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Optimality and stability properties of PD-controllers for a class of nonlinear SISO network systems

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***Abstract*—In this paper the stability and optimality properties of a class of PD controlled nonlinear SISO network systems is analyzed. The considered system class embraces network systems with input-output relative degree two and asymptotically stable zero dynamics. A set of semi-global stability conditions is derived which can be interpreted as tuning guidelines. The PD-controller optimality conditions are derived on the basis of a passivity property with respect to a suitable (control) output using inverse optimality.**

I. Introduction

For linear systems it is well-known that Proportional-Derivative (PD) controllers lead to smooth responses, given the feedback of the slope with respect to time. For second-order mechanical systems this is equivalent to the injection of damping into the closed-loop dynamics, while the proportional action can be seen as application of an external force, yielding a different shape of the associated potential energy function. Theese ideas motivate the name energy-shaping plus damping injection control [9]. For n-dimensional nonlinear systems, the idea of energy-shaping plus damping injection can be applied, unless it is more complicated to ensure the nonlinear stability feature of the closed-loop system with a linear controller. Different case studies, nevertheless reported asymptotic stability of the associated closed-loop dynamics (see e.g. [1]–[4], [8]). Extensions to nonlinear PD control were reported in [5], [6].

For relative degree-one systems, with asymptotically stable zero dynamics it is well-known [10] that the system is feedback equivalent to a passive system, i.e. there exists a stabilizing passivity-based feedback controller. Furthermore, this controller is optimal with respect to a meaningful objective function penalizing output-deviation agains control effort, which can be determined using inverse optimality [7], [11]. When the system has relative degree two, i.e. it can be transformed into a Byrnes-Isidori normal form with $n-2$-dimensional zero dynamics, then some structural properties may be exploited to design stabilizing controllers, e.g. using back-stepping-based control design [7], [12]. The resulting controller ensures stability and normally will have a complex nonlinea structure

In this paper these properties are analyzed for the particular case of PD controllers for minimum phase nonlinear network systems with relative-degree two. In particular, closed-loop semiglobal stability conditions are derived and the associated optimality properties are analyzed by drawing a passive output for the particular control structure. The obtained results show that PD controllers are optimal for this class of nonlinear systems with respect to a compromise penalization of the state deviation with respect to a manifold depending on the measurement function, and the control effort. In comparison with the general backstepping approach [7], [12], the proposed controller is *a priori* set as a linear one and thus ensures a very simple, model-independent structure. On the other hand, in contrast to non-model-based neural-network and fuzzy controllers, the proposed control scheme has a rigorous stability and optimality assessment.

The paper is organized as follows. In Section 2, the control problem is formulated and the main assumptions, defning the system class, are presented. In Section 3, closed-loop stability conditions are derived. In Section 4, the optimality of the PD controller for the considered class of systems is discussed and an explicit objective function is derived using inverse optimality. In Section 5, a representative case example is presented with a nonlinear van der Pol oscilator in interconnection with a network with nonlinear dynamics. Results are summarized and the main conclusions are presented in Section 6.

II. Problem formulation

Consider the following class of nonlinear SISO systems

$$\dot{x} = f(x,u), \quad x(0) = x_0, \quad y = h(x), \tag{1}$$

with $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}$, $h : X \to \mathbb{R}$, and $f : X \to X$ such that the following assumptions are satisfed:

Assumption 1: The origin $x = 0$ is a steady-state for $u = 0$, i.e., $f(0,0) = 0$.

Assumption 2: There exists a diffeomorphism

$$\zeta = \Phi(x), \qquad \text{such that } \zeta_1 = h(x) = y, \tag{2}$$

and so that the dynamics (1) are equivalent to

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \varphi_{21}(\zeta_1)\zeta_2 + \varphi_{22}(\zeta_1) + \gamma(\zeta_1)u \\ \dot{\psi} &= \varphi_0(\zeta, u) \\ y &= \zeta_1, \end{aligned} \tag{3}$$

where

$$\psi = [\zeta_3, \ldots, \zeta_n]', \quad \gamma(\zeta_1) \neq 0. \tag{4}$$

Assumption 3: The dynamical control system (3) is minimum-phase over a compact set X_0, i.e. its associated zero-dynamics

$$\dot{\psi} = \varphi_0([0,0,\psi']',0) \tag{5}$$

are asymptotically stable for all $\psi \in X_0$.

We are interested in analyzing the stability and optimality properties of the classical PD OF controller

$$u = k_p y + k_d \dot{y}, \quad y = h(x) \tag{6}$$

for the dynamical control system (3). For this purpose a Lyapunov function approach based on a physical interpretation of the PD controller action in terms of energy shaping plus damping injection [9] is employed, and the optimality properties are analyzed using inverse-optimality [7], [11].

III. Stability

The closed-loop dynamics of the dynamic control system (3) with the PD controller (6) are given by

$$\dot{x} = f(x,u), \quad u = k_p h(x) + k_d L_f h(x), \quad y = h(x). \tag{7}$$

We have the following result for the stability properties of the closed-loop dynamics (7), the proof of which is quite standard (cp. [3], [8]).

Theorem 1: Consider the dynamical control system (1) with the PD controller (6). Let Assumptions 1 to 3 hold. The closed-loop dynamics (7) are asymptotically stable with domain of attraction $\Omega \subset \mathbb{R}^n$ if

$$\begin{array}{l} k_p[\gamma(\zeta_1) + \gamma'(\zeta_1)\zeta_1] < -\varphi'_{21}(\zeta_1), \\ k_d\gamma(\zeta_1) < -\varphi_{21}(\zeta_1), \ \zeta_1 = h(x) \end{array} \tag{8}$$

holds for all $x \in \Omega$.

Proof: In virtue of Assumption 3 the dynamics of ζ_1 and ζ_2 can be written in the combined second-order form

$$\ddot{\zeta}_1 = -\delta(\zeta_1\dot{\zeta}_1 + \phi(\zeta_1) \tag{9}$$

where

$$\begin{array}{l} \delta(\zeta_1) = -[\varphi_{21}(\zeta_1) + k_d\gamma(\zeta_1)] \\ \phi(\zeta_1) = \varphi_{22}(\zeta_1) + k_p\gamma(\zeta_1)\zeta_1. \end{array} \tag{10}$$

Consider the potential

$$V = -\int_0^{\zeta_1} \phi(\sigma)d\sigma. \tag{11}$$

The function V has a unique minimum at $\zeta_1 = 0$ over a compact set Γ_1 (to be defined) if V' (i.e., $-\phi$) is isotonically increasing over Γ_1, or equivalently, if $V''(\zeta_1) > 0$, $\zeta_1 \in \Gamma_1$. By (10) this condition is equivalent to

$$k_p[\gamma(\zeta_1) + \gamma'(\zeta_1)\zeta_1] < -\varphi'_{21}(\zeta_1). \tag{12}$$

Next, Γ_1 is defined as the set over which inequality (12) is satisfied

$$\Gamma_1 = \{\zeta_1 \in \mathbb{R} | k_p[\gamma(\zeta_1) + \gamma'(\zeta_1)\zeta_1] < -\varphi'_{21}(\zeta_1)\}. \tag{13}$$

Note that in virtue of (12) over Γ_1 the following implication holds:

$$\phi(\zeta_1) = 0 \quad \Leftrightarrow \quad \zeta_1 = 0. \tag{14}$$

Now, introduce the Lyapunov function candidate

$$W(\zeta_1, \dot{\zeta}_1) = \frac{1}{2}\dot{\zeta}_1^2 + V(\zeta_1). \tag{15}$$

The rate of change of W along trajectories of (9) is given by

$$\dot{W} = -\delta(\zeta_1)\dot{\zeta}_1^2. \tag{16}$$

If $\delta > 0$ over a given set Γ_2 (to be defined), then the rate of change of W (16) is negative semi-definite, implying $\dot{\zeta}_1 \to 0$. On the other hand,

$$\delta(\zeta_1) > 0 \quad \Leftrightarrow \quad k_d\gamma(\zeta_1) < -\varphi_{21}(\zeta_1) \tag{17}$$

giving rise to the definition of the set

$$\Gamma_2 = \{\zeta_1 \in \mathbb{R} | k_d\gamma(\zeta_1) < -\varphi_{21}(\zeta_1)\}. \tag{18}$$

Accordingly, for all $\zeta_1 \in \Gamma_2$ $\dot{\zeta}_1 \to 0$, implying that $\ddot{\zeta}_1 = 0$, and by virtue of (9) $\phi(\zeta_1) = 0$. From (14) this implies $\zeta_1 = 0$ over the set Γ_1, meaning that

$$\forall \zeta_1 \in \Gamma_1 \cup \Gamma_2 : \lim_{t\to\infty} \zeta_1 = 0. \tag{19}$$

Let ζ_{1*} denote the smallest value of ζ_1 at the boundary of the set $\Gamma_1 \cup \Gamma_2$. Given that (16) is negative semidefinite (actually negative definite) over $\Gamma_1 \cup \Gamma_2$, it follows that the compact set

$$\Gamma_c = \{\zeta_1 \in \Gamma_1 \cup \Gamma_2 | V(\zeta_1) \le V(\zeta_{1*})\} \tag{20}$$

is positively invariant. Furthermore, the set

$$\Gamma_c \times [\zeta_{2min}, \zeta_{2max}] \tag{21}$$

is positively invariant for any values $\zeta_{2min}, \zeta_{2max}$. Denote by Σ_0 the subset of X_0 (Assumption 3) with $\zeta_1, \zeta_2 = 0$, and denote by X_{0c} the set

$$X_{0c} = X_0 \cup \{\Gamma_c \times \mathbb{R}^{n-2}\}. \tag{22}$$

Clearly, Σ_0 separates X_0 and is a closed-loop attractor for the compact set X_{0c} (22). Since the dynamics on $\Sigma_0 \cup X_{0c}$ are asymptotically stable, it follows from Seibert's Reduction Principle [14], that $\zeta = 0$ is a closed-loop attractor for the compact set X_{0c}. Finally, define Ω as the transformation of X_{0c} into $x-$coordinates to complete the proof. $\Diamond$

Remark 1: Note that the conditions (8) requiere that γ and k_p, k_d are of opposite signs, i.e. the product $\gamma(\zeta_1)k_p < 0$ and $\gamma(\zeta_1)k_d < 0$ for all $\zeta_1 \in \Gamma_c$. This observation will be crucial for the analysis of optimality.

IV. Optimality

In this section, the optimality properties of the PD controller for the considered system class are analyzed following the framework of inverse optimality for passive (relative-degree one) controllers. Given that the system has relative-degree two, first, an adecuate control output has to be determined with respect to which the system has relative-degree one and the associated zero-dynamics remains asymptotically stable.

A. Passivity

In order to analyze the optimality properties of the PD controller (6) for the class of systems (1) recall from constructive control theory that optimal controllers are passive with respect to some control output z [15]. Given that the PD controller is of relative degree two with respect to the measured output $y = h(x)$ it is not passive with respect to y.

This fact motivates to f rst address the question with respect to which control output z the PD controller (6) is passive.

For this purpose, consider the (diffeomorphic) state transormation

$$\zeta_1 = \zeta_1, \quad z = \zeta_1 + \frac{k_d}{k_p}\zeta_2, \tag{23}$$

and write the dynamics in the new variables

$$\begin{aligned} \dot{\zeta}_1 &= -\frac{k_p}{k_d}(\zeta_1 - z) \\ \dot{z} &= f(\zeta_1, z) + g(\zeta_1)u \\ \dot{\psi} &= \varphi_0\left(\left[\zeta_1, \frac{k_p}{k_d}(z-\zeta_1), \psi'\right]', 0\right), \end{aligned} \tag{24}$$

where

$$\begin{aligned} f(\zeta_1, z) &= \left(\frac{k_p}{k_d} + \varphi_{21}(\zeta_1)(z-\zeta_1)\right) + \frac{k_d}{k_p}\varphi_{22}(\zeta_1) \\ g(\zeta_1) &= \frac{k_d}{k_p}\gamma(\zeta_1). \end{aligned}$$

For $k_d \neq 0$, the relative degree between the control input u and the control output z is

$$\text{rd}(z, u) = 1. \tag{25}$$

The zero dynamics associated to (24),

$$\begin{aligned} \dot{\zeta}_1 &= -\frac{k_p}{k_d}\zeta_1 \\ \dot{\psi} &= \varphi_0\left[\zeta_1, -\frac{k_p}{k_d}\zeta_1, \psi\right], \end{aligned} \tag{26}$$

are asymptotically stable by virtue of the asymptotic stability of the original zero dynamics (Assumption 3). It follows that the system (24) is passive in a set Ω, if the controller u is designed such that $z \to 0$ asymptotically in Ω. According to Theorem 1, this can be achieved using the PD controller (6), implying that the PD controller (6) is a passive controller with respect to the control output z. This result is summarized in the following lemma.

Lemma 1: Consider the dynamical control system (1). The PD controller (6) is passive with respect to the control output

$$z = h(x) + \frac{k_d}{k_p}\Phi_2(x), \tag{27}$$

where Φ is the Byrnes-Isidori diffeomorphism taking the dynamics (1) into the form (3).

B. Inverse optimality

Having the passivity property of the PD controller (6) as point of departure (Lemma 1), it follows [7], [11] that the PD controller is optimal with respect to a certain objective function of the form

$$J[z(t)] = \int_t^\infty \{l[\zeta_1(\tau), z(\tau)] + r[\zeta_1(\tau), z(\tau)]u^2(\tau)\}d\tau, \tag{28}$$

with $l \geq 0$ and $r > 0$. The objective function (34) can be determined using inverse optimality [].

For the purpose of determining analytic expressions for the functions l and r, recall the associated Hamilton-Jacobi-Bellmann (HJB) equations

$$l(z) + L_f V(z) - \frac{[L_g V(z)]^2}{4r(z)} = 0, \quad u = -\frac{L_g V(z)}{2r(z)}. \tag{29}$$

Note that in terms of the output z the PD control simply reads

$$u = k_p z. \tag{30}$$

Let the cost function V be given by

$$V(z) = \frac{1}{2}z^2, \tag{31}$$

and rewrite the HJB equations

$$l(\zeta_1, z) + zf(\zeta_1, z) - \frac{z^2 g^2(\zeta_1)}{4r(\zeta_1, z)} = 0, \quad k_p z = -\frac{zg(\zeta_1)}{2r(z)}. \tag{32}$$

From the preceding equation pair the following solution for l and r are determined uniquely as

$$\begin{aligned} r(\zeta_1, z) &= \frac{g(\zeta_1)}{2k_p} > 0, \\ l(\zeta_1, z) &= -\left[k_p g\frac{z^2}{2} - zf(z, \zeta_1)\right] \geq 0 \, \forall z \in \Omega, \end{aligned} \tag{33}$$

where the set Ω is equivalent to the one def ned in (21) in ζ-coordinates. Note that the positivity of the preceding functions l and r is ensured by *Remark 1* of the last section.

The preceding result is summarized in the next theorem.

Theorem 2: The PD controller (6) is optimal for the dynamical control system (1) with respect to the objective function

$$J[z(t)] = \int_t^\infty \{-\left[k_p g\frac{z^2}{2} - zf(z, \zeta_1)\right] + \frac{g(\zeta_1)}{2k_p}u^2(\tau)\}d\tau. \tag{34}$$

This result shows that the PD controller is optimal in the sense that it ensures a compromise between controller speed (weighted with l) and control effort (weighted with r) over the set Ω. Note that the control speed is measured with respect to the control output z, being determined according to the manifold $h(x)$, or equivalently, the states $\zeta_1 = y$ and $\zeta_2 = \dot{y}$.

V. Case Study

To illustrate the above general results, consider the six-state networked system illustrated in Figure 1, with interconnected van-der-Pol oscilator (the states x_1, x_2), forcing the interconnected states $x_3, \ldots, x_6$. The corresponding interaction matrix for the network is given by the weighted adjacency matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & -1 & 0 & -0.5 & -0.6 \\ 0 & 0.1 & 0.2 & -1 & -0.2 & -0.3 \\ 0 & 0.2 & 0 & 0 & -1 & -0.2 \\ 0 & 0 & 0 & 0.25 & 0.8 & -1 \end{bmatrix} \tag{35}$$

and the nonlinear source term (φ), control (B) and measurement (C) matrix are given by

$$\varphi(x) = \begin{bmatrix} 0 - x_2^2 x_1 \\ 0 \\ -x_3^3 \\ -x_4^3 \\ -x_5^3 \\ -x_6^3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (36)$$
$$C = [\ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0\ ]$$

The system dynamics are given by

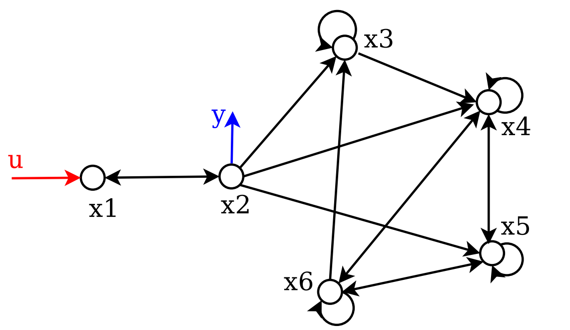

Fig. 1. Illustration of the underlying graph for system (37).

$$\dot{x} = Ax + Bu + \varphi(x), \quad x(0) = x_0, \quad y = Cx \qquad (37)$$

Note that the dynamics of the state x_2 can be written in the form

$$\ddot{x}_2 = (1 - x_2^2)\dot{x}_2 - x_2$$

which is just the van-der-Pol oscilator [13] with nonlinear oscilating behavior.

The relative degree between input u and output y is two, and when $x_1, x_2 = 0$, the remaining dynamics are asymptotically stable, given the eigenvalues of A corresponding to the zero-dynamics have negative real part, and the nonlinearity is stabilizing. Thus the zero-dynamics are asymptotically stable.

The open-loop unstable dynamics of this system are illustrated in Figure 2, showing the classical nonlinear oscillation of the van-der-Pol oscilator for x_1 and x_2, which forces the network states $x_3, \ldots, x_6$ to follow the oscillation in a attenuated fashion, given the linear plus cubic stabilizing terms in the dynamics of each node.

In closed-loop with the PD-controller (6) using the gains

$$k_p = -6, \quad k_d = -2 \qquad (38)$$

the dynamics are asymptotically stable, as illustrated in Figure 3. It can be seen that the oscillatory system part (x_1, x_2) reaches the SS $x = 0$ in about 10 time units, while the remaining part (x_3, to x_6) are not excited and reach the SS about 5 time units.

VI. Conclusion

The stability and optimality properties of the classical PD controller have been analyzed for a class of network nonlinear SISO systems with relative degree two. A domain of closed-loop asymptotic stabilization has been identif ed and suff cient

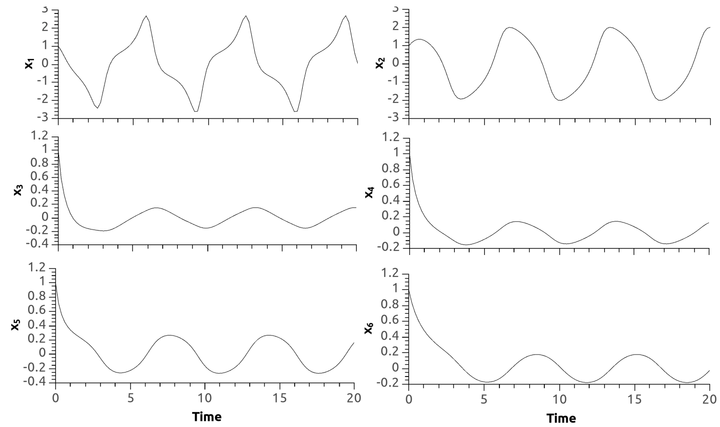

Fig. 2. Open-loop time-responses for network system (37) with initial state $x_0 = [1, 1, 1, 1, 1, 1]'$.

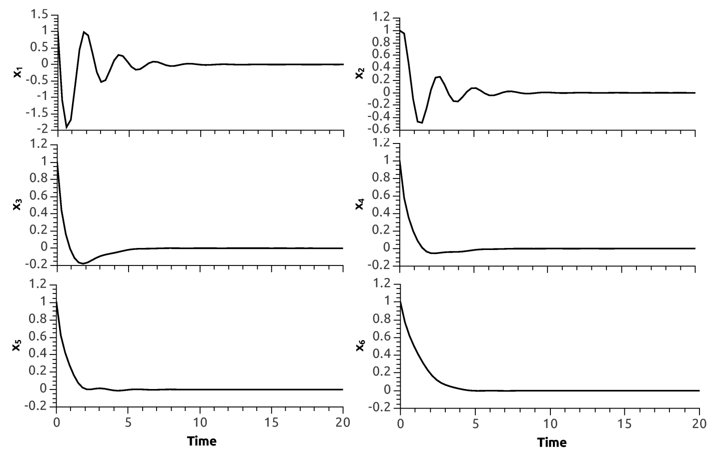

Fig. 3. Closed-loop time-responses for network system (37) with initial state $x_0 = [1, 1, 1, 1, 1, 1]'$ and PD-control (6) with gains $k_p = -6, k_d = -2$.

conditions for semi-global stability have been derived. The passivity properties of the PD controller have been studied, identifying a gain dependent output, which attains a simple geometric interpretation in terms of the f rst two states in the Byrnes-Isidori normalforn. Using inverse optimality, it has been shown that the PD controller is optimal with respect to a meaningful objective function.

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