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Solvability of a Hadamard fractional differential equation with Hybrid Hadamard integral boundary value conditions

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***Abstract*—The goal of this paper is to investigate the existence of solutions for a nonlinear Hadamard fractional differential equations involving the Hadamard fractional derivative with hybrid Hadamard integral boundary conditions. The existence results are obtained by using the generalization of Darbo's fixed point theorem combined with the technique of measures of noncompactness in the Banach algebras.**

***Index Terms*—Integral boundary conditions, Measure of noncompactness, Hadamard fractional derivative, upper semicontinuous function**

I. Introduction

Differential equations with fractional-order have gained considerable importance due to their application in many fields of science and technology as the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data (see [5], [7], [15], [18], [20], [23] and the references therein). Recently, the fractional-order differential equations with integral boundary value problems have attracted a great deal of attention and interests.

Existence of solutions for fractional differential equations has been investigated by many authors in various types. We quote some techniques of nonlinear analysis used to establish the uniqueness, existence and multiplicity of solutons for this kind of differential equations with complicated boundary value conditions, such as, Banach fixed point theorem, Leray-Schauder nonlinear alternative, Krosnoselskii's fixed point theorem on cones and Legget-Williams fixed point and other approaches [1], [2], [10], [11], [16]

In the works mentioned above, compactness and Lipschitz condition are satisfied, if not these techniques cannot be used. Hence, there have been many published papers, which are devoted to the existence of solutions of nonlinear integral equations by using the technique of a suitable measure of noncompactness in Banach algebras. We refer the readers to [4], [6], [17], [19], [21], [24] and references therein.

Motivated and inspired by the works mentioned above, we are concerned with the existence of solutions for the following nonlinear fractional differential hybrid equations with with hybrid Hadamard integral boundary conditions

$$D^q\left[\frac{u(t)}{f(t,u(t))}\right]=g(t,u(t)),\quad 1<t<e,\ 2<q\leq 3 \tag{I.1}$$

subject to the boundary conditions

$$\begin{aligned} &u(1)=0,\\ &\left.\left(\frac{u(t)}{f(t,u(t))}\right)''\right|_{t=1}=0,\\ &\left.\left(\frac{u(t)}{f(t,u(t))}\right)\right|_{t=e}=\lambda\left(I^pu\right)(\eta) \end{aligned} \tag{I.2}$$

where D^q is the Hadamard fractional derivative, I^p is the Hadamard fractional integral of order $p>0$, $f\in C([1,e]\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g\in C([1,e]\times\mathbb{R},\mathbb{R})$.

The content of this paper is organised as follows. In Section 2, some required concepts are presented that will be used in the sequel. In Section 3, we establish our main results by using generalization of Darbos fixed point theorem combined with the technique of measures of noncompactness in the Banach algebras.

II. Basic results

In this section, we give a collection of auxiliary facts which will be needed further on (details can be found, e.g., in). Let $C(I,\mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm

$$\|u\|_\infty=\sup\{|u(t)|:\ t\in I\}$$

We begin by defining Hadamard fractional integrals and derivatives, and we introduce some properties that can be used thereafter.

Definition II.1. *[22] The Hadamard fractional integral of order $q\in\mathbb{R}^+$ for a function $f\in C[a,b]$, $0\leq a\leq t\leq b\leq\infty$, is defined as*

$$I^qf(t)=\frac{1}{\Gamma(q)}\int_a^t\left(\log\frac{t}{s}\right)^{q-1}f(s)\frac{ds}{s},$$

where $\Gamma(.)$ *is the Gamma function and* $\log(.) = \log_e(.)$.

Definition II.2. *[22] Let* $0 < a < b < \infty$ *and* $\delta = t\frac{d}{dt}$. *The Hadamard derivative of fractional order* $q \in \mathbb{R}^+$ *for a function* $f \in C^{n-1}([a,b],\mathbb{R})$ *is defined as*

$$\begin{aligned} D^q f(t) &= \delta^n \left(I^{n-q}\right)(t) \\ &= \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s} ds, \end{aligned}$$

where $n-1 < q \leq n \in \mathbb{Z}^+$, $n = [q]+1$ *denotes the integer part of the real number* q.

Lemma II.3. *([22], Property 2.24) If* $a, \alpha, \beta > 0$, *then*

$$\left(D^q \left(\log\frac{t}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-q)} \left(\log\frac{t}{a}\right)^{\beta-q-1},$$

$$\left(I^q \left(\log\frac{t}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+q)} \left(\log\frac{t}{a}\right)^{\beta+q-1}.$$

Lemma II.4. *([22]) Let* $q > 0$ *and* $u \in [1,\infty) \cap L^1[1,\infty)$. *Then the solution of Hadamard fractional differential equation* $D^\alpha u(t) = 0$ *is given by*

$$u(t) = \sum_{i=1}^n c_i (\log t)^{q-i},$$

and the following formula holds:

$$I^q D^q u(t) = u(t) + \sum_{i=1}^n c_i (\log t)^{q-i},$$

for some $c_i \in \mathbb{R}$, $i = 1,2,...,n$, *where* $n = [q]+1$.

Next, we present some definitions and properties of the non-compactness measure. Let E be a real Banach space with norm $\|.\|$ and the zero element 0. We write $B(u,r)$ to denote the closed ball centered at u with radius r and especially, we write B_r in case of $u = 0$. If X is non-empty subset of E , then $\overline{X}$ and $\mathrm{Conv}X$ denote the closure and the closed convex closure of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty bounded subsets of E and let $\mathfrak{N}_E$ indicate its subfamily of all relatively compact sets.

We use the following definition of the measure of noncompactness given in [8].

Definition II.5. *A mapping* $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ *is said to be a measure of noncompactness in* E *if it satisfies the following conditions:*
1. The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is non-empty and* $\ker\mu \in \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(X) = \mu\left(\overline{X}\right) = \mu(\mathrm{Conv}X)$
4. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
5. If (X_n) *is a sequence of closed sets from* $\mathfrak{M}_E$ *such that* $X_{n+1} \subset X_n$ $(n = 1,2,...)$ *and* $\lim_{n\to\infty}\mu(X_n) = 0$, *then the intersection set* $\cap_{n=1}^\infty X_n$ *is nonempty.*

In the sequel, we assume that the space E has the structure of Banach algebras. For given subsets X, Y of a Banach algebra E let us denote

$$XY = \{xy : x \in E, y \in E\}$$

The following definition contains a useful concept [9]

Definition II.6. *Let* E *be a Banach algebra. A measure of non-compactness* μ *in* E *said to satisfy condition* (m) *if it satisfies the following condition:*

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X),$$

for any X, $Y \in \mathfrak{M}_E$.

It is known that the family of all real-valued and continuous functions defined on the interval. $I = [1,e]$ is denoted by $C[1,e]$. Also, $C[1,e])$ is a Banach space with the standard norm

$$\|u\|_\infty = \sup\{|u(t)| : t \in [1,e]\}.$$

Obviously, space $C(I)$ also has the structure of the Banach Algebra.

Further, fix arbitrarily $X \in \mathfrak{M}_{C[1,e]}$ and $\epsilon > 0$. For $u \in X$ denote by $\omega(u,\epsilon)$ the modulus of continuity of u, i.e.,

$$\omega(u,\epsilon) = \sup\{|u(t) - u(s)| : t,s \in [1,e], |t-s| \leq \epsilon\}.$$

Further, let us put

$$\omega(X,\epsilon) = \sup\{\omega(u,\epsilon) : u \in X\}$$

and

$$\omega_0(X) = \lim_{\epsilon\to 0}\omega(X,\epsilon).$$

Then, function ω_0 is a measure of noncompactness in space $C[1,e]$ (see [8]).

Proposition II.1. *[12], [14] The measure of noncompactness* ω_0 *on* $C[1,e]$ *satisfies condition* (m).

Theorem II.7. *[13] Let* C *be a nonempty, bounded, closed and convex subset of a Banach space* E *and let* $T : C \longrightarrow C$ *be a continuous mapping. Suppose that there exists* $k \in [0,1)$ *such that*

$$\mu(TX) \leq k\mu(X),$$

for any non-empty subset X *of* C, *where* μ *is a measure of non-compactness in* E. *Then* T *has a fixed point in* C.

The next result known as a generalization of Darbos fixed point theorem will play a pivotal role in the development of the results in this paper.

Theorem II.8. *([3]) Let* C *be a nonempty, bounded, closed and convex subset of a Banach space* E *and let* $T : C \longrightarrow C$ *be a continuous operator satisfying*

$$\mu(TX) \leq \varphi(\mu(X)),$$

for any non-empty subset X *of* C, *where* μ *is a measure of non-compactness in* E *and* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a nondecreasing function such that* $\lim_{n\to+\infty}\varphi^n(t) = 0$ *for each* $t \in \mathbb{R}_+$,

$\varphi^n(t)$ denotes the n-iteration of φ. Then T has a fixed point in C.

Moreover, in [3] the authors proved the following lemma which will be useful in our considerations.

Lemma II.9. *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following conditions are equivalent:*
(i) $\lim_{n\to+\infty} \varphi^n(t) = 0$, for any $t \geq 0$,
(ii) $\varphi(t) < t$, for any $t > 0$.

By commodity, we will denote by $\mathscr{A}$ the class of functions given by

$$\mathscr{A} = \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+ : \varphi\, is\, nondecreasing\, and \\ \lim_{n\to+\infty} \varphi^n(t) = 0,\ for\, any\, t \in \mathbb{R}_+\}$$

where $\varphi^n(t)$ denotes the n-iteration of φ.

III. Main result

In this section , we intend to state our main theoretical findings on the existence results. For convenience we put

$$\Omega = 1 - \frac{\lambda\Gamma(q-1)}{\Gamma(p+q-1)} (\log\eta)^{p+q-2}. \tag{III.1}$$

Lemma III.1. *Let $h \in C([1,e], \mathbb{R})$. The solution function u_0 of the for the hybrid Hadamard equation*

$$D^q\left[\frac{u(t)}{f(t,u(t))}\right] = h(t),\quad 1 < t < e,\ 2 < q \leq 3 \tag{III.2}$$

subject to the boundary conditions

$$\begin{aligned} &u(1) = 0,\\ &\left(\frac{u(t)}{f(t,u(t))}\right)''\Bigg|_{t=1} = 0,\\ &\left(\frac{u(t)}{f(t,u(t))}\right)\Bigg|_{t=e} = \lambda(I^p u)(\eta) \end{aligned} \tag{III.3}$$

if and only if the function u_0 is a solution for the following Hadamard integral equation:

$$\begin{aligned} u(t) = f(t,u(t)) \Bigg\{ &\frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s,u(s))}{s} ds \\ &+ \frac{(\log t)^{q-2}}{\Omega} \left(\frac{\lambda}{\Gamma(q+p)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{p+q-1} \frac{h(s)}{s} ds \right.\\ &\left. - \frac{1}{\Gamma(q)} \int_1^e \left(\log\frac{e}{s}\right)^{q-1} \frac{h(s)}{s} ds \right) \Bigg\}. \end{aligned} \tag{III.4}$$

Proof. Let u_0 be a solution for hybrid equation (III.2) By virtue of the lemma II.4, there exist constants c_1, c_2, $c_3 \in \mathbb{R}$ provided that

$$\begin{aligned} u_0(t) = f(t,u(t)) \Bigg\{ &\frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{h(s)}{s} ds \\ &+ c_1(\log t)^{q-1} + c_2(\log t)^{q-2} + c_3(\log t)^{q-3} \Bigg\}, \end{aligned} \tag{III.5}$$

The conditions $u(1) = 0$, $\left(\frac{u(t)}{f(t,u(t))}\right)''\Big|_{t=1} = 0$ imply that $c_1 = c_3 = 0$. Taking the Hadamard fractional integral of order $p > 0$ for (III.6) and using Lemmas **??**- II.3, we get that

$$\begin{aligned} I^p\left(\frac{u(t)}{f(t,u(t))}\right) = &\frac{1}{\Gamma(q+p)} \int_1^t \left(\log\frac{t}{s}\right)^{p+q-1} \frac{h(s)}{s} ds \\ &+ c_2 \frac{\Gamma(q-1)}{\Gamma(p+q-1)} (\log t)^{p+q-2}. \end{aligned}$$

By using the Hadamard integral boundary condition $\left(\frac{u(t)}{f(t,u(t))}\right)\Big|_{t=e} = \lambda(I^p u)(\eta)$, we get

$$\begin{aligned} c_2 = \frac{1}{\Omega} \Bigg(&\frac{\lambda}{\Gamma(q+p)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{p+q-1} \frac{h(s)}{s} ds \\ &- \frac{1}{\Gamma(q)} \int_1^e \left(\log\frac{e}{s}\right)^{q-1} \frac{h(s)}{s} ds \Bigg) \end{aligned}$$

where Ω is defined in (III.1).

By inserting the values c_i for $i = 1, 2, 3$ in (III.6), we get

$$\begin{aligned} u_0(t) = f(t,u_0(t)) \Bigg\{ &\frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s,u(s))}{s} ds \\ &+ \frac{(\log t)^{q-2}}{\Omega} \left(\frac{\lambda}{\Gamma(q+p)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{p+q-1} \frac{h(s)}{s} ds \right.\\ &\left. - \frac{1}{\Gamma(q)} \int_1^e \left(\log\frac{e}{s}\right)^{q-1} \frac{h(s)}{s} ds \right) \Bigg\}. \end{aligned}$$

This means that u_0 is a solution for integral equation (III.4). Conversely, one can easily see that u_0 is a solution function for the hybrid boundary value problem of fractional order (III.2)(III.3) whenever u_0 is a solution function for the fractional integral equation (III.4). □

Theorem III.2. *Suppose that $f \in C([1,e] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([1,e] \times \mathbb{R}, \mathbb{R})$. Also, we have the following assumptions:*
(A1) There exists an upper semi-continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\varphi(t) < t$ for any $t > 0$, φ is nondecreasing, and

$$|f(t,u) - f(t,v)| \leq \varphi(|u-v|),\ t\,[1,e],\ u,v \in \mathbb{R},$$

(A2) There are a continuous nondecreasing function $\Psi : [0,\infty) \longrightarrow [0,\infty)$ and a continuous function $\xi : [1,e] \longrightarrow \mathbb{R}_+$ such that

$$|g(t,u)| \leq \xi(t)\,\Psi(|u|),$$

(A3) There exists $\rho > 0$ such that

$$\rho \geq \frac{[\varphi(\rho) + F_0]\,\|\xi\|\,\Psi(\rho)}{\Gamma(q+1)} \left\{1 + \frac{1}{|\Omega|}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)} + 1\right)\right\},$$

and

$$\frac{\|\xi\|\,\Psi(\rho)}{\Gamma(q+1)} \left\{1 + \frac{1}{|\Omega|}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)} + 1\right)\right\} \leq 1,$$

where

$$F_0 = \sup\{|f(t,0)| : t \in [1,e]\}.$$

Then, the fractional hybrid BVP (I.1)(I.2) has at least one solution in $C[1,e]$.

Proof. In view of Lemma III.1, we define the operator T on $C[1,e]$ by

$$Tu(t) = f(t,u(t))\left\{\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds + \frac{(\log t)^{q-2}}{\Omega}\left(\frac{\lambda}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{g(s,u(s))}{s}ds - \frac{1}{\Gamma(q)}\int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right)\right\}. \tag{III.6}$$

Obviously, $u\in C[1,e]$ as a solution for hybrid (1)-(2) satisfies the operator equation $Tu=u$. Now, we define two operators G and H on $C[1,e]$ by

$$Gu(t) = f(t,u(t)),$$

and

$$Hu(t) = \frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds + \frac{(\log t)^{q-2}}{\Omega}\left(\frac{\lambda}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{g(s,u(s))}{s}ds - \frac{1}{\Gamma(q)}\int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right)\Bigg\},$$

for $t\in[1,e]$. Then $Tu=(Gu).(Hu)$ for any $u\in C[1,e]$.

We divide the rest of the proof into five steps.

Step 1. T applies $C[1,e]$ into itself.

In order to show that $Tu\in C[1,e]$, it is sufficient to show that $Gu, Hu\in C[1,e]$ for any $u\in C[1,e]$. The continuity of G arise from the continuity of f. Next, we will prove that if $u\in C[1,e]$, then $Gu\in C[1,e]$. To do this, let $t\in C[1,e]$ be fixed and $\{t_n\}$ be a sequence in $[1,e]$ such that $t_n\longrightarrow t$ as $n\longrightarrow\infty$. Without loss of generality, we may assume $t_n>t$.

Then,

$$\begin{aligned}
&\Gamma(q)\left|Hu(t_n)-Hu(t)\right| \\
&= \Bigg|\left(\int_1^{t_n}\left(\log\frac{t_n}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds - \int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right) \\
&\quad + \frac{\left((\log t_n)^{q-2}-(\log t)^{q-2}\right)}{\Omega}\times \\
&\quad \left(\frac{\lambda\Gamma(q)}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{g(s,u(s))}{s}ds - \int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right)\Bigg| \\
&\le \left(\int_1^{t_n}\left[\left(\log\frac{t_n}{s}\right)^{q-1}-\left(\log\frac{t}{s}\right)^{q-1}\right]|g(s,u(s))|\frac{ds}{s} + \int_{t_n}^t\left(\log\frac{t}{s}\right)^{q-1}|g(s,u(s))|\frac{ds}{s}\right) \\
&\quad + \frac{\left((\log t_n)^{q-2}-(\log t)^{q-2}\right)}{\Omega}\times \\
&\quad \left(\frac{\lambda\Gamma(q)}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}|g(s,u(s))|\frac{ds}{s} + \int_1^e\left(\log\frac{e}{s}\right)^{q-1}|g(s,u(s))|\frac{ds}{s}\right).
\end{aligned} \tag{III.7}$$

In view of $(A2)$, we obtain

$$\begin{aligned}
&\left|Hu(t_n)-Hu(t)\right| \\
&\le \frac{\|\xi\|\,\Psi(\|u\|)}{\Gamma(q)}\times \\
&\quad \Bigg\{\int_1^{t_n}\left[\left(\log\frac{t_n}{s}\right)^{q-1}-\left(\log\frac{t}{s}\right)^{q-1}\right]\frac{ds}{s} + \int_{t_n}^t\left(\log\frac{t}{s}\right)^{q-1}\frac{ds}{s} \\
&\quad + \frac{\left((\log t_n)^{q-2}-(\log t)^{q-2}\right)}{\Omega}\times \\
&\quad \left(\frac{\lambda\Gamma(q)}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{ds}{s} + \int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{ds}{s}\right)\Bigg\}.
\end{aligned} \tag{III.8}$$

Since the functions $t\longmapsto(\log t)^q$, $t\longmapsto(\log t)^{q-2}$ are continuous on $[1,e]$, we have

$$\lim_{n\to+\infty}\frac{\|\xi\|\,\Psi(\|u\|)}{\Gamma(q+1)}\Bigg\{(\log t_n)^q-(\log t)^q + \frac{\left((\log t_n)^{q-2}-(\log t)^{q-2}\right)}{\Omega}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)}(\log\eta)^{q+p}+1\right)\Bigg\}=0,$$

which yields

$$\lim_{n\to+\infty}\left|(Hu)(t_n)-(Hu)(t)\right|=0.$$

Then Hu is continuous at t. Therefore, $Hu\in[1,e]$, for all $u\in C[1,e]$. This proves that if $u\in C[1,e]$, then $Tu\in C[1,e]$.

Step 2. An estimate of $\|Tu\|$ for $u \in C[1,e]$.
Fix $u \in C[1,e]$ and $t \in [1,e]$. By assumptions $(A2)$ and $(A3)$, one can write

$$\begin{aligned}
&|(Tu)(t)| = |(Gu)(t)|\,|(Hu)(t)| \\
&\frac{|f(t,u(t))|}{\Gamma(q)}\left|\left\{\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right.\right. \\
&+\frac{(\log t)^{q-2}}{|\Omega|}\left(\frac{\lambda\Gamma(q)}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{g(s,u(s))}{s}ds\right. \\
&\left.\left.\left.-\int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right)\right\}\right| \\
&\le\frac{|f(t,u(t))-f(t,0)+f(t,0)|}{\Gamma(q)}\times \\
&\left\{\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{|g(s,u(s))|}{s}ds\right. \\
&+\frac{(\log t)^{q-2}}{|\Omega|}\left(\frac{\lambda\Gamma(q)}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{|g(s,u(s))|}{s}ds\right. \\
&\left.\left.+\int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{|g(s,u(s))|}{s}ds\right)\right\} \\
&\le\frac{\varphi(|u(t)|)+F_0}{\Gamma(q+1)}\times \\
&\left\{\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{\xi(s)\Psi(|u(s)|)}{s}ds\right. \\
&+\frac{(\log t)^{q-2}}{|\Omega|}\left\{\frac{\lambda\Gamma(q)}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{\xi(s)\Psi(|u(s)|)}{s}ds\right. \\
&\left.+\int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{\xi(s)\Psi(|u(s)|)}{s}ds\right\} \\
&\le\frac{[\varphi(\|u\|)+F_0]\,\|\xi\|\,(\Psi(\|u\|))}{\Gamma(q+1)}\times \\
&\qquad\left\{1+\frac{1}{|\Omega|}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)}+1\right)\right\}.
\end{aligned}$$

Hence,

$$\|Tu\|\le\frac{[\varphi(\|u\|)+F_0]\,\|\xi\|\,(\Psi(\|u\|))}{\Gamma(q+1)}\times \left\{1+\frac{1}{|\Omega|}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)}+1\right)\right\} \tag{III.9}$$

In view of assumption $(A3)$, we deduce that operator T applies B_ρ into itself. In addition, from (III.9), it follows that

$$\|GB_\rho\|\le\varphi(\rho)+F_0,$$

and

$$\|HB_\rho\|\le\frac{\|\xi\|\,(\Psi(\rho))}{\Gamma(q+1)}\left\{1+\frac{1}{|\Omega|}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)}+1\right)\right\}.$$

Step 3. We show that G and H are continuous on the ball B_ρ.
First, we check that G is continuous on B_ρ. To do this, we fix $\epsilon > 0$ and take $u, v \in B_\rho$ with $\|u-v\|\le\epsilon$. Then, for $t\in[1,e]$,

$$\begin{aligned}
|(Gu)(t)-(Gv)(t)| &= |f(t,u(t))-f(t,v(t))| \\
&\le\varphi(|u(t)-v(t)|) \\
&\le\varphi(\|u-v\|) \\
&\le\|u-v\| \\
&\le\epsilon,
\end{aligned}$$

and, since ϵ tends to 0, we have checked that G is continuous in B_ρ.

Now, we prove that H is continuous in B_ρ. To do this, we fix $\epsilon > 0$ and take $u, v \in B_\rho$ with $\|u-v\|\le\epsilon$. Then, for $t\in[1,e]$,

$$\begin{aligned}
&|(Hu)(t)-(Hv)(t)| \\
&=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{|g(s,u(s))-g(s,v(s))|}{s}ds \\
&\quad+\frac{(\log t)^{q-2}}{\Omega}\times \\
&\quad\left(\frac{\lambda}{\Gamma(q+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+q-1}\frac{|g(s,u(s))-g(s,v(s))|}{s}ds\right. \\
&\quad\left.-\frac{1}{\Gamma(q)}\int_1^e\left(\log\frac{e}{s}\right)^{q-1}\frac{|g(s,u(s))-g(s,v(s))|}{s}ds\right) \\
&\le\frac{\omega_g(J,\epsilon)}{\Gamma(q+1)}\left\{1+\frac{1}{|\Omega|}\left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)}+1\right)\right\},
\end{aligned} \tag{III.10}$$

where

$$\omega_g(J,\epsilon)=\sup\{|g(t,u)-g(t,v)|: t\in J: u,v\in[-\rho,\rho];\ |u-v|\le\epsilon\}$$

Since g is uniformly continuous on the compact $[1,e]\times[-\rho,\rho]$, we have $\omega_g(J,\epsilon)\longrightarrow 0$ as $\epsilon\longrightarrow 0$. Then, from (III.10), we conclude that operator G is continuous on B_ρ. As $T=G.H$, it follows that T is continuous on B_ρ.

Step 4. . Estimate $\omega_0(GX)$ and $\omega_0(HX)$ for $\emptyset\ne X\subset B_\rho$.

First, we estimate $\omega_0(GX)$. Fix $\epsilon>0$, let $u\in X$ and let $t_1,t_2\in J$ with $|t_2-t_1|\le\epsilon$. Then

$$\begin{aligned}
|(Gu)(t_2)-(Gu)(t_1)| &= |f(t_2,u(t_2))-f(t_1,u(t_1))| \\
&\le|f(t_2,u(t_2))-f(t_2,u(t_1))| \\
&\quad+|f(t_2,u(t_1))-f(t_1,u(t_1))| \\
&\le\varphi(|u(t_2)-u(t_1)|)+\omega(f,\epsilon) \\
&\le\varphi(\omega(u,\epsilon))+\omega(f,\epsilon),
\end{aligned}$$

where

$$\omega(f,\epsilon)=\sup\{|f(t_2,u)-f(t_1,u)|: t_1,t_2\in J,\ |t_2-t_1|\le\epsilon,\ u\in[-\rho,\rho]\}.$$

Therefore

$$\omega(GX,\epsilon)\le\varphi(\omega(u,\epsilon))+\omega(f,\epsilon). \tag{III.11}$$

Since $f(t,u)$ is uniformly continuous on the compact $J \times [-\rho,\rho]$, $\omega(f,\epsilon) \longrightarrow 0$ when $\epsilon \longrightarrow 0$. Then, from (III.11) we get

$$\omega_0(GX) \leq \varphi(\omega_0(X)).$$

Next, we estimate $\omega_0(HX)$. Fix $\epsilon > 0$, take $u \in X$ and $t_1, t_2 \in J$ with $|t_2 - t_1| \leq \epsilon$. Without loss of generality, we can suppose that $t_1 < t_2$. Then

$$\begin{aligned} &|Hu(t_2) - Hu(t_1)| \\ &\leq \frac{\|\xi\|\,\Psi(\|u\|)}{\Gamma(q)} \left\{ \int_1^{t_2} \left[\left(\log\frac{t_2}{s}\right)^{q-1} - \left(\log\frac{t_1}{s}\right)^{q-1} \right] \frac{ds}{s} \right. \\ &\quad + \int_{t_2}^{t_1} \left(\log\frac{t_1}{s}\right)^{q-1} \frac{ds}{s} \\ &\quad + \frac{\left((\log t_2)^{q-2} - (\log t_1)^{q-2}\right)}{|\Omega|} \times \\ &\quad \left(\frac{\lambda\Gamma(q)}{\Gamma(q+p)} \int_1^{\eta} \left(\log\frac{\eta}{s}\right)^{p+q-1} \frac{ds}{s} \right. \\ &\quad \left.\left. + \int_1^{e} \left(\log\frac{e}{s}\right)^{q-1} \frac{ds}{s} \right) \right\} \\ &\leq \frac{\|\xi\|\,\Psi(\|u\|)}{\Gamma(q+1)} \left\{ (\log t_2)^q - (\log t_1)^q \right. \\ &\quad + \frac{\left((\log t_2)^{q-2} - (\log t_1)^{q-2}\right)}{|\Omega|} \times \\ &\quad \left. \left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)} (\log\eta)^{q+p} + 1 \right) \right\}. \end{aligned}$$

Let $l(t) = (\log t)^q$. Function l is continuously differentiable on $[1,e]$. Hence, for all $t_1, t_2 \in [1,e]$, without loss of generality, let $t_1 < t_2$. Then there exist positive constants M_1 such that

$$\left| \frac{l(t_2) - l(t_1)}{t_2 - t_1} \right| = |l(\tau)| \leq M_1, \quad \tau \in (t_1, t_2). \tag{III.12}$$

On the other hand , $t \longmapsto \log t$ is 1-Lipschitz fuction on $[1,e]$ and $0 < q-2 \leq 1$, then $t \longmapsto \log t$ is a Holderian function with exponent $q-2$. That is

$$\left| (\log t_2)^{q-2} - (\log t_1)^{q-2} \right| \leq |\log t_2 - \log t_1|^{q-2} \leq |t_2 - t_1|^{q-2}, \tag{III.13}$$

for all $t_1, t_2 \in J$, with $t_2 < t_1$.
From (III.12) and (III.13), we deduce

$$\begin{aligned} &|Hu(t_2) - Hu(t_1)| \\ &\leq \frac{\|\xi\|\,\Psi(\|u\|)}{\Gamma(q+1)} \times \\ &\left\{ M_1\epsilon + \frac{\epsilon^{q-2}}{|\Omega|} \left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)} (\log\eta)^{q+p} + 1 \right) \right\}, \end{aligned}$$

this gives us $\omega_0(HX) = 0$.

Step 5. Estimate $\omega_0(TX)$ for $\emptyset \neq X \subset B_\rho$.

From definition II.6, we have $\omega_0(XY) \leq \|X\|\,\omega_0(Y) + \|Y\|\,\omega_0(X)$ and by using the estimates obtained in Steps 2 and 4, we deduce

$$\begin{aligned} \omega_0(TX) &= \omega_0(GX.HX) \leq \|GX\|\,\omega_0(HX) + \|HX\|\,\omega_0(GX) \\ &\leq \|GB_\rho\|\,\omega_0(HX) + \|HB_\rho\|\,\omega_0(GX) \\ &\leq \frac{\|\xi\|\,(\Psi(\rho))}{\Gamma(q+1)} \left\{ 1 + \frac{1}{|\Omega|} \left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)} + 1 \right) \right\} \varphi(\omega_0(X)) \end{aligned} \tag{III.14}$$

In view of assumption $(A3)$, $\frac{\|\xi\|(\Psi(\|u\|))}{\Gamma(q+1)} \left\{ 1 + \frac{1}{|\Omega|} \left(\frac{\lambda\Gamma(q+1)}{\Gamma(q+p+1)} + 1 \right) \right\} \leq 1$, and from (III.14), we conclude that

$$\omega_0(TX) \leq \varphi(\omega_0(X)).$$

Then by Theorem II.8, operator T has at least one fixed point in B_ρ. which is a solution of problem (I.1)-(I.2). This completes the proof. □

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