

Analysis of functional output-controllability of time-invariant Composite linear systems

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Abstract—This paper deals with the generalization of a test for calculating the functional output-controllability character of finite-dimensional linear continuous-time-invariant systems for composite systems either in series or in parallel case. It is computed by means of the rank of a certain constant matrix which can be associated to the composite system.

Index Terms—Linear systems, serial and parallel composite systems output-observability.

1. INTRODUCTION

It is well known that many physical problems as for example electrical networks, multibody systems, chemical engineering, semidiscretized Stokes equations, convolutional codes among others (see [2] [6] [8] for example), use for its description, the state space representation in the form

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \right\} \quad (1)$$

where $A \in M_n(\mathbb{C})$, $B \in M_{n \times m}(\mathbb{C})$, $C \in M_{p \times n}(\mathbb{C})$, or, in a more general form

$$\left. \begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \right\} \quad (2)$$

where $E, A \in M_n(\mathbb{C})$, $B \in M_{n \times m}(\mathbb{C})$, $C \in M_{p \times n}(\mathbb{C})$ and E can be singular.

These linear systems can be described with an input-output relation called transfer function obtained by applying Laplace transformation to equation (1)

$$\left. \begin{aligned} sX &= AX + BU \\ Y &= CX, \end{aligned} \right\},$$

obtaining the following relation

$$H(s) = C(sI - A)^{-1}B, \quad (3)$$

or applying Laplace transformation to equation (2)

$$\left. \begin{aligned} sEX &= AX + BU \\ Y &= CX, \end{aligned} \right\},$$

obtaining the following relation

$$\bar{H}(s) = C(sE - A)^{-1}B. \quad (4)$$

In engineering problems, a system is sometimes built by interconnecting some other systems, this kind of systems are called composite systems.

Let $\dot{x}_i = A_i x_i + B_i u_i$, $y_i = C_i x_i$ for $i = 1, 2$, be two systems that can be connected in different ways. The most common are the following:

i) serialized one after the other, so that the input information $u_2 = y_1(t)$. Consequently

$$\left. \begin{aligned} \dot{x} &= \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \right\} \quad (5)$$

ii) The second model presented is the parallel connection. This type of connection is of special interest especially the so-called interleaver parallel

concatenation (see [3], and [4] for example).

$$\left. \begin{aligned} \dot{x} &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= (C_1 \ C_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \right\} \quad (6)$$

The controllability concept of a dynamical standard system is largely studied by several authors and under many different points of view (see [1], [2], [6] for example). Nevertheless, controllability for the output vector of a system has been less treated (see [7], [8], [11] for example).

The functional output-controllability generally means, that the system can steer output of dynamical system along the arbitrary given curve over any interval of time, independently of its state vector. A similar but least essentially restrictive condition is the pointwise output-controllability.

In this paper functional output-controllability for singular systems is analyzed generalizing the study realized for standard systems and a test to study this property is presented.

2. PRELIMINARIES

In this paper, it is considered the state space system introduced in equation (1)

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \right\},$$

where x is the state vector, y is the output vector, u is the input (or control) vector, $A \in M_n(\mathbb{C})$ is the state matrix, $B \in M_{n \times m}(\mathbb{C})$ is the input matrix and $C \in M_{p \times n}(\mathbb{C})$ is the output matrix.

For simplicity we will write the systems by a triple of matrices (A, B, C) .

A way to understand the properties of the system is treating it by purely algebraic techniques. The main aspect of this approach is defining an equivalence relation preserving these properties.

The equivalence relation considered is such that derived after to make the following elementary transformations: basis change in the state space, basis change in the input space, basis change in the output space, feedback and output injection.

More concretely.

Definition 2.1: Two systems (A_i, B_i, C_i) , $i = 1, 2$, are equivalent if and only if there exist matrices

$P \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F^B \in M_{m \times n}(\mathbb{C})$, $F^C \in M_{p \times q}(\mathbb{C})$ such that

$$\begin{aligned} A_2 &= P^{-1}A_1P + PB_1F^B + F^CC_1P, \\ B_2 &= P^{-1}B_1R, \\ C_2 &= SC_1P. \end{aligned} \quad (7)$$

Having defined an equivalence relation, the standard procedure then is to look for a canonical form. That is to say to look for a triple of matrices which is equivalent to a given triple and which has a simple form from which we can directly read off the properties and invariants of the corresponding system. For a better understanding, we will give the following notations: I_ℓ denotes the ℓ -order identity matrix, $N_i = \text{diag}(N_{i_1}, \dots, N_{i_t}) \in M_{n_i}(\mathbb{C})$, $i = 1, 2, 3$, $N_{i_j} = \begin{pmatrix} 0 & I_{n_{i_j}-1} \\ 0 & 0 \end{pmatrix} \in M_{n_{i_j}}(\mathbb{C})$, $J = \text{diag}(J_1, \dots, J_t) \in M_{n_4}(\mathbb{C})$, $J_i = \text{diag}(J_{i_1}, \dots, J_{i_s})$, $J_{i_j} = \lambda_i I_{i_j} + N$.

Proposition 2.1: A system (A, B, C) can be reduced to (A_r, B_r, C_r) where:

$$A_r = \text{diag}(N_1, N_2, N_3, J)$$

$$B_r = \begin{pmatrix} B_1^t & 0 & 0 & 0 \\ 0 & B_2^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^t \quad \text{and} \quad C_r = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.1: Not all parts i),..., iv), necessarily appears in the decomposition of the system.

A. Functional output-controllability

The output-controllability generally means, that the system can steer output of dynamical system independently of its state vector. Concretely:

Definition 2.2: A system is functional output-controllable if and only if its output can be steered along the arbitrary given curve over any interval of time. It means that if it is given any output $y_d(t)$, $t \geq 0$, there exists t_1 and a control u_t , $t \geq 0$, such that for any $t \geq t_1$, $y(t) = y_d(t)$.

Proposition 2.2 ([2]): A system is functional output-controllable if and only

$$\text{rank } C(sI - A)^{-1}B = p$$

in the field of rational functions.

A necessary and sufficient condition for functional output-controllability is

Proposition 2.3 ([2], [12]):

$$\text{rank} \begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix} = n + p,$$

1) *Test for functional output-controllability for standard systems:* The functional output-controllability can be computed by means of the rank of a constant matrix in the following manner

Theorem 2.1 ([8]): The system (A, B, C) is functional output-controllable if and only if

$$\text{rank } oC_f(A, B, C) =$$

$$\text{rank} \begin{pmatrix} C & & & & & \\ CA & CB & & & & \\ CA^2 & CAB & CB & & & \\ \vdots & & & \ddots & & \\ CA^n & CA^{n-1}B & \dots & CAB & CB & \end{pmatrix} = (n + 1)p.$$

The null terms are not written in the matrix.

In order to proof this theorem we make use of the equivalence relation defined in 2.1 that permit us to consider an equivalent simple reduced form for the system.

Remark 2.2: We call $oC_i(A, B, C)$ or simply oC_i if confusion is not possible, the following matrix

$$oC_i = \begin{pmatrix} C & & & & & \\ CA & CB & & & & \\ CA^2 & CAB & CB & & & \\ \vdots & & & \ddots & & \\ CA^i & CA^{i-1}B & \dots & CAB & CB & \end{pmatrix}, \forall i \geq 1.$$

- i) If the system (A, B, C) is functional output-controllable, then the matrices oC_i have full row rank for all $0 \leq i \leq n$.
- ii) If the matrix oC_{n-1} has full row rank, it is not necessary that the matrix oC_n has full row rank.

Example 2.1: Let (A, B, C) a system with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$

$$\text{rank} \begin{pmatrix} C \\ CA \\ CB \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2,$$

$$\text{rank} \begin{pmatrix} CA & CB \\ CA^2 & CAB & CB \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2.$$

B. Functional output-controllability for singular systems

The output-controllability character can be generalized to the singular systems (2), in the following manner.

Definition 2.3: A singular system is functional output-controllable if and only if its output can be

steered along the arbitrary given curve over any interval of time. It means that if it is given any output $y_d(t)$, $t \geq 0$, there exists t_1 and a control u_t , $t \geq 0$, such that for any $t \geq t_1$, $y(t) = y_d(t)$.

Proposition 2.4 ([10]): A singular system is functional output-controllable if and only

$$\text{rank } C(sE - A)^{-1}B = p$$

in the field of rational functions.

A necessary and sufficient condition for functional output-controllability is

Proposition 2.5:

$$\text{rank} \begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix} = n + p.$$

Remark 2.3: Notice that for $E = I$ the proposition coincides with proposition 2.3

Remark 2.4: If $\text{rank } C < p$ the system is not functional output-controllable. Then, henceforth and without lost of generality, we will suppose that $\text{rank } C = p$.

1) *Test for functional output-controllability for singular systems:* The functional output-controllability can be computed by means of the rank of a certain constant matrix defined in the following manner.

Definition 2.4: For each system (E, A, B, C) we consider the following matrices.

$$\begin{aligned} M_0 &= C \\ M_1 &= \begin{pmatrix} A & B & -E \\ C & 0 & 0 \end{pmatrix} \in M_{(n+2p) \times (2n+m)}(\mathbb{C}) \\ M_2 &= \begin{pmatrix} A & B & -E \\ C & 0 & 0 \\ 0 & 0 & A & B & -E \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C \end{pmatrix} \in M_{(2n+3p) \times (3n+2m)}(\mathbb{C}) \\ &\vdots \\ M_i &= \begin{pmatrix} A & B & -E & 0 & 0 & 0 & \dots & 0 \\ C & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & A & B & -E & 0 & & \\ 0 & 0 & C & 0 & 0 & 0 & & \\ \vdots & & & & & \ddots & & \\ & & \dots & & & & A & B & -E \\ & & \dots & & & & C & 0 & 0 \\ & & \dots & & & & 0 & 0 & C \end{pmatrix} \\ &\in M_{in+(i+1)p \times ((i+1)n+im)}(\mathbb{C}) \end{aligned}$$

Calling now $M_n = oC_f(E, A, B, C)$, we have the following result.

Theorem 2.2: The system (E, A, B, C) is functional output-controllable if and only if

$$\text{rank } oC_f(E, A, B, C) =$$

$$\text{rank} \begin{pmatrix} A & B & -E & 0 & 0 & 0 & \dots & 0 \\ C & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & A & B & -E & 0 & & \\ 0 & 0 & C & 0 & 0 & 0 & & \\ \vdots & & & & & & \ddots & \\ & & & & & & & A & B & -E \\ & & & & & & & C & 0 & 0 \\ & & & & & & & 0 & 0 & C \end{pmatrix} =$$

$$(n+1)p + n^2.$$

The null terms are not written in the matrix.

Remark 2.5: For $E = I$, the test coincides with the test for standard systems. It suffices to make block elementary row and columns transformations to the matrix $oC_f(I, A, B, C)$:

$$\text{rank} \begin{pmatrix} A & B & -I & 0 & 0 & 0 & 0 & \dots & 0 \\ C & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & A & B & -I & 0 & & & \\ 0 & 0 & C & 0 & 0 & 0 & & & \\ \vdots & & & & & & & & \\ & & & & & & & & A & B & -I \\ & & & & & & & & C & 0 & 0 \\ & & & & & & & & 0 & 0 & C \end{pmatrix} =$$

$$\text{rank} \begin{pmatrix} I & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & I & & & & & & & & \\ & & & C & & & & & & & \\ & & & CA & & CB & & & & & \\ & & & CA^2 & & CAB & & CB & & & \\ & & & \vdots & & \ddots & & & & & \\ & & & CA^n & & CA^{n-1}B & & CB & & & \end{pmatrix}.$$

Example 2.2: Let (E, A, B, C) be a system with $E = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $C = (1 \ 0 \ 0)$

$$oC_f(E, A, B, C) =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using Matlab, it is easy to computing the rank of this matrix, we have

$$\text{rank } oC_f(E, A, B, C) = 13.$$

Then the system it is functional output-controllable.

But if we consider the system (E_1, A_1, B_1, C_1) with $E_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $C_1 = (1 \ 0 \ 0)$

$$oC_f(E_1, A_1, B_1, C_1) =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

As before, using Matlab, it is easy to computing the rank of this matrix,

$$\text{rank } oC_f(E_1, A_1, B_1, C_1) = 11.$$

Then the system it is not functional output-controllable.

Remark 2.6: i) If the singular system (E, A, B, C) is functional output-controllable, then the matrices M_i has full row rank for all $0 \leq i \leq n$.

ii) If the matrix M_{n-1} has full row rank, the matrix M_n does not necessarily has full row rank, as it can be seen in the following example.

Example 2.3: Let (E, A, B, C) with $E = I$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $C = (1 \ 0)$.

$$\text{rank} \begin{pmatrix} A & B & -I \\ C & 0 & 0 \\ 0 & 0 & C \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$4 = n + 2p,$$

but

$$\text{rank} \begin{pmatrix} A & B & -I \\ C & 0 & 0 \\ 0 & 0 & A & B & -I \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C \end{pmatrix} =$$

$$\text{rank} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$6 < 7.$$

Corollary 2.1: i) The system (E, A, B, C) is functional output-controllable if and only if the matrices M_i for all i has full row rank.

ii) For all $\ell \geq n$ we have that

$$\text{rank } M_{\ell+1} - \text{rank } M_\ell = \text{rank } M_{\ell+2} - \text{rank } M_{\ell+1}.$$

This corollary provides an iterative method to compute functional output-controllability in the following manner.

Step 1: Compute $\text{rank } M_0$. If $\text{rank} < p$ the system is not functional output-controllable,

If $\text{rank} = p$, then

Step 2: Compute $\text{rank } M_\ell$. If $\text{rank} < (\ell + 1)p + \ell n$ the system is not output observable.

If $\text{rank} = (\ell + 1)p + \ell n$ and $\ell = n$ the system is functional output-controllable, and if $\ell < n$ go to step 2.

3. FUNCTIONAL OUTPUT-CONTROLLABILITY FOR COMPOSITE SYSTEMS

A. Serial composite case

We consider the composite system (5), it is easy to compute that the transfer function of the serial composite system is the product of the transfer functions of the component systems

$$H(s) = H_2(s) \cdot H_1(s).$$

And a necessary and sufficient condition for functional output-controllability is as follows.

Theorem 3.1: A serial composite system (5) is output-controllable if and only if

$$\text{rank } H_2(s) \cdot H_1(s) = p_2$$

as a rational matrix.

Using the extension test to singular systems for the case $E = I$ we obtain a necessary and sufficient condition for functional output-controllability in which only has to calculate a product of matrices concretely B_2C_1 .

It is important to remark that the functional output controllability of the systems does not ensure the functional output controllability of the parallel concatenated system, as we can see with the following example

Example 3.1: We consider (A_1, B_1, C_1) and (A_2, B_2, C_2) with

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C_1 = (1 \quad 1)$$

and

$$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, C_2 = -C_1.$$

It is easy to observe that (A_i, B_i, C_i) are functional output-controllable but the the serial concatenated system is not functional output-controllable.

And, the serial composite system can be functional output-controllable without necessarily being the component systems, as we can see in the following example

Example 3.2: Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be two systems with

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = (-1 \quad 1)$$

and

$$A_2 = (0), B_2 = (1), C_2 = (1),$$

the serial composite system is (A, B, C) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = (0 \quad 0 \quad 1).$$

Clearly the system (A, B, C) is functional output-controllable but the system (A_1, B_1, C_1) is not functional output-controllable.

Theorem 3.2: The serial system (5) is functional output-controllable if, and only if, the matrix

$$\left(\begin{array}{c|c} \mathbf{X} & \mathbf{Y} \\ \hline \mathbf{Z} & 0 \end{array} \right) \in M_{a \times b}(\mathbb{R}),$$

with

$$\mathbf{X} = \begin{pmatrix} A_1 & -I_1 & 0 & 0 & 0 \\ & A_1 & -I_1 & 0 & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \dots & A_1 & -I_1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} B_1 \\ B_1 \\ \vdots \\ B_1 \end{pmatrix}$$

$$\mathbf{Z} = \begin{pmatrix} B_2C_1 & 0 & 0 & A_2 & -I_2 & \dots & \dots \\ & \ddots & & & & \ddots & \ddots \\ & & B_2C_1 & 0 & C_2 & 0 & A_2 & -I_2 \\ 0 & 0 & & & & \ddots & \ddots \\ & \ddots & & & & 0 & C_2 \\ 0 & 0 & & & & 0 & C_2 \\ 0 & 0 & & & & 0 & C_2 \end{pmatrix}$$

and $a \times b = ((n_1 + n_2)^2 + (n_1 + n_2 + 1)p_2) \times ((n_1 + n_2 + 1)(n_1 + n_2)m_1)$, has full row rank $(n + 1)p + n^2$.

Example 3.3: Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be two systems with

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = (0 \ 1)$$

and

$$A_2 = (2), B_2 = (3), C = (1).$$

The serial composite system is (A, B, C) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, C = (0 \ 0 \ 1).$$

Then,

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Z} = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{rank} \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & 0 \end{pmatrix} = 13$$

Then, the serial concatenated system is functional output-controllable.

B. Parallel composite case

We consider the composite system (6), it is easy to compute that the transfer function of the serial composite system is the addition of the transfer functions of the component systems

$$H(s) = H_2(s) + H_1(s).$$

And a necessary and sufficient condition for functional output-controllability is as follows.

Theorem 3.3: A parallel composite system (6) is output-controllable if and only if

$$\text{rank } H_2(s) + H_1(s) = p_2 = p_1$$

as a rational matrix.

As well as in the serial case, it is important to remark that the functional output controllability of the systems does not ensure the functional output controllability of the parallel concatenated system, as we can see with the following example

Example 3.4: Consider (A_1, B_1, C_1) and (A_2, B_2, C_2) with

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C_1 = (1 \ 1)$$

and

$$A_2 = A_1, B_2 = B_1, C_2 = -C_1$$

It is easy to observe that (A_i, B_i, C_i) are functional output-controllable but the the parallel concatenated system is not functional output-controllable.

And, the parallel composite system can be functional output-controllable without necessarily being the component systems, as we can see in the following example

Example 3.5: Let (A_1, B_1, C_1) and (A_2, B_2, C_2) with

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = (1 \ 0)$$

and

$$A_2 = (1), B_2 = (1), C_2 = (1),$$

the parallel composite system is (A, B, C) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, C = (1 \ 0 \ 1).$$

Clearly the system (A, B, C) is functional output-controllable but the system (A_1, B_1, C_1) is not functional output-controllable.

Proposition 3.1: A sufficient condition for functional output-controllability is that

$$\text{rank} (oC_{n_1+n_2}(A_1, B_1, C_1) + oC_{n_1+n_2}(A_2, B_2, C_2)) = (n_1 + n_2 + 1)p.$$

Proof: Applying Theorem 2.1 we have that for $n = n_1 + n_2$,

$$\text{rank} \begin{pmatrix} C_1 + C_2 & & & & & \\ C_1 A_1 + C_2 A_2 & C_1 B_1 + C_2 B_2 & & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ C_1 A_1^n + C_2 A_2^n & C_1 A_1^{n-1} B_1 + C_2 A_2^{n-1} A_2 & \dots & C_1 B_1 + C_2 B_2 & & \end{pmatrix} \leq$$

$$\text{rank} \begin{pmatrix} C_1 + C_2 & C_2 & C_1 B_1 + C_2 B_2 & & & \\ C_1 A_1 + C_2 A_2 & C_2 A_2 & C_1 B_1 + C_2 B_2 & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ C_1 A_1^n + C_2 A_2^n & C_2 A_2^n & \sum_1^2 C_i A_i^{n-1} B_i & \dots & C_1 B_1 + C_2 B_2 & \end{pmatrix}$$

$$= \text{rank} \begin{pmatrix} C_1 & C_2 & & & & \\ C_1 A_1 & C_2 A_2 & C_1 B_1 + C_2 B_2 & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ C_1 A_1^n & C_2 A_2^n & C_1 A_1^{n-1} B_1 + C_2 A_2^{n-1} A_2 & \dots & C_1 B_1 + C_2 B_2 & \end{pmatrix} \cdot \blacksquare$$

Example 3.6: Let (A_1, B_1, C_1) and (A_2, B_2, C_2) two systems with

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = (1 \quad -1)$$

and

$$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, C = (1 \quad 1).$$

Then, the parallel composite system is (A, B, C) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix},$$

$$C = (1 \quad -1 \quad 1 \quad 1)$$

$$oC_4(A_1, B_1, C_1) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$oC_{n_1+n_2}(A_2, B_2, C_2) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{rank} (oC_4(A_1, B_1, C_1) + oC_{n_1+n_2}(A_2, B_2, C_2)) =$$

$$\text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} = 5 = (2 + 2 + 1).$$

So, the parallel composite system is functional output-controllable.

4. CONCLUSION

In this paper a generalization of the test for calculating the functional output-controllability character of finite-dimensional linear continuous-time-invariant systems for composite systems either in series or in parallel case is obtained. It is computed by means of the rank of a certain constant matrix which can be associated to the composite systems.

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