Volume 3, 2021 International Journal of Electrical Engineering and Computer Science (EEACS)

Received: May 16, 2020. Revised: December 18, 2020. Accepted: January 13, 2021. Published: January 17, 2021.

Zernike Polynomials and their Spectral Representation

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***Abstract*—The Zernike polynomials $Z_n^m(\rho,\phi)$ are known in optical physics, and they are used for the various diffractions and aberrations problems of lenses. They are defined on a circle, so that their representation decouples radial and axial coordinates. It is know that the Zernike radial polynomials $R_n^m(\rho)$ are represented through Jacobi polynomials. This paper deals with Chebyshev expansions for Jacobi polynomials. We have developed the recursive evaluation for spectral coefficients used in these expansions. These consequently provide a straightforward interpretation of Fourier transform of Zernike polynomials.**

I. Introduction

The conventional representation of Zernike radial polynomials gives unsatisfactory results for large values of degree n [3]. Some methods employ the recurrence relations [6], the other [2] suggests an algorithm in the form of discrete cosine transform which overcomes other methods in terms of accuracy. Nevertheless, the final formula represents an integral whose evaluation is performed using uniform sampling of the integrand. The proposed Chebyshev expansion of Jacobi polynomials $P_n^{\alpha,\beta}(x)$ results directly in spectral coefficients of $R_n^m(\rho)$ without need of using discrete Fourier transform.

Jacobi polynomials $P_n^{\alpha,\beta}(x)$ are defined as [7]

$$P_n^{\alpha,\beta}(x) = \frac{1}{2^n}\sum_{m=0}^{n}\begin{pmatrix} n+\alpha \\ m \end{pmatrix}\begin{pmatrix} n+\beta \\ n-m \end{pmatrix} \times (x-1)^{n-m}(x+1)^m \tag{1}$$

and they satisfy the self-adjoined differential equation

$$\frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}P_n^{\alpha,\beta}(x)\right] + n\,(n+1+\alpha+\beta)(1-x)^{\alpha}(1+x)^{\beta}\,P_n^{\alpha,\beta}(x) = 0. \tag{2}$$

Later, we will use Jacobi polynomials for radial part $R_n^m(\rho)$ of the Zernike polynomials

$$Z_n^m(\rho,\phi) = \begin{cases} R_n^m(\rho)\cos m\phi \\ R_n^m(\rho)\sin m\phi, \end{cases} \tag{3}$$

Based on (2) we can derived for $y(x) \equiv P_n^{\alpha,\beta}(x)$ a standard form of the Jacobi differential equation [7] as

$$(1-x^2)y''(x) - [(\alpha+\beta+2)x+\alpha-\beta]y'(x) + n(n+1+\alpha+\beta)y(x) = 0. \tag{4}$$

In order to represent Jacobi polynomials through the Chebyshev expansions

$$P_n^{\alpha,\beta}(x) = \sum_{\ell=0}^{n} a^{(\alpha,\beta)}(\ell)T_\ell(x). \tag{5}$$

$$P_n^{\alpha,\beta}(x) = \sum_{\ell=0}^{n} b^{(\alpha,\beta)}(\ell)U_\ell(x), \tag{6}$$

where $T_\ell(x)$, and $U_\ell(x)$ are Chebyshev polynomials of the first, and second kind, respectively, we have developed recursive evaluation of the spectral coefficients $a^{(\alpha,\beta)}(\ell)$ and $b^{(\alpha,\beta)}(\ell)$.

A. Recursive algorithm-I

Inserting in (4)

$$y(x) = \sum_{\ell=0}^{n} a^{(\alpha,\beta)}(\ell)T_\ell(x), \tag{7}$$

after considerable algebra we obtain the three-point recursive formulae which are concisely summarized in Table I.

The algorithm for the coefficients $a^{(\alpha,\beta)}(\ell)$ was used for evaluating Jacobi polynomials (5). Two numerical results for Jacobi polynomials $P_8^{(-0.15,0.75)}(x)$ and $P_{11}^{(17,0)}(x)$ are summarized in Table II. Note, the second polynomial is closely related to Zernike radial polynomial $R_{39}^{17}(\rho)$, that we introduce later.

B. Recursive algorithm-II

Inserting in (4)

$$y(x) = \sum_{\ell=0}^{n} b^{(\alpha,\beta)}(\ell)U_\ell(x), \tag{8}$$

we need to perform quite involved algebra to obtain the five-point recursive formulae - Table III. The algorithm for the

TABLE II
THE COEFFICIENTS $a^{(-0.15,0.75)}(\ell)$ AND $a^{(17,0)}(\ell)$ OF JACOBI POLYNOMIALS.

ℓ	$a^{(-0.15,0.75)}(\ell)$	$a^{(17,0)}(\ell)$
0	0.382162212452087	2.796583266044617 e06
1	-0.688698483046875	5.318721431858063 e06
2	0.756058665304687	4.568044430351257 e06
3	-0.666385898718750	3.530190682125092 e06
4	0.729511893632812	2.438413190643311 e06
5	-0.614211136875000	1.489855783063889 e06
6	0.677729102812500	0.792833667308807 e06
7	-0.502127525250000	0.359071999025345 e06
8	0.578841452718750	0.133550766944885 e06
9	-	0.038451398126602 e06
10	-	0.007664178707123 e06
11	-	0.000799205801010 e06

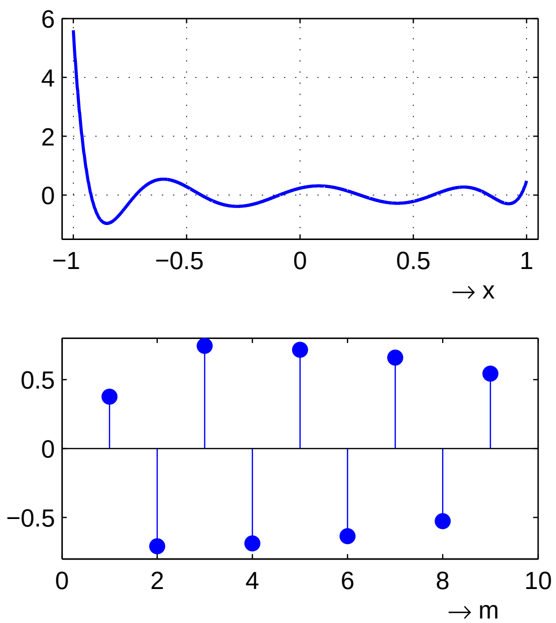

Fig. 1. Jacobi polynomial $P_8^{(-0.15,0.75)}(x)$ and its coefficients $a^{(-0.15,0.75)}(\ell)$ from equation (5).

TABLE IV
THE COEFFICIENTS $b^{(-0.15,0.75)}(\ell)$ AND $b^{(17,0)}(\ell)$ OF JACOBI POLYNOMIALS.

ℓ	$b^{(-0.15,0.75)}(\ell)$	$b^{(17,0)}(\ell)$
0	0.004132879799744	0.512561050868988 e06
1	-0.011156292164062	0.894265374866486 e06
2	0.013273385835937	1.064815619853973 e06
3	-0.026087380921875	1.020167449530602 e06
4	0.025891395410156	0.822789761667252 e06
5	-0.056041805812500	0.565391892019272 e06
6	0.049443825046875	0.329641450181961 e06
7	-0.251063762625000	0.160310300449371 e06
8	0.289420726359375	0.062943294118881 e06
9	-	0.018826096162796 e06
10	-	0.003832089353561 e06
11	-	0.000399602900505 e06

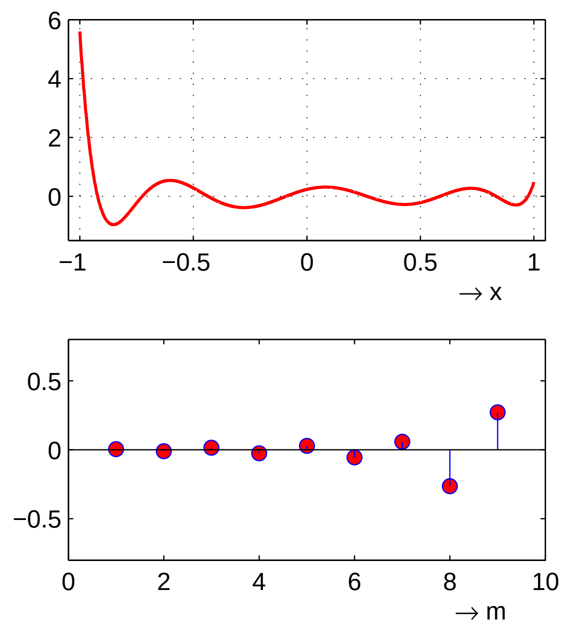

Fig. 2. Jacobi polynomial $P_8^{(-0.15,0.75)}(x)$ and its coefficients $b^{(-0.15,0.75)}(\ell)$ from equation (6).

coefficients $b^{(\alpha,\beta)}(\ell)$ was used for evaluating Jacobi polynomials (6). Two numerical results for Jacobi polynomials $P_8^{(-0.15,0.75)}(x)$ and $P_{11}^{(17,0)}(x)$ are summarized in Table IV.

II. ZERNIKE AND JACOBI POLYNOMIALS

The radial function $R_n^m(\rho)$ (3) of a Zernike polynomial is related to Jacobi polynomials $P_n^{\alpha,\beta}(x)$ [5] as

$$R_n^m(\rho) = (-1)^k \rho^m P_k^{m,0}(1-2\,\rho^2), \tag{9}$$

where $k = \frac{n-m}{2}$ must be an integer. Now, we use the recursive algorithm for evaluating the spectral coefficients $a^{(\alpha,\beta)}(\ell)$

$$P_k^{\alpha,\beta}(1-2\,\rho^2) = \sum_{\ell=0}^{k} a^{(\alpha,\beta)}(\ell) T_\ell(1-2\,\rho^2), \tag{10}$$

and then for radial function we have

$$R_n^m(\rho) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell a^{(m,0)}(\ell)\,\rho^m\,T_{2\ell}(\rho). \tag{11}$$

We can also use the spectral coefficients $b^{(\alpha,\beta)}(\ell)$ to represent

$$P_k^{\alpha,\beta}(1-2\,\rho^2) = \sum_{\ell=0}^{k} b^{(\alpha,\beta)}(\ell) U_\ell(1-2\,\rho^2). \tag{12}$$

Chebyshev polynomials of the second kind satisfy the identity

$$U_\ell(1-2\,\rho^2) = (-1)^\ell U_\ell(2\,\rho^2-1) = (-1)^\ell \frac{U_{2\ell+1}(\rho)}{2\rho}, \tag{13}$$

and then for radial function we have alternatively

$$R_n^m(\rho) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell b^{(m,0)}(\ell)\,\rho^{m-1}\,U_{2\ell+1}(\rho). \tag{14}$$

Using in (11) identity

$$\rho^m T_{2\ell}(\rho) = 2^{-m} \sum_{\mu=0}^{m} \binom{m}{\mu} T_{2\ell+m-2\mu}(\rho), \qquad (15)$$

we can evaluate spectral representation $c^{(m,0)}$ of the radial polynomial

$$R_n^m(\rho) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell c^{(m,0)}(\ell) T_{2\ell+m}(\rho). \qquad (16)$$

Similarly, using identity

$$\rho^{m-1} U_{2\ell+1}(\rho) = 2^{-(m-1)} \sum_{\mu=0}^{m-1} \binom{m-1}{\mu} U_{2\ell+m-2\mu}(\rho), \qquad (17)$$

in (14), we obtain spectral representation $d^{(m,0)}$ of the radial polynomial

$$R_n^m(\rho) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell d^{(m,0)}(\ell) U_{2\ell+m}(\rho). \qquad (18)$$

As the first Jacobi polynomials are

$$P_0^{(m,0)}(1-2\rho^2) = 1, \qquad (19)$$
$$P_1^{(m,0)}(1-2\rho^2) = (m+1) - (m+2)\rho^2, \qquad (20)$$

it is not difficult to make the crosscheck of radial polynomial representations (9) either with (16), or (18). For illustration, we present an explicit method for finding spectral coefficients $c^{(3,0)}(\ell)$ for Zernike radial polynomial $R^3_{2k+3}(\rho)$. Actually,

$$R^3_{2k+3}(\rho) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell a^{(3,0)}(\ell)\, \rho^3\, T_{2\ell}(\rho), \qquad (21)$$

then we substitute for

$$\rho^3 T_{2\ell}(\rho) = \frac{1}{8} \left[T_{2\ell-3}(\rho) + 3T_{2\ell-1}(\rho) + 3T_{2\ell+1}(\rho) + T_{2\ell+3}(\rho)\right]. \qquad (22)$$

If we collect the coefficients $a^{(3,0)}(m)$ that correspond to the same degree ℓ, we obtain a simple algorithm for spectral coefficients $c^{(3,0)}(\ell)$ for $\ell = 0$ to $k+1$

$$c^{(3,0)}(\ell) = \frac{1}{8}\left[a^{(3,0)}(\ell-1) - 3a^{(3,0)}(\ell) + 3a^{(3,0)}(\ell+1) - a^{(3,0)}(\ell+2)\right]. \qquad (23)$$

III. Conclusion

We have derived a robust evaluation of Zernike radial polynomials which derives its numerical stability from evaluation of spectral coefficients $a^{(\alpha,\beta)}(\ell)$, $b^{(\alpha,\beta)}(\ell)$. Instead of using discrete Fourier transform we have employed the Chebyshev expansion for a solution of the differential equation for Jacobi polynomials. Suggested method does not require any transformation (DCT or DFT) to get a spectral coefficients and therefore it is computationally efficient and numerically stable. We have presented numerical calculation of Jacobi polynomials $P_8^{(-0.15,0.75)}(x)$, $P_{11}^{(17,0)}(x)$, and $P_4^{(3,0)}(1-2\rho^2)$. Using these algorithms for Jacobi polynomials we could easily evaluate corresponding Zernike radial polynomials - Figure 4 and 5. Our algorithms provide Zernike radial polynomials of a considerable high degree n.

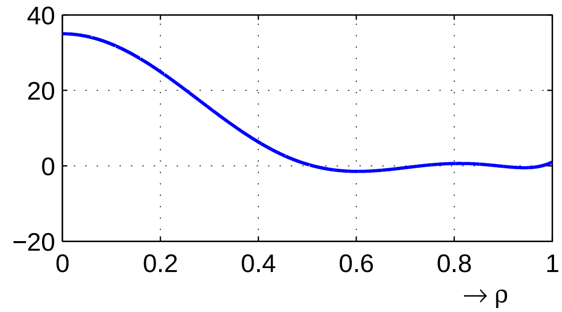

Fig. 3. Jacobi polynomial $P_4^{(3,0)}(1-2\rho^2)$ according to eq. (11) is closely related to Zernike radial polynomial $R^3_{11}(\rho)$. Corresponding spectral coefficients are $(-1)^\ell a^{(3,0)}(\ell) = [+6.234375, -11.375000, +9.187500, -5.625000, +2.578125]$.

For computation of Zernike radial polynomials Janssen and Dirksen [2] used a discrete Fourier (cosine) transform of Chebyshev polynomial of the second kind

$$R_n^m(\rho) = \frac{1}{2\pi} \int_0^{2\pi} U_n(\rho\cos\theta)\, \cos m\theta\, d\theta. \qquad (24)$$

Equations (11) and (14) suggest that the spectral coefficients $a^{(\alpha,\beta)}(\ell)$, $b^{(\alpha,\beta)}(\ell)$ can be an alternative representation of the above integral (24).

Acknowledgment

The authors wish to thank the Grant Agency of the Czech Republic for supporting this research through project GAP102/11/1795: *Novel selective transforms for non-stationary signal processing*.

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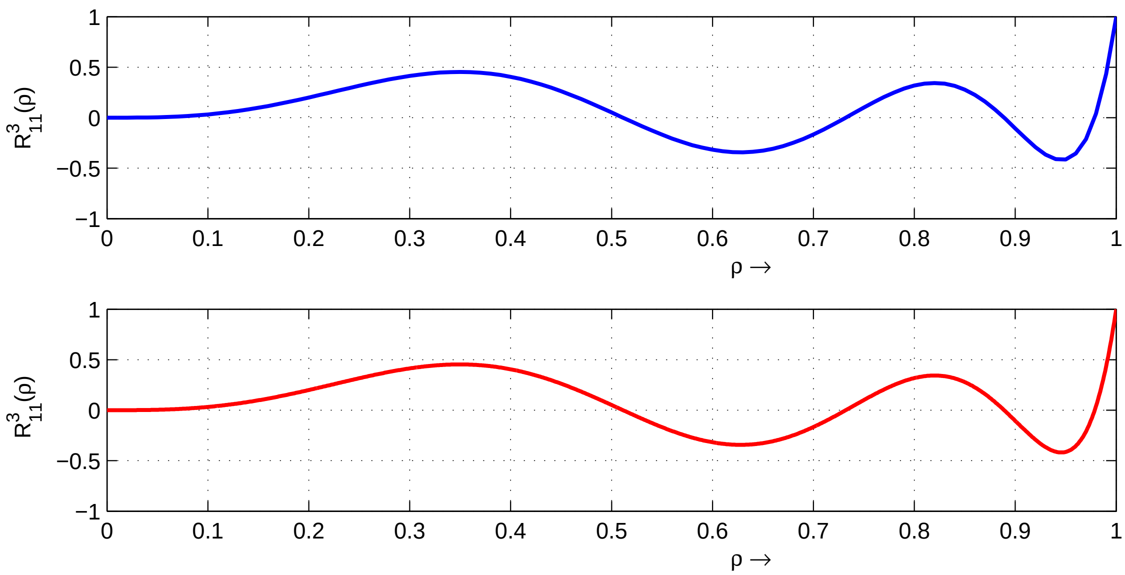

Fig. 4. Zernike radial polynomial $R_{11}^{3}(\rho)$ evaluated by cosine transform as in [2] and corresponding evaluation using spectral representation of Jacobi polynomial $P_4^{(3,0)}(1-2\rho^2)$

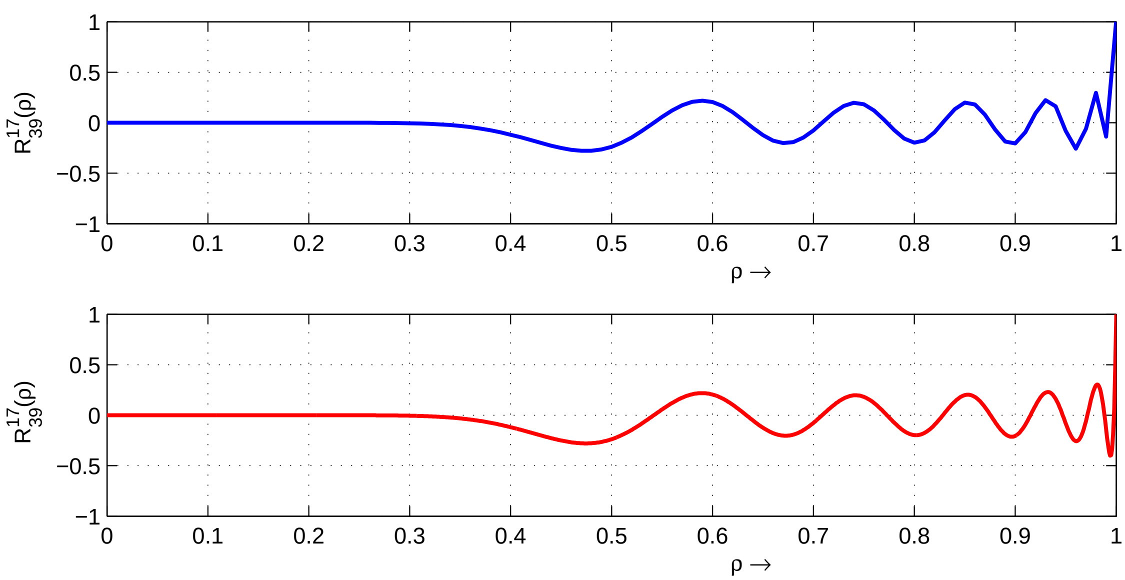

Fig. 5. Zernike radial polynomial $R_{39}^{17}(\rho)$ evaluated by cosine transform as in [2] and corresponding evaluation using spectral representation of Jacobi polynomial $P_{11}^{(17,0)}(1-2\rho^2)$. It is worth noting, that for higher degree n the evaluation of Zernike radial polynomial through spectral representation is free of disturbances for $\rho \to 1$.

TABLE I

RECURSIVE EVALUATION OF SPECTRAL COEFFICIENTS $a^{(\alpha,\beta)}(\ell)$ FOR A GENERAL JACOBI POLYNOMIAL $P_n^{\alpha,\beta}(x) = \sum_{\ell=0}^{n} a^{(\alpha,\beta)}(\ell) T_\ell(x)$.

given	$n,\ \alpha,\ \beta$
initial values	$a^{(\alpha,\beta)}(n) = 2^{-(2n-1)} \dfrac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$
	$a^{(\alpha,\beta)}(n-1) = \dfrac{2\,(\alpha-\beta)n}{\alpha+\beta+2n}\, a^{(\alpha,\beta)}(n)$
body	
(for	$k = 0 \quad to \quad n-2$
	$a^{(\alpha,\beta)}(n-k-2) = \dfrac{2\,(\alpha-\beta)\,(n-k-1)}{(k+2)\,(\alpha+\beta+2n-k-1)} a^{(\alpha,\beta)}(n-k-1) + \dfrac{(2n-k)\,(\alpha+\beta+k+1)}{(k+2)\,(\alpha+\beta+2n-k-1)} a^{(\alpha,\beta)}(n-k)$
end)	
	$a^{(\alpha,\beta)}(0) \to a^{(\alpha,\beta)}(0)/2$

TABLE III

RECURSIVE EVALUATION OF SPECTRAL COEFFICIENTS $b^{(\alpha,\beta)}(\ell)$ FOR A GENERAL JACOBI POLYNOMIAL $P_n^{\alpha,\beta}(x) = \sum_{\ell=0}^{n} b^{(\alpha,\beta)}(\ell) U_\ell(x)$

given	$n,\ \alpha,\ \beta$
initial values	$b^{(\alpha,\beta)}(n+1) = b^{(\alpha,\beta)}(n+2) = 0$
	$b^{(\alpha,\beta)}(n) = 2^{-(2n)} \dfrac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$
	$b^{(\alpha,\beta)}(n-1) = \dfrac{2\,(\alpha-\beta)n}{\alpha+\beta+2n}\, b^{(\alpha,\beta)}(n)$
body	
(for	$k = -2 \quad to \quad n-4$
	$b^{(\alpha,\beta)}(n-k-4)\ [n(n+2) - (n-k-4)(n-k-2) + (k+4)(\alpha+\beta-1)] =$
	$+2\, b^{(\alpha,\beta)}(n-k-3)\ (n-k+3)(\alpha-\beta)$
	$+2\, b^{(\alpha,\beta)}(n-k-2)\ [n(n+2) - (n-k-2)(n-k) + (k+1)(\alpha+\beta-1)]$
	$-2\, b^{(\alpha,\beta)}(n-k-1)\ (n-k+1)(\alpha-\beta)$
	$+\ \ b^{(\alpha,\beta)}(n-k)\ [n(n+2) - (n-k)(n-k+2) + (2n-k+2)(\alpha+\beta-1)]$
end)	