Received: May 17, 2021. Revised: October 5, 2021. Accepted: October 29, 2021. Published: November 2, 2021.

Solvability of a fractional differential equation with nonlocal boundary conditions

Noureddine Bouteraa
Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO),
University of Oran1, Ahmed Benbella.
Algeria
bouteraa-27@hotmail.fr

***Abstract*—In this work, we establish the existence and uniqueness of solutions for of nonlinear fractional differential equations with nonlocal boundary conditions. Our results are obtained by using Leray-Schauder nonlinear alternative and Banach contraction principle. To show the applicability of our results, an example is presented at the end.**

***Index Terms*—Fractional differential equations; existence; nonlocal boundary; Leray-Schauder nonlinear alternative; Banach contraction principle.**

I. Introduction

In this paper, we are interested in the existence of solutions for nonlinear fractional difference equations, $t \in J = [0, T]$

$$^{c}D_{0+}^{\alpha} u(t) - A^{c}D_{0+}^{\beta} u(t) = f\left(t, u(t), ^{c}D_{0+}^{\beta} u(t), ^{c}D_{0+}^{\alpha} u(t)\right), \tag{I.1}$$

subject to the three-point boundary conditions

$$\begin{cases} \lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\ \lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l, \end{cases} \tag{I.2}$$

where $T > 0$, $0 \leq \eta \leq T$, $\lambda \neq \mu + \gamma$, $d, l, \lambda, \mu, \gamma \in \mathbb{R}$, $\beta + 1 < \alpha$, A is an $\mathbb{R}^{n\times n}$ matrix and $^{c}D_{0+}^{\alpha}$, $^{c}D_{0+}^{\beta}$ are the Caputo fractional derivatives of ordre $1 < \alpha \leq 2$, $0 < \beta \leq 1$ respectively.

Fractional differential equation theory have recieved increasing attention. This theory has been developed very quickly and attracted a considerable interest from researches (see [**?**], [**?**], [**?**]). The motivation for those works stems from both the developpement of the theory and the applications of such construction in various sciences such as physics, mechanics, chemistry, engineering, and so on. For an extensive collection of such results, we refer the reader to the monographs of Kilbas et al. [**?**], Miller and Ross [**?**] and Podlubny [17].

In most of the available literature, fractional integral inequalities play an important role in the qualitative analysis of the solutions for fractional diferential equations (see [9, 16, 22].

As one of the focal topics in the research, some kinds of fractional equation with specific configurations have been presented. More specifically, in [5], the authors investigated the existence of positive solutions of the nonlinear fractional differential equation

We establish the existence results for the nonlocal boundary value problem $(1.1) - (1.2)$. by using Leray-Schauder nonlinear alternative and the Banach fixed point theorem. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, deals with main results and we give an example to illustrate our results.

II. Basic results

In this paper, we are interested in the existence of solutions for nonlinear fractional difference equations, $t \in J = [0, T]$

$$^{c}D_{0+}^{\alpha} u(t) - A^{c}D_{0+}^{\beta} u(t) = f\left(t, u(t), ^{c}D_{0+}^{\beta} u(t), ^{c}D_{0+}^{\alpha} u(t)\right), \tag{II.1}$$

subject to the three-point boundary conditions

$$\begin{cases} \lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\ \lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l, \end{cases} \tag{II.2}$$

where $T > 0$, $0 \leq \eta \leq T$, $\lambda \neq \mu + \gamma$, $d, l, \lambda, \mu, \gamma \in \mathbb{R}$, $\beta + 1 < \alpha$, A is an $\mathbb{R}^{n\times n}$ matrix and $^{c}D_{0+}^{\alpha}$, $^{c}D_{0+}^{\beta}$ are the Caputo fractional derivatives of ordre $1 < \alpha \leq 2$, $0 < \beta \leq 1$ respectively.

Fractional differential equation theory have recieved increasing attention. This theory has been developed very quickly and attracted a considerable interest from researches (see [1, 10, 18]). The motivation for those works stems from both the developpement of the theory and the applications of such construction in various sciences such as physics, mechanics, chemistry, engineering, and so on. For an extensive collection of such results, we refer the reader to the monographs of Kilbas et al. [14], Miller and Ross [15] and Podlubny [17].

In most of the available literature, fractional integral inequalities play an important role in the qualitative analysis of the solutions for fractional diferential equations (see [9, 16, 22].

As one of the focal topics in the research, some kinds of fractional equation with specific configurations have been presented. More specifically, in [5], the authors investigated the existence of positive solutions of the nonlinear fractional differential equation

$$^{c}D_{0+}^{\alpha}u(t) = f(t, u(t), u'(t)), \quad t \in [0,1],$$

subject to the three-point boundary conditions

$$\begin{cases} \beta u(0) + \gamma u(1) = u(\eta), \\ u(0) = \int_0^1 u(\eta), \\ \beta^c D_{0+}^{p} u(0) + \gamma^c D_{0+}^{p} u(1) =^c D_{0+}^{p} u(\eta), \end{cases}$$

where $2 < \alpha \leq 3$, $1 < p \leq 2$, $0 < \eta < 1$, $\beta, \gamma \in \mathbb{R}^{+}$, $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$ and $^{c}D_{0+}^{\alpha}$ denotes the Caputo fractional derivative of order α. Note that in the papers [3, 4, 6, 7, 8, 11] the authors have deal with the problem of existence and uniqueness of solution of nonlinear fractional differential equations. The existence results obtained by differents approches.

In 2016, Wang et al. [20] have formulated and proved uniqueness of global solutions to the equation similar to (**??**) (see, Section 3) by using the generalized Gronwall inequality.

In 2017, J. Sheng and W. Jiang [19] studied the existence and uniqueness of the solutions for fractional damped dynamical systems

$$\begin{cases} ^{c}D_{0+}^{\alpha}u(t) - A^{c}D_{0+}^{\beta}u(t) = f(t, u(t)), \quad t \in [0,T], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $0 < T < \infty$, $u \in \mathbb{R}^n$, A is an $\mathbb{R}^{n\times n}$ matrix , $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ jointly continuous function and $^{c}D_{0+}^{\alpha}$, $^{c}D_{0+}^{\beta}$ are the Caputo derivatives of order α, β respectively.

In 2018, Abbes et al. [2] studied the existence and uniqueness of the solutions for fractional damped dynamical systems, for $t \in [0,T]$

$$\begin{cases} ^{c}D_{0+}^{\alpha}u(t) - A^{c}D_{0+}^{\beta}u(t) = f\left(t, u(t), ^{c}D_{0+}^{\beta}u(t), ^{c}D_{0+}^{\alpha}u(t)\right), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $0 < T < \infty$, $u \in \mathbb{R}^n$, A is an $\mathbb{R}^{n\times n}$ matrix and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ jointly continuous.

In 2019, Tao Zhu [23] studied the existence and uniqueness of positive solutions of the following fractional differential equations, for $t \in [0,T)$, $0 < \beta < \alpha < 1$

$$\begin{cases} ^{c}D_{0+}^{\alpha}u(t) - A^{c}D_{0+}^{\beta}u(t) = f(t, u(t)), \\ u(0) = u_0. \end{cases}$$

Inspired and motivated by the works mentioned above, we establish the existence results for the nonlocal boundary value problem $(1.1) - (1.2)$. by using Leray-Schauder nonlinear alternative and the Banach fixed point theorem. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, deals with main results and we give an example to illustrate our results.

III. Preliminaries

Let as introduce notations, definitions and preliminary facts that will be need in the sequel. For more details, see for example [13, 15, 15, 17].

Definition III.1. *The Caputo fractional derivative of order α for the function $u \in C^n([0,\infty), \mathbb{R})$ is defined by*

$$^{c}D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds.$$

where $\Gamma(\cdot)$ is the Eleur gamma function and $\alpha > 0$, $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α.

Definition III.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Eleur gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma III.3. [14] *Let $u \in AC^n[0,T]$, $n \in \mathbb{N}$ and $u(\cdot) \in C[0,T]$. Then, we have*
(i) $^{c}D_{0+}^{\beta}\left(I_{0+}^{\alpha}u(t)\right) = I_{0+}^{\alpha-\beta}u(t)$,
(ii) $I_{0+}^{\alpha}\left(^{c}D_{0+}^{\alpha}u(t)\right) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!}u^{(k)}(0)$, $t > 0$, $n-1 < \alpha < n$,
Especially, when $1 < \alpha < 2$, then we have

$$I_{0+}^{\alpha}\left(^{c}D_{0+}^{\alpha}u(t)\right) = u(t) - u(0) - tu'(0).$$

Lemma III.4. ([10]) *Let $0 < \beta < 1 < \alpha < 2$, then we have*

$$I_{0+}^{\alpha}\left(^{c}D_{0+}^{\beta}u(t)\right) = I_{0+}^{\alpha-\beta}u(t) - \frac{u(0)\, t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.$$

IV. Existence results

Let $C(J, \mathbb{R}^n)$ be the Banach space for all continuous function from J into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{\infty} = sup\{\|u(t)\| : t \in J\},$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. Denote $L^1(J, \mathbb{R}^n)$ the Banach space of the measurable functions $u : J \to \mathbb{R}^n$ that are Lebesgue integrable with norm

$$\|u\|_{L^1} = \int_0^T \|u(t)\|\, dt.$$

Let $AC(J, \mathbb{R}^n)$ be the Banach space of absolutely continuous valued functions on J and set

$$AC^n(J) = \left\{u : J \to \mathbb{R}^n : u, u', u'', \ldots, u^{(n-1)} \in C(J, \mathbb{R}^n)\right\}$$

and

$$u^{(n-1)} \in AC\left(J, \mathbb{R}^n\right).$$

By

$$C^1(J) = \left\{u : J \to \mathbb{R}^n \; where \; u' \in C\left(J, \mathbb{R}^n\right)\right\},$$

we denote the Banach space equipped with the norm

$$\|u\|_1 = max\left\{\|u\|_\infty, \|u'\|_\infty\right\}.$$

For the sake of brevity, we set

$\delta = (\lambda - \mu - \gamma)\Gamma(\alpha - \beta + 1) + A\left(\mu T^{\alpha-\beta} + \gamma\eta^{\alpha-\beta}\right), \quad \Delta = \frac{\Gamma(\alpha-\beta+1)}{\delta},$

$\sigma = A(\alpha - \beta)\left(\mu T^{\alpha-\beta-1} + \gamma\eta^{\alpha-\beta-1}\right), \quad \Lambda = (\lambda - \mu - \gamma) - (\mu T + \gamma\eta)\left(\frac{\sigma}{\delta}\right),$

$R_1 = \left(1 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)M_1 + TM_2 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha L_1\Gamma(2-\beta) + T^{1-\beta+\alpha}(L_3\|A\|+L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_2)},$

$R_2 = M_2 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{T^{\alpha-1}L_1\Gamma(2-\beta) + T^{\alpha-\beta}(L_3\|A\|+L_2)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_2)},$

$M_1 = \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T + \gamma\eta) + 1\right]\Phi + \Lambda^{-1}(\mu T + \gamma\eta)\Theta\right\},$

and

$M_2 = \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\Phi + \Theta\right\},$

with

$\Phi = \frac{\|A\|}{\Gamma(\alpha-\beta+1)}\left(\mu T^{\alpha-\beta} + \gamma\eta^{\alpha-\beta}\right) + \frac{(\mu T^\alpha + \gamma\eta^\alpha)\left(L_1\Gamma(2-\beta) + T^{1-\beta}(L_3\|A\|+L_2)\right)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)},$

$\Theta = \frac{\|A\|}{\Gamma(\alpha-\beta)}\left(\mu T^{\alpha-\beta-1} + \gamma\eta^{\alpha-\beta-1}\right) + \frac{\left(\mu T^{\alpha-1} + \gamma\eta^{\alpha-1}\right)\left(L_1\Gamma(2-\beta) + T^{1-\beta}(L_3\|A\|+L_2)\right)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)}.$

Lemma IV.1. *Let $y(\cdot) \in C\left(J, \mathbb{R}^n\right)$. The function $u(\cdot) \in C^1\left(J, \mathbb{R}^n\right)$ is a solution of the fractional differential problem*

$$^cD^\alpha_{0+}u(t) - A^cD^\beta_{0+}u(t) = y(t), \quad t \in J = [0, T], \quad \text{(IV.1)}$$

subject to the three-point boundary conditions

$$\begin{cases} \lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\ \lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l, \end{cases} \quad \text{(IV.2)}$$

if and only if, u is a solution of the fractional integral equation

$$u(t) = \left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)}\right)u(0) + tu'(0)$$

$$+\frac{A}{\Gamma(\alpha - \beta)}\int_0^t (t-s)^{\alpha-\beta-1}u(s)\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds, \quad \text{(IV.3)}$$

with

$$u(0) = \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T + \gamma\eta) + 1\right]\right.$$

$$\times\left[AI^{\alpha-\beta}_{0+}(\mu u(T) + \gamma u(\eta)) + I^\alpha_{0+}(\mu y(T) + \gamma y(\eta)) + d\right]$$

$$+\Lambda^{-1}(\mu T + \gamma\eta)\left[AI^{\alpha-\beta-1}_{0+}(\mu u(T) + \gamma u(\eta)) + I^{\alpha-1}_{0+}(\mu y(T) + \gamma y(\eta)) \quad \text{(IV.4)}$$

and

$$u'(0) = \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\left[AI^{\alpha-\beta}_{0+}(\mu u(T) + \gamma u(\eta)) + I^\alpha_{0+}(\mu y(T) + \gamma y(\eta))\right.\right.$$

$$\left.+\left[AI^{\alpha-\beta-1}_{0+}(\mu u(T) + \gamma u(\eta)) + I^{\alpha-1}_{0+}(\mu y(T) + \gamma y(\eta))\right] + l\right\}, \quad \text{(IV.5)}$$

where

$$I^\alpha_{0+}u(T) = \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}u(s)\,ds,$$

$$I^\alpha_{0+}u(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}u(s)\,ds,$$

$$I^{\alpha-\beta}_{0+}u(T) = \frac{1}{\Gamma(\alpha - \beta)}\int_0^T (T-s)^{\alpha-\beta-1}u(s)\,ds,$$

$$I^{\alpha-\beta}_{0+}u(\eta) = \frac{1}{\Gamma(\alpha - \beta)}\int_0^\eta (\eta-s)^{\alpha-\beta-1}u(s)\,ds.$$

Proof. From Lemma 2.3, we have

$$u(t) = u(0) + tu'(0) - \frac{At^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)}u(0)$$

$$+\frac{A}{\Gamma(\alpha - \beta)}\int_0^t (t-s)^{\alpha-\beta-1}u(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds$$

Applying conditions (**??**), we obtain (**??**) and (**??**).
Conversely, assume that u satisfies the fractional integral (**??**), and using the facts that $^cD^\alpha_{0+}$ is the left inverse of I^α_{0+} and the fact that $^cD^\alpha_{0+}C = 0$, where C is a constant, we get

$$^cD^\alpha_{0+}u(t) - A^cD^\beta_{0+}u(t) = f\left(t, u(t), u'(t)\right), \quad t \in J = [0, T].$$

Also, we can easily show that

$$\begin{cases} \lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\ \lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l. \end{cases}$$

The proof is complete.

□

To simplify the proofs in the forthcoming theoreme, we etablish the bounds for the integrals and the bounds for the term arising in the sequel.

Lemma IV.2. *For $y(\cdot) \in C(J, \mathbb{R}^n)$, we have*

$$\left|I_{0+}^{\alpha} y(\eta)\right| = \left|\int_0^{\eta} \frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\, d\tau\right| \leq \frac{\eta^{\alpha}}{\Gamma(\alpha+1)} \|y\|.$$

Proof. Obviously,

$$\int_0^{\eta} \frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\, d\tau = \left[-\frac{(\eta-\tau)^{\alpha}}{\alpha\Gamma(\alpha)}\right]_0^{\eta} = \frac{\eta^{\alpha}}{\alpha\Gamma(\alpha)} = \frac{s^{\alpha}}{\Gamma(\alpha+1)}$$

□

Hence

$$\left|\int_0^{\eta} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\, d\tau\right| \leq \frac{\eta^{\alpha}}{\Gamma(\alpha+1)} \|y\|.$$

Lemma IV.3. *For $u(\cdot) \in C^1(J, \mathbb{R}^n)$ and $0 < \beta \leq 1$, we have*

$$\left\|{}^c D_{0+}^{\beta} u(t)\right\|_{\infty} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)} \|u'\|_{\infty},$$

and, so

$$\left\|{}^c D_{0+}^{\beta} u(t)\right\|_{\infty} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)} \|u\|_1.$$

Proof. Clearly, when $\beta = 1$, the conclusion are true. So, consider the case $0 < \beta < 1$. By Definition 2.1, for each $u(\cdot) \in C^1(J, \mathbb{R}^n)$ and $t \in J$, we have

$$\begin{aligned}
\left|D_{0+}^{\beta} u(t)\right| &= \frac{1}{\Gamma(1-\beta)} \left|\int_0^t (t-s)^{-\beta} u'(s)\, ds\right| \\
&\leq \|u'\|_{\infty} \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\, ds \\
&= \|u'\|_{\infty} \frac{t^{1-\beta}}{\Gamma(1-\beta)} \\
&\leq \frac{T^{1-\beta}}{\Gamma(1-\beta)} \|u'\|_{\infty} \\
&\leq \frac{T^{1-\beta}}{\Gamma(1-\beta)} \|u'\|_1.
\end{aligned}$$

□

We need to give the following hypothesis:
(H_1) there existe a constants $L_1,\ L_2 > 0$ and $0 < L_3 < 1$ such that

$$\begin{aligned}
|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})| &\leq L_1 \|u-\overline{u}\| \\
&+ L_2 \|v-\overline{v}\| + L_3 \|w-\overline{w}\|,
\end{aligned}$$

for any $u,\ v,\ \overline{u},\ \overline{v},\ \overline{w} \in \mathbb{R}^n$ and $t \in J$.

Now we are in a position to present the first main result of this paper. The existence results is based on Banach contraction principle.

Theorem IV.4. ([12]) *(Banach's fixed point theorem) Let C be a non-empty closed subset of a Banach space E, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem IV.5. *Assume that (H_1) holds. If*

$$max(R_1, R_2) < 1, \tag{IV.6}$$

then the boundary value problem $(1.1)-(1.2)$ has a unique solution on J.

Proof. We transform the problem $(1.1)-(1.2)$ into fixed point problem. Let $N : C^1(J, \mathbb{R}^n) \to C^1(J, \mathbb{R}^n)$ the operator defined by

$$\begin{aligned}
(Nu)(t) &= \left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) B + tD \\
&+ \frac{A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} u(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,
\end{aligned} \tag{IV.7}$$

with

$$\begin{aligned}
B &= \Delta \left\{ \left[\Lambda^{-1} \left(\frac{\sigma}{\delta}\right) (\mu T + \gamma\eta) + 1\right] \right. \\
&\times \left[A I_{0+}^{\alpha-\beta} (\mu u(T) + \gamma u(\eta)) + I_{0+}^{\alpha} (\mu y(T) + \gamma y(\eta)) + d\right] \\
&+ \Lambda^{-1} (\mu T + \gamma\eta) \\
&\left. \times \left[A I_{0+}^{\alpha-\beta-1} (\mu u(T) + \gamma u(\eta)) + I_{0+}^{\alpha-1} (\mu y(T) + \gamma y(\eta)) + l\right] \right\},
\end{aligned}$$

and

$$\begin{aligned}
D &= \Lambda^{-1} \left\{ \left(\frac{\sigma}{\delta}\right) \left[A I_{0+}^{\alpha-\beta} (\mu u(T) + \gamma u(\eta)) + I_{0+}^{\alpha} (\mu g(T) + \gamma g(\eta)) + d\right] \right. \\
&\left. + \left[A I_{0+}^{\alpha-\beta-1} (\mu u(T) + \gamma u(\eta)) + I_{0+}^{\alpha-1} (\mu g(T) + \gamma g(\eta))\right] + l \right\},
\end{aligned}$$

where $g \in C(J, \mathbb{R}^n)$ be such that

$$g(t) = f\left(t, u(t), {}^c D_{0+}^{\beta} u(t), g(t) + A\, {}^c D_{0+}^{\beta} u(t)\right)$$

For every $u \in C^1(J, \mathbb{R}^n)$ and any $t \in J$, we have

$$\begin{aligned}
(Nu)'(t) &= D - \frac{(\alpha-\beta) At^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)} B \\
&+ \frac{A}{\Gamma(\alpha-\beta-1)} \int_0^t (t-s)^{\alpha-\beta-2} u(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} g(s)\, ds.
\end{aligned} \tag{IV.8}$$

Clearly, the fixed points of operator N are solutions of problem $(1.1)-(1.2)$.
It is clear that $(Nu)' \in C(J, \mathbb{R}^n)$, consequently, N is well defined.

Let $u,\ v \in C(J, \mathbb{R}^n)$. Then for $t \in J$, we have

$$\|(Nu)(t)-(Nv)(t)\| \leq \left(1+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|B-B_1\|$$

$$+T\|D-D_1\|+\frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|u(s)-v(s)\| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s)-h(s)\| ds,$$

with $B_1=u(0), D_1=u'(0)$ are defined above.

From (H), for any $t\in J$, we have

$$\|g(t)-h(t)\| = L_1\|u(t)-v(t)\|$$

$$+L_2\left\|{}^cD^\beta_{0+}u(t)-{}^cD^\beta_{0+}v(t)\right\|$$

$$+L_3\left\|g(t)+A^cD^\beta_{0+}u(t)-h(t)-A^cD^\beta_{0+}v(t)\right\|$$

$$\leq L_1\|u(t)-v(t)\|+L_2\left\|{}^cD^\beta_{0+}u(t)-{}^cD^\beta_{0+}v(t)\right\|$$

$$+L_3\|g(t)-h(t)\|+L_3\|A\|\left\|{}^cD^\beta_{0+}u(t)-{}^cD^\beta_{0+}v(t)\right\|$$

$$\leq L_1\|u(t)-v(t)\|+L_3\|g(t)-h(t)\|$$

$$+(L_3\|A\|+L_2)\left\|{}^cD^\beta_{0+}(u(t)-v(t))\right\|.$$

Thus

$$\|g(t)-h(t)\| \leq \frac{L_1}{1-L_3}\|u(t)-v(t)\|$$

$$+\frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^\beta_{0+}(u(t)-v(t))\right\|$$

$$\leq \frac{L_1}{1-L_3}\|u-v\|_\infty+\frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^\beta_{0+}(u-v)\right\|_\infty$$

Then, according to the Lemma **??**, we get

$$\|g(t)-h(t)\| \leq \frac{L_1}{1-L_3}\|u-v\|_1$$

$$+\frac{T^{1-\beta}(L_3\|A\|+L_2)}{\Gamma(2-\beta)(1-L_3)}\|u-v\|_1$$

$$=\frac{L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)}{\Gamma(2-\beta)(1-L_3)}\|u-v\|_1. \quad \text{(IV.9)}$$

By employing (**??**) and Lemma **??**, we get

$$\|B_1-B_2\| \leq \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right]\Phi\right.$$

$$\left.+\Lambda^{-1}(\mu T+\gamma\eta)\Theta\right\}\|u-v\|_1$$

$$=M_1\|u-v\|_1.$$

and

$$\|D_1-D_2\| \leq \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\Phi+\Theta\right\}\|u-v\|_1$$

$$=M_2\|u-v\|_1,$$

where Φ and Θ defined above.
Thus, for $t\in J$, we have

$$\|(Nu)(t)-(Nv)(t)\| \leq \left[\left(1+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)M_1\right.$$

$$+TM_2+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$$

$$\left.+\frac{T^\alpha L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}(L_3\|A\|+L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_2)}\right]\|u-v\|_1$$

$$=R_1\|u-v\|_1.$$

Also

$$\left\|(Nu)'(t)-(Nv)'(t)\right\| \leq \|D_2-D_1\|$$

$$+\frac{(\alpha-\beta)\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\|B_1-B_2\|$$

$$+\frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}\|u(s)-v(s)\| ds$$

$$+\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s)-h(s)\| ds.$$

By employing (**??**) and Lemma **??**, we get

$$\left\|(Nu)'(t)-(Nv)'(t)\right\| \leq \left[M_2+\frac{(\alpha-\beta)\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1+\right.$$

$$+\frac{\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}$$

$$\left.+\frac{T^{\alpha-1}L_1\Gamma(2-\beta)+T^{\alpha-\beta}(L_3\|A\|+L_2)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_2)}\right]\|u-v\|_1$$

$$=R_2\|u-v\|_1.$$

Therefore

$$\|(Nu)(t)-(Nv)(t)\| \leq max\{R_1,R_2\}\|u-v\|_1.$$

Thus, by (**??**) the operator N is a contraction. Hence it follows by Banach's contraction principle that the boundary value problem (**??**) $-$ (**??**) has a unique solution on J. □

Now we are in a position to present the second main result of this paper. The existence results is based on Leray-Schauder nonlinear alternative.

Theorem IV.6. . [12] *(Nonlinear alternative for single valued maps). Let E be a Banach space, C a closed, convex subset of E and U an open subset of C with $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous and compact (that is $F(\overline{U})$ is relatively compact subset of C) map. Then either*
(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.

Set

$$l_1 = M_3+TM_4+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}M_3+TM_4+\frac{\|A\| rT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} +\frac{T^{\alpha}}{\Gamma(\alpha+1)}M,$$

and

$$l_2 = M_4+\frac{(\alpha-\beta)\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_3+\frac{\|A\| rT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{T^{\alpha}M}{\Gamma(\alpha)}.$$

Theorem IV.7. *Assume that (H_1) holds and there exists a positive constant $M > 0$ such that $max\{l_1, l_2\} = l < M$. Then the boundary value problem $(1.1) - (1.2)$ has at least one solution on J.*

Proof. Let N be the operator defined in (**??**).

N is continuous. Let (u_n) be a sequence such that $u_n \to u$ in $C(J, \mathbb{R}^n)$. Then for $t \in J$, we have

$$\|(Nu)(t)-(Nu_n)(t)\| \le \left(1+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|B_1-B_{n2}\| +\frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|u(s)-u_n(s)\|ds$$

$$+T\|D_1-D_{n2}\|+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s)-g_n(s)\|ds,$$

where B_{n2}, $D_{n2} \in \mathbb{R}^n$, with

$$B_{n2} = \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right]\right.$$
$$\times\left[AI_{0^+}^{\alpha-\beta}(\mu u_n(T)+\gamma u_n(\eta))+I_{0^+}^{\alpha}(\mu g_n(T)+\gamma g_n(\eta))+d\right]$$
$$+\Lambda^{-1}(\mu T+\gamma\eta)\left[AI_{0^+}^{\alpha-\beta-1}(\mu u_n(T)+\gamma u_n(\eta))\right.$$
$$\left.\left.+I_{0^+}^{\alpha-1}(\mu g_n(T)+\gamma g_n(\eta))+l\right]\right\},$$
$$D_{n2} = \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\left[AI_{0^+}^{\alpha-\beta}(\mu u_n(T)+\gamma u_n(\eta))\right.\right.$$
$$\left.+I_{0^+}^{\alpha}(\mu g_n(T)+\gamma g_n(\eta))+d\right]$$
$$\left.+AI_{0^+}^{\alpha-\beta-1}(\mu u_n(T)+\gamma u_n(\eta))+I_{0^+}^{\alpha-1}(\mu g_n(T)+\gamma g_n(\eta))+l\right\},$$

and

$$g_n(t) = f\left(t, u_n(t), {}^cD_{0^+}^{\beta}u_n(t), g_n(t)+A{}^cD_{0^+}^{\beta}u_n(t)\right).$$

From (H), for any $t \in J$, we have

$$\|g(t)-g_n(t)\| = L_1\|u(t)-u_n(t)\|$$
$$+L_3\left\|g(t)+A{}^cD_{0^+}^{\beta}u(t)-g_n(t)-A{}^cD_{0^+}^{\beta}u_n(t)\right\|$$
$$+L_2\left\|{}^cD_{0^+}^{\beta}u(t)-{}^cD_{0^+}^{\beta}u_n(t)\right\|$$
$$\le L_1\|u(t)-u_n(t)\|+L_2\left\|{}^cD_{0^+}^{\beta}u(t)-{}^cD_{0^+}^{\beta}u_n(t)\right\|$$
$$+L_3\|g(t)-g_n(t)\|+L_3\|A\|\left\|{}^cD_{0^+}^{\beta}u(t)-{}^cD_{0^+}^{\beta}u_n(t)\right\|$$
$$\le L_1\|u(t)-u_n(t)\|+L_3\|g(t)-g_n(t)\|$$
$$+(L_3\|A\|+L_2)\left\|{}^cD_{0^+}^{\beta}(u(t)-u_n(t))\right\|.$$

Thus

$$\|g(t)-g_n(t)\| \le \frac{L_1}{1-L_3}\|u(t)-u_n(t)\|$$
$$+\frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD_{0^+}^{\beta}(u(t)-u_n(t))\right\|$$
$$\le \frac{L_1}{1-L_3}\|u-u_n\|_{\infty}+\frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD_{0^+}^{\beta}(u-u_n)\right\|_{\infty}.$$

Then, according to the Lemma **??**, we get

$$\|g(t)-g_n(t)\| \le \frac{L_1}{1-L_3}\|u-u_n\|_1+\frac{T^{1-\beta}(L_3\|A\|+L_2)}{(1-L_3)\Gamma(2-\beta)}\|u-u_n\|_1$$
$$=\frac{L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)}{(1-L_3)\Gamma(2-\beta)}\|u-u_n\|_1.$$

By employing (**??**) and Lemma **??**, we get

$$\|B_1-B_{n2}\| \le \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right]\Phi\right.$$
$$\left.+\Lambda^{-1}(\mu T+\gamma\eta)\Theta\right\}\|u-v\|_1,$$

$$= M_1\|u-v\|_1.$$

and

$$\|D_1-D_{n2}\| \le \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\Phi+\Theta\right\}\|u-v\|_1$$

$$= M_2\|u-v\|_1,$$

Thus, for $t \in J$, we have

$$\|(Nu)(t)-(Nv)(t)\| \le \left[\left(1+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)M_1\right.$$
$$+TM_2+\frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$$
$$\left.+\frac{T^{\alpha}L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}(L_3\|A\|+L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_2)}\right]\|u-v\|_1$$
$$= R_1\|u-u_n\|_1.$$

Also

$$\left\|(Nu)'(t)-(Nu_n)'(t)\right\| \le \|D_{n2}-D_1\|$$
$$+\frac{(\alpha-\beta)\|A\| T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\|B_1-B_{n2}\|$$
$$+\frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}\|u(s)-u_n(s)\|ds$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s)-g_n(s)\|ds.$$

By employing (??), we get

$$\left\|(Nu)'(t)-(Nu_n)'(t)\right\| \leq \left[M_2+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{T^{\alpha-1}L_1\Gamma(2-\beta)(L_3\|A\|+L_2)T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1\right]\|u-u_n\|_1 .$$
$$= R_2\|u-u_n\|_1 .$$

Thus $\|Nu-Nu_n\|_1 \to 0$ as $n\to\infty$, which implies that the operator N is continuous.

Now, we show N maps bounded sets into bounded sets in $C(J,\mathbb{R}^n)$. For a positive number r, let $B_r = \left\{u\in C^1(J,\mathbb{R}^n) : \|u\|_1 \leq r\right\}$ be a bounded set in $C(J,\mathbb{R}^n)$. Then we have

$$\|g(t)\| \leq \left\|f\left(t,u(t),g(t)+A^cD^{\beta}_{0+}u(t),D^{\beta}_{0+}u(t)\right) - f(t,0,0,0)\right\| + \|f(t,0,0,0)\|$$
$$\leq L_1\|u(t)\| + L_3\left\|g(t)+A^cD^{\beta}_{0+}u(t)\right\| + L_2\left\|D^{\beta}_{0+}u(t)\right\| + \|f(t,0,0,0)\|$$
$$\leq L_1\|u\|_\infty + L_3\|g(t)\| + (L_3\|A\|+L_2)\left\|D^{\beta}_{0+}u\right\|_\infty + f^*,$$

where $\sup_{t\in J}|f(t,0,0,0)| = f^* < \infty$. Thus

$$\|g(t)\| \leq \frac{L_1}{1-L_3}\|u\|_\infty + \frac{L_3\|A\|+L_2}{1-L_3}\left\|D^{\beta}_{0+}u\right\|_\infty + \frac{f^*}{1-L_3}.$$

Then, By Lemma ??, we have

$$\|g(t)\| \leq \frac{L_1}{1-L_3}\|u\|_\infty + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|u'\|_\infty + \frac{f^*}{1-L_3}$$
$$\leq \frac{L_1}{1-L_3}\|u\|_1 + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|u\|_1 + \frac{f^*}{1-L_3}$$
$$\leq \frac{L_1 r}{1-L_3} + \frac{(L_3\|A\|+L_2)rT^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} + \frac{f^*}{1-L_3} = M, \tag{IV.10}$$

which implies that

$$\|B\| \leq r\|A\|\Delta\left[\left(\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)\right) \times((\mu T+\gamma\eta)+1)\left(\frac{\mu T^{\alpha-\beta}+\gamma\eta^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) + \Lambda^{-1}(\mu T+\gamma\eta)\left(\frac{\mu T^{\alpha-\beta-1}+\gamma\eta^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right)\right]$$
$$+M\Delta\left[\left(\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right)\left(\frac{\mu T^{\alpha}+\gamma\eta^{\alpha}}{\Gamma(\alpha+1)}\right) + \Lambda^{-1}(\mu T+\gamma\eta)\left(\frac{\mu T^{\alpha-1}+\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right]$$
$$+\Delta\Lambda^{-1}(\mu T+\gamma\eta)\left[l+d\left(\left(\frac{\sigma}{\delta}\right)+1\right)\right] = M_3,$$

and

$$\|D\| \leq r\|A\|\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)\frac{\mu T^{\alpha-\beta}+\gamma\eta^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\mu T^{\alpha-\beta-1}+\gamma\eta^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right]$$
$$+M\Lambda^{-1}\left[\left(\frac{\sigma}{\delta}\right)\left(\frac{\mu T^{\alpha}+\gamma\eta^{\alpha}}{\Gamma(\alpha+1)}\right) + \left(\frac{\mu T^{\alpha-1}+\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right]$$
$$+\Lambda^{-1}\left[\left(\frac{\sigma}{\delta}\right)d+l\right] = M_4.$$

Thus (??) implies

$$\|(Nu)(t)\| \leq M_3 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}M_3 + TM_4 + \frac{\|A\|rT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}M = l_1,$$

and

$$\left\|(Nu)'(t)\right\| \leq M_4 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_3 + \frac{\|A\|rT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{T^{\alpha}M}{\Gamma(\alpha)} = l_2.$$

Therefore

$$\|(Nu)\|_1 \leq \max\{l_1, l_2\} = l. \tag{IV.11}$$

Now, we show that N maps bounded sets into equicontinuous sets of $C^1(J,\mathbb{R}^n)$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in B_r$ is bounded sets of $C^1(J,\mathbb{R}^n)$. Then

$$\|(Nu)(t_2)-(Nu)(t_1)\| \leq M_4(t_2-t_1) + \left(1+\frac{\|A\|M_3}{\Gamma(\alpha-\beta+1)}\right)\left(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right)$$
$$+\frac{\|A\|r}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}ds$$
$$+\frac{\|A\|r}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right]ds$$
$$+\frac{M_1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds + \int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds\right]$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2 \to t_1$.
Similarly, we have

$$\left\|(Nu)'(t_2)-(Nu)'(t_1)\right\| \leq \frac{(\alpha-\beta)\|A\|M_3}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}\right) + \frac{\|A\|r}{\Gamma(\alpha-\beta-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-2}ds$$

$$+\frac{\|A\| r}{\Gamma(\alpha-\beta-2)} \int_0^{t_1} \left[(t_2-s)^{\alpha-\beta-2}-(t_1-s)^{\alpha-\beta-2}\right] ds$$

$$+\frac{M}{\Gamma(\alpha-1)} \left[\int_{t_1}^{t_2} (t_2-s)^{\alpha-2} ds\right.$$

$$\left.+\int_0^{t_1} \left[(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\right] ds\right]$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, $\|(Nu)(t_2)-(Nu)(t_1)\| \to 0$, as $t_2 \to t_1$. This shows that the operator N is completely continuous, by the Ascoli-Arzela theorem. Thus, the operator N satisfies all the conditions of Theorem **??**, and hence by its conclusion, either condition (i) or condition (ii) holds. We show that the condition (ii) is not possible.

Let $U=\left\{u \in C^1(J, \mathbb{R}^n) : \|u\|<M\right\}$ with $\max\{l_1, l_2\}=l<M$. In view of condition $l<M$ and by (**??**), we have

$$\|Nu\| \le max\{l_1, l_2\}<M.$$

Now, suppose there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u=\lambda Nu$. Then for such a choice of u and the constant λ, we have

$$M=\|u\|=\lambda\|Nu\|<\max\{l_1, l_2\}<M,$$

which is a contradiction. Consequently, by the Leray-Schauder alternative, we deduce that F has a fixed point $u \in \overline{U}$ which is a solution of the boundary value problem (**??**) $-$ (**??**). The proof is completed.

□

We construct an example to illustrate the applicability of the results presented.

Example IV.1. *Consider the following fractional differential equation, for $t \in J=[0,1]$*

$$^cD_{0+}^{\alpha}u(t)-A\,^cD_{0+}^{\beta}u(t)=f\left(t, u(t), ^cD_{0+}^{\beta}u(t), ^cD_{0+}^{\alpha}u(t)\right), \quad \text{(IV.12)}$$

subject to the three-point boundary conditions

$$\begin{cases} u(0)-u(1)-u\left(\frac{1}{2}\right)=1, \\ u'(0)-u'(1)-u'\left(\frac{1}{2}\right)=1, \end{cases} \quad \text{(IV.13)}$$

where $\alpha=2,\ \beta=1,\ \lambda=\mu=d=l=1,\quad A=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ *and*

$$f_i(t,u,v,w)=\frac{c_i t}{8} arctan(|u|+|v|+|w|), \ i=1,\ 2,$$

such that $f=(f_1, f_2)$ *with* $0<c_i<1$, $i=1,\ 2$.
For every $u_i,\ v_i \in \mathbb{R}^2$, $i=1,\ 2,\ 3$, *we have*

$$|f_i(t,u_1,u_2,u_3)-f_i(t,v_1,v_2,v_3)| \le \frac{c_i}{8}(|u_1-v_1|$$

$$+|u_2-v_2|+|u_3-v_3|),\ i=1,\ 2,$$

where $L_1=L_2=L_3=\frac{c_i}{8}$ *for appropriate choice of constants* $c_i,\ i=1,2.$ *we check the condition of Theorem* **??**. *Clearly, assumption* (H_1) *holds. A simple computations of* $R_1,\ R_2,\ l_1$ *and* l_2 *shows tha the second condition of Theorems* **??** *and* **??** *is satisfied. Thus the conclusion of Theorems* **??** *and* **??** *applies, and hence the problem* (**??**) $-$ (**??**) *has a unique solution and at least one solution on* $[0,1]$.

REFERENCES

[1] O. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, Fract. Cal. Appl. Anal. 15 (2012), 700-711.

[2] S. Abbas, M. Benchohra, S. Bouriah and J. J. Nieto, Periodic solution for nonlinear fractional differential systems, Diff. Equ. Appl. 10, 3 (2018), 299-316.

[3] S. Asawasamrit, W. Nithiarayaphaks, S. K. Ntouyas and J. Tariboon, Existence and stability analysis for fractional differential equations with mixed nonlocal conditions, Mathematics. 7, 117 (2019), 11 pages.

[4] N. Bouteraa and S. Benaicha, Existence of solutions for three-point boundary value problem for nonlinear fractional equations, Analele Universitatii Oradia Fasc. Mathematica, Tom XXIV (2017), Issue No. 2, 109-119.

[5] S. Benaicha and N. Boutetaa, Existence of solutions for three-point boundary value problem for nonlinear fractional differential equations, Bulletin of the Transilvania University of Brasov, Serie III: Mathematics, Informatics, Physics. Volume 10 (59), No. 2-2017, 31-48.

[6] N. Bouteraa, S. Benaicha and H. Djourdem, Positive solutions for nonlinear fractional equation with nonlocal boundary conditions, Universal journal of Mathematics and Applications. 1 (1) (2018), 39-45.

[7] N. Bouteraa and S. Benaicha, The uniqueness of positive solution for higher-order nonlinear fractional differential equation with nonlocal boundary conditions, Advances in the Theory of Nonlinear and it Application, 2(2018) No 2, 74-84.

[8] N. Bouteraa and S. Benaicha, The uniqueness of positive solution for nonlinear fractional differential equation with nonlocal boundary conditions, Analele Universitatii Oradia Fasc. Mathematica, Tom XXIV (2018), Issue No. 2, 53-65.

[9] Z . Denton and A. S. Vatsala, Fractional integral inequalities and applications, Comput. Math. Appl. 59 (2010), 1087-1094.

[10] X. L. Dind, Controllability and optimality of linear time-invariant neutral control systems with different fractional orders. Acta Math. Sci. 35 B(5) (2015), 1003-1013.

[11] S. Etemad, S. K. Ntouyas, J. Tariboon, Existence results for three-point boundary value problems for nonlinear fractional diffrential equations. J. Nonlinear Sci. Appl. 9 (2016).

[12] A. Granas and J. Dugundji, Fixed point theory. Springer, New-York(2003).

[13] V. Kac, P. Cheung, Quantum Calculus, Springer, New York, (2002).

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trijullo, Theory and applications of fractional differential equations, Elsevier Science b. V, Amsterdam, (2006).

[15] S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons, Inc. New York, (1993).

[16] Q. Ma and J. Pecaric, Some new explicit bounds for weakly singular integral inequality with applications to fractional differential equations and integral equations. J. Math. Anal. Appl. 341 (2008), 894-905.

[17] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering. Academic Press New-York 1993.

[18] P. M. Rajkovic, S. D. Marancovic, M. S. Stankovic, On q-analogues of Caputo derivative and Mettag-Leffter function, Frac. Calc. Appl. Anal. 10(2007), 359-373.

[19] J. Sheng and W. Jiang, Existence and uniqueness of the solutions for fractional damped dynamical systems, Advances in Difference Equations. 2017:16 (2017), 14 pages.

[20] F. Wang, D. Chen, X. Zhang and Y.Wu. The existence and unique- ness theorem of the solution to a class of nonlinear fractional order system with time delay. Applied Mathematics Letters, 53 (2016), 45–51.

[21] H. Ye, J. Gao and Y. Deng, A generalized Gronwell inequality and its applications to a fractional differential equation, J. Math. Anal. Appl. 328 (2007), 1075-1081.

[22] T. Zhu, New Henry-Gronwall integral inequalities and their applications, Bull. Braz. Math. Soc. 49 (2018), 647-657
[23] T. Zhu, Existence and uniqueness of positive solutions for fractional differential equations, Boundary Value Problems, 2019:22, (2019), 11 pages.