New Types of Multisoliton Solutions of Some Integrable Equations via Direct Methods

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Abstract: -
Exact explicit solutions, which describe new multisoliton dynamics, have been identified for some KdV type equations using direct methods devised for this purpose. It is found that the equations, having multi-soliton solutions in terms of the KdV-type solitons, possess also an alternative set of multi-soliton solutions which include localized static structures that behave like (static) solitons when they collide with moving solitons. The alternative sets of solutions include the steady-state solution describing the static soliton itself and unsteady solutions describing mutual interactions in a system consisting of a static soliton and several moving solitons. As distinct from common multisoliton solutions those solutions represent combinations of algebraic and hyperbolic functions and cannot be obtained using the traditional methods of soliton theory.

Key-Words: - integrability, solitons, KdV-type equations, direct methods, steady-state solitons

1 Introduction

The Korteweg-de Vries (KdV) equation arises in many physical contexts as the leading order equation governing weakly nonlinear long waves when nonlinearity and dispersion are in balance. To address the higher effects the asymptotic expansion can be extended to the next order in the wave amplitude yielding higher order KdV equations. Some equations of this class proved to be completely integrable and, consequently, possessing all the remarkable properties characteristic of such equations [1, 2]. Chief among these properties is the existence of multi-soliton solutions describing the "elastic" collisions of solitary waves. Some of higher order KdV equations, although not proven integrable, possess multi-soliton solutions too.

In the present paper, some unknown effects in soliton dynamics governed by the higher order KdV-type equations are inferred from new exact explicit solutions of the equations. The solutions have been obtained using direct methods developed in [3, 4]. It is found that some higher order KdV equations, having multi-soliton solutions in terms of the KdV-type solitons, possess also an alternative set of multi-soliton solutions which include localized static structures that behave like solitons when they collide with moving solitons. Both the moving soliton and and the static pattern retain their original shapes, with the only effect of collision on the static pattern being a phase (position) shift. This implies that these steady state patterns may be termed *static solitons*. The alternative sets of solutions include the steady-state solution describing the static soliton itself and unsteady solutions describing mutual interactions in a system consisting of a static soliton and several moving solitons. As distinct from common multisoliton solutions those solutions represent combinations of algebraic and hyperbolic functions. The alternative multi-soliton solutions including static solitons can be algorithmically derived using the methods of [3, 4]. It also seems that, in some cases, such (static soliton + multi-soliton) solutions can be constructed from the common multi-soliton solutions using some rules as an example of the Sawada-Kotera equation [5] shows. The (static soliton + multi-soliton) solutions cannot be obtained using the traditional methods of soliton theory.

2 Methods

2.1 Direct method using a "potential" variable

The method represents an enhanced version of the method developed in [3].

The solution $u(x,t)$ of an evolution equation is sought as a function of new variables with one of them being the 'potential' p, as follows

$$u(x,t) = G\left(p\left(x,t\right), \xi\left(x,t\right)\right); \qquad p(x,t) = \int u(x,t)dx \tag{1}$$

where the second argument, in general, can be chosen arbitrarily. Note that the choice of the limits of integration in the definition of p in (1) is of no importance as it does not influence the solution for $u = p_x$. The use of the potential p as one of the independent variables is essential if the procedure is aimed at defining the solitary wave solutions. For such solutions, u vanishes for (at least) two values of the 'potential' p and so the corresponding choice of the function $G(p, \xi)$ ('Ansatz') provides obtaining solitary wave solutions in the original variables.

The first step of the procedure is determining arbitrary functions of ξ contained in the form of $G(p, \xi)$. To implement this step, the transformation of independent variables $(x, t) \rightarrow (p(x, t), \xi(x, t))$ is made in the equation and then the chosen form of $u = G(p, \xi))$ is substituted into the transformed equation. This results in the relation, usually having the form of a polynomial with coefficients dependent on the unknown functions of ξ and their derivatives, and the requirement of vanishing the coefficients of the polynomial yields an overdetermined system of ordinary differential equations for the unknown functions. Having solutions of that system defined one can implement the next step which is solving the first order PDE for $p(x, t)$

$$p_x = G\left(p, \xi\left(x, t\right)\right) \tag{2}$$

To specify the arbitrary function $F(t)$ contained in the solution of (2) the solution is substituted into the integrated form of the original equation which results in an ODE for $F(t)$. Solving this ODE specifies the expression for $p(x, t)$ completely so that the solution $u(x, t)$ of the original equation can be calculated as $u = p_x$.

The form of the Ansatz (1) for obtaining solutions including static solitons can be inferred from a one-soliton solution of an equation under consideration. Such a solution is represented as a function $u(\xi)$ of a travelling wave argument $\xi = k(x - Vt)$. Eliminating ξ from expressions for $u(\xi)$ and $p(\xi)$ yields the Ansatz $u = G_1(p)$. To include a static soliton the constant coefficients in (4) are replaced by functions of x, as follows

$$G(p, x) = a(x)p^2 + b(x)p + c(x) \tag{3}$$

In particular, for the KdV-like sech^2 solitons, $G_1(p)$ is a quadratic function

$$G_1(p) = ap^2 + bp + c \tag{4}$$

where a, b and c are constants. Replacing the constant coefficients in (4) by functions of x yields the Ansatz

$$G(p, x) = a(x)p^2 + b(x)p + c(x) \tag{5}$$

2.2 Modified Hirota's direct method

The procedure starts from the Hirota transformation

$$u(x, t) = M\frac{\partial^2 \ln f(x, t)}{\partial x^2} \tag{6}$$

where M is a constant to be determined in the course of calculations. Next, a solution of type

$$f(x, t) = f^{(0)}(x) + \sum_{n=1}^{n=\infty} \epsilon^n f^{(n)}(x, t) \tag{7}$$

is sought where ϵ serves as a book-keeping parameter (not a small quantity). Substituting (6) and (7) into the evolution equation and equating to zero the terms with different powers of ϵ yields a system of equations for $f^{(0)}$, $f^{(1)}$ and so on. Although, in general, the system includes an infinite number of equations it can be terminated at some order m if the equations in the higher orders $n > m$ do not contain inputs from the lower orders.

A modification of the traditional Hirota representation for $f(x, t)$ concerns the forms of the functions $f^{(0)}$, $f^{(1)}$ and so on. First of all, as it is already indicated in (7), $f^{(0)}$ is a function of x instead of a constant (unity) of the traditional representation which implies seeking a steady-state pattern as the lowest order solution. In the next orders, the constant coefficients of the exponents with the travelling wave argument are also replaced by functions of x.

3 Alternative set of solutions of the Sawada–Kotera equation

Consider first the integrable Sawada–Kotera (SK) equation [5, 6]

$$u_t + u_{5x} + \beta u u_{3x} + \beta u_x u_{2x} + \frac{\beta^2}{5}u^2 u_x = 0 \tag{8}$$

which, being one of the simplest integrable scalar evolution equations of order higher than 3, is among the equations of major mathematical and physical importance. The solitary wave solution of the SK equation has the form

$$u = \frac{30k^2}{\beta}\text{sech}^2(kx - 16k^5 t - \phi) \tag{9}$$

Besides the KdV-like sech^2 solitons (9) the SK equation (8) admits steady-state solutions of the form [4]

$$u = \left(\frac{60k^2}{\beta}\right)\frac{a - k^2(x + b)^2}{(a + k^2(x + b)^2)^2} \tag{10}$$

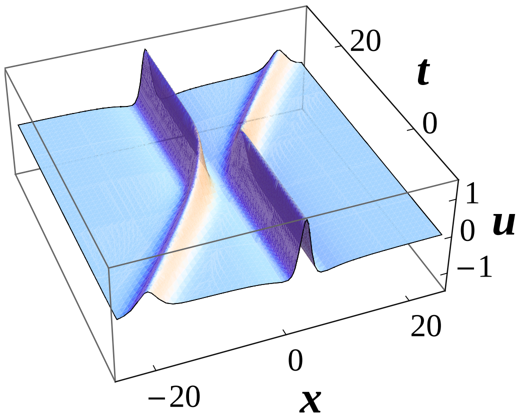

Figure 1: Solution (11) of the SK equation (8): $\beta = 15$, $k = 0.5$, $\phi = -5$.

where a, b and k are constants. Of course their property of being localized is not sufficient for naming those patterns as 'solitons' but it appears that they really behave as (static) solitons when they collide with regular (moving) solitons – their shape remains unchanged after the collision, only a phase (position) shift is observed.

A solution of equation (8) describing collision of a static soliton with a regular (moving) sech^2 soliton is obtained using the method described in Section 2.1 in the form

$$
\begin{aligned}
&u = \left(\frac{30k^2}{\beta}\right)\Big(-3 + k^2x^2 + k^4x^4\mathrm{sech}^2\eta \\
&+6kx\tanh\eta - 3k^2x^2\tanh^2\eta\Big)\Big/ \\
&\Big(3 + k^2x^2 - 3kx\tanh\eta\Big)^2, \\
&\eta = kx - 16k^5t - \phi \qquad (11)
\end{aligned}
$$

The solution is shown in Fig. 1.

Solutions describing mutual interactions in a system of a static soliton and several moving solitons can be algorithmically derived using Hirota's method with a modified solution form described in Section 2.2. Thus, it seems that, for the SK equation, static solitons fit naturally into the KdV type multisoliton dynamics and, in parallel with the common N-soliton solutions, there exist the (static soliton + N-soliton) solutions. An additional support for this view is provided by that, for the SK equation, there exists a simple rule which allows direct construction of the (static soliton + N-soliton) solutions from the common N-soliton solutions. According to this rule, the solution including a static soliton is obtained from a common N-soliton solution represented via Hirota's transformation by supplying each exponent in the function $f(x,t)$ with a corresponding multiplier dependent on x, as follows

$$
\begin{aligned}
&e^{\theta_{i_1}+\theta_{i_2}+\cdots+\theta_{i_m}} \Rightarrow \\
&\left(1 + c_2\left(x - 3\left(\frac{1}{k_{i_1}} + \frac{1}{k_{i_2}} + \cdots + \frac{1}{k_{i_m}}\right)\right)^2\right) \\
&\times e^{\theta_{i_1}+\theta_{i_2}+\cdots+\theta_{i_m}} \qquad (12)
\end{aligned}
$$

where c_2 is an armitrary constant and $0 \le m \le N$ with $m = 0$ corresponding to a constant term (unity). This rule is found to be valid for one-, two-, and three-soliton solutions which makes it plausible that it holds for N-soliton solutions with any N, and that mutual interactions in such a system including a static soliton are elastic, as they are in the one-, two-, and three-soliton cases.

4 Concluding remarks

The alternative sets of solutions including static solitons have been identified for some higher order KdV-type equations. Solutions, which describe mutual interactions in a system consisting of a static soliton and several moving solitons can be algorithmically derived using the Hirota method with a modified representation form. Moreover,for the SK equation, the simple rule (11) which allows direct construction of the (static soliton + multi-soliton) solutions from the common multi-soliton solutions can be found. So it seems that static solitons fit naturally into the KdV-type multi-soliton dynamics. One more example of the KdV type system, in which there exists the static soliton phenomenon, is provided by the KdV-KK equation, which is a combination of the KdV equation with the Kaup-Kupershmidt (KK) equation [7], [8] differential polynomial. The KdV-KK equation attracted attention only recently when it was discovered [3] that it admitted multi-soliton solutions in terms of the so-called Generalized Kaup-Kupershmidt (GKK) solitary waves possessing some features not found among other KdV type equations. Multi-soliton solutions of the KdV-KK equation expressed in terms of the GKK solitons can be defined algorithmically [3] using the Hirota method. The KdV-KK equation also admits steady-state solutions similar to (10). Solution of the KdV-KK equation describing interaction of moving GKK solitons with a static soliton can be obtained using the methods described in Section 2.

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