frac{\ln\left(S - \sqrt{S^2 - 1}\right)^{-1}}{\pi f} = -\frac{1}{\pi f}\ln\left(\frac{1}{S - \sqrt{S^2 - 1}}\right). \quad (50)$$

Finally, it is clear that this form of $\Delta t''$ is identical to the form of the novel link as

$$\Delta t'' = -\frac{\ln\left(S + \sqrt{S^2 - 1}\right)}{\pi f}. \quad (51)$$

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