On the field energy of two charges with application to electric dipoles

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Abstract: - This paper revisits the topic of electrostatic field energy due to a pair of the electric charges. Not only point-like charges but also charged spheres of radius R are studied. The self-energies as well as the interaction field energy are discussed in full detail. By combining two alternative didactic paths (one mathematical and the other based on energy conservation principles), it is shown that the interaction field energy (which is a volume integral of the energy density over the infinite space) is always equal to the potential energy, regardless of the nature of the electrical charges. For these two characteristic cases of electric charges (i.e., point-like and uniformly charged spheres), the location and the amount of the major part of the interaction field energy is discussed; the relevant factoids are documented in the form of two compact theorems in the Appendix. The case of non-uniformly charged spheres (i.e., spherical conductors) is also discussed to some extent. In addition to this general presentation, the particular case of the electric dipole is discussed, as a special case, accompanied with many numerical results.

Key-Words: - Electric dipole, Work from Infinity, Potential energy, Charged spheres, Analytical methods

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1 Introduction

Not many years ago, it was proposed that 'given the ambiguities and complexities associated with field energy, a traditional approach focusing on potential energy is more appropriate for introductory physics in secondary schools, colleges, and universities', [1]. To support his position, the same author refers to a claim of Feynman that '… The concepts of simple [point-like] charged particles and the electromagnetic field are in some way inconsistent.' (see [2], Vol. II, p. 28-1).

Actually, the abovementioned difficulty comes from the fact that, according to Coulomb's law, the electric intensity exactly at a point-like charge becomes *infinite*, thus the total electric field energy *cannot* be uniquely determined. Although Coulomb's law is quite analogous to Newton's gravitational law (both of them are inverse square laws), this difficulty does not appear in the gravitational theory because (leaving General Relativity aside for the moment) a point mass particle (of zero kinetic energy) has zero potential energy at infinity (i.e., not any additional self field energy at infinity). In other words, the gravitational field around two point mass particles separated by a distance d, is characterized by only the potential energy (also called binding energy), which by definition is the work done to bring one of these masses from infinity to that point, while keeping the other mass fixed.

According to Maxwell (1873), [3], the total energy of the electric field is proportional to the square of electric intensity ($\left|\vec{\boldsymbol{E}}\right|^{\mathbf{2}}$). In 2014, Hilborn, [1], and Tort, [4], reported their independent studies on the superposition of the electric fields due to two *point-like* charges (q_1 and q_2), i.e., $\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2$ (see, Fig. 1). Both of the aforementioned authors have split the field energy in three parts, applying the identity: $\left|\vec{\boldsymbol{E}}\right|^{\mathbf{2}} = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{E}} = \left|\vec{\boldsymbol{E}}_1\right|^{\mathbf{2}} + \left|\vec{\boldsymbol{E}}_2\right|^{\mathbf{2}} + 2\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2$. As we shall see in Section 2, the first term $\left|\vec{\boldsymbol{E}}_1\right|^{\mathbf{2}}$, leads to a self-energy exclusively related to the electric charge q_1, and becomes singular at this point. Similarly, the second term, $\left|\vec{\boldsymbol{E}}_2\right|^{\mathbf{2}}$, leads to a self-energy related to the charge q_2, and becomes singular at this point. The third term, $\left(2\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2\right)$, is twice the dot-product of both constituent point-like

charge fields and (after extensive manipulation) has been proven (see [1] and [4]) to be equal to the potential energy U, which –by definition– is the work done from infinity to bring the two electric charges together separated by a distance d. Regarding the regular integral of the term $(2\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2)$, Hilborn, [1], has proposed the evaluation of this integral in three alternative ways, i.e., through the use of trigonometric substitutions, integral tables, or symbolic manipulation packages such as MATHEMATICA®.

To deal with the problem of the abovementioned infinities at point-like charged particles, following Corbò, [5], Tort, [4], introduced a regularization and renormalization scheme, [6]. A more advanced technique which identifies the infinities as constants in a perturbative series of Coulomb's potential, was proposed by Mundarain, [7]. In any case, a rough solution is to ignore the field energy related to the squares of the electric intensity components $|\vec{\boldsymbol{E}}_1|^2$ and $|\vec{\boldsymbol{E}}_2|^2$, and deal only with the interaction field energy which is exclusively related to the dot product $(2\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2)$. An argument in favour of this approach is that we are mainly interested in energy *changes* and not in the absolute value of the field energy.

My objection to fully ignore the intensity components $|\vec{\boldsymbol{E}}_1|^2$ and $|\vec{\boldsymbol{E}}_2|^2$, comes from previous experience, however related to the field energy of magnetic dipoles, and particularly with the magnetic field of the Earth. Clearly, if we calculate the *potential energy* which corresponds to the well known magnetic moment (for the year 2020 is estimated to $M = 7.88 \times 10^{22}\, Am^2 = md$, $m =$ pole strength, $d = 2 \times 6{,}371{,}200\, m$ Earth's diameter thus separation distance of the magnetic dipole) we will find it to be equal to $U = \frac{k_m m^2}{d} = \frac{k_m m^2}{d} = 3 \times 10^{17}$ Joule, which is about 2.67 times *less than* the total field energy that fills the infinite exterior space (about 8×10^{17} Joule), without considering the interior of the Earth at all (which is anticipated to be equally large, if not larger than the exterior one). Therefore, in this particular case, the influence of the two self-energy terms in the total field energy becomes imperative.

The counter argument might be that when considering magnetostatics, there is a question as to whether a potential energy is defined. In the older view of Gilbert, Coulomb, Poisson, etc., magnetism is due to "magnetic charges"/poles, and the formalism of magnetostatics is the same as that of electrostatics. A magnetic potential does exist, and is defined at a point to be the work done to bring unit poles from infinity to that point, while keeping all other poles fixed. However, in 1820, Ampere made the conjecture that magnetism is *not* due to "poles", but to electric currents. In this view, there is no "magnetic" scalar potential, but rather a magnetic vector potential A (which is *not* an energy). As far as we can tell today, Ampere was right, and Gilbert, Coulomb, Poisson, and others, were wrong (a comment that of course does not reduce their value in the process of science). This (experiment) issue is hard to settle because the two hypotheses as to the nature of magnetism imply the same results for the magnetic field outside a magnetic dipole (at distances large compared to size of the distribution of poles or currents of the dipole). Therefore, in the above paragraph I have taken the Gilbertian view of magnetism, although this view is now disfavoured, [8].

Within this context, this paper is a contribution to the estimation of the total electrostatic field energy, from the abovementioned Maxwellian point of view, and its aim is threefold. First it discusses simple remedies for overcoming the shortcoming of the singularity at the point-like electric charges by considering two small spheres of charge, which can be understood by the average student in secondary schools, colleges and universities. Second, it applies the law of energy conservation to make clear the meaning of the total field energy. Third, it discusses in adequate length the distribution of the interaction field energy in the surrounding three-dimensional space, as well as the *self*-energies of the electric charges, and presents closed-form analytical expressions in characteristic zones, where possible. The study is completed by numerical results regarding an electric dipole. The first two out of the five Appendices are in the form of compact theorems.

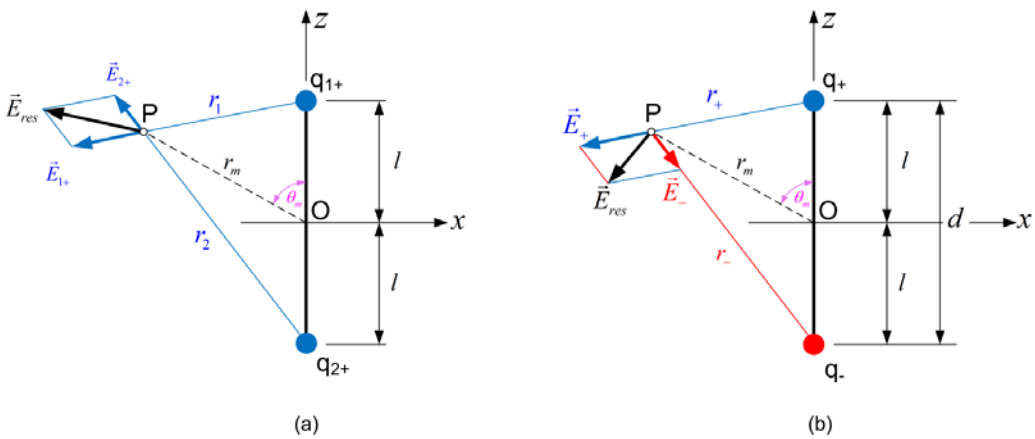

Fig. 1: Electric intensity induced due to two charges of (a) same (here, positive) sign and (b) opposite sign.

2 Basic equations

2.1 Definitions

We recall that the concept of electrostatic potential energy was first formulated by Poisson (1812), as an analog of gravitational potential energy, as in both cases the basic force law has an inverse square $(1/r^2)$ behavior. By definition, the electric potential energy U of two charges q_1 and q_2 separated by distance d equals the work needed to bring these charges together from infinity (where their potential energy is zero).

The electric scalar potential V of a point charge q is given by

$$V = k_e \frac{q}{r}, \tag{1}$$

where $k_e = 1/4\pi\varepsilon_0$ in SI units is the Coulomb's constant $(8.9875517923(14)\times10^9\ \mathrm{kg{\cdot}m^3{\cdot}s^{-4}{\cdot}A^{-2}})$, with ε_0 denoting the vacuum permittivity, and r is the distance from the charge to the observer. It is always useful to remember that $\varepsilon_0 = 1/4\pi k_e$.

The potential energy of two point charges, q_1 and q_2 (either opposite-sign or like-sign), separated by distance d is

$$U = k_e \frac{q_1 q_2}{d} = q_1 V_2 = q_2 V_1 . \tag{2}$$

Clearly, the potential energy U equals the work $W_{infinite}$ done while bringing q_2 from infinity to distance d from charge q_1, keeping q_1 fixed in space, and vice-versa. In the first case that q_1 is fixed, V_1 is the potential of q_1. In the second case that q_2 is fixed, V_2 is the potential of q_2.

Supposing that electric charge is continuously distributed with density ρ, instead of Eq. (2) we write $U = \frac{1}{2}\iiint \rho V \, \mathrm{d}\upsilon$, where $V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{Volume}^{3D} \frac{\rho(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{r}'|} d\upsilon$ is the scalar electric potential and $\mathrm{d}\upsilon$ is the elementary volume in which the charges exist. This may be useful when we deal with distributed charges instead of point charges.

We recall that Isaac Newton showed that the form (2) holds for the gravitational interaction energy of spherical shells of mass, as well as for point masses. By analogy, electric charges uniformly distributed on the outer surface of a sphere obey Coulomb's law with respect to their centers as well, thus Eq. (2) will be again valid.

So far, we have not mentioned fields. The previous knowledge was advocated by Coulomb, Poisson, etc.

It is worthy to mention that, Charles-Augustin de Coulomb (1785) also gave the force law, $\vec{\boldsymbol{F}}$ on q_1 due to q_2:

$$\vec{F} = k_e \frac{q_1 q_2 \vec{r}}{|\vec{r}|^3} = k_e \frac{q_1 q_2 \vec{e}_r}{r^2}, \tag{3}$$

where vector $\vec{\boldsymbol{r}}$ points from 2 to 1, and $\vec{\boldsymbol{e}}_r$ is the corresponding unit vector.

The electric field in electrostatics can be defined at a point as the force that would be exerted on unit charge at that point (due to other charges):

$$\vec{E} = \sum_{i=1}^{2} k_e \frac{q_i \vec{r}_i}{|\vec{r}_i|^3} = \sum_{i=1}^{2} k_e \frac{q_i \vec{e}_{ri}}{r_i^2} \tag{4}$$

where vector $\vec{\boldsymbol{r}}_{\boldsymbol{i}}$ points from charge q_i to the observation point P.

For a continuous charge distribution, we can write $\boldsymbol{E}(\boldsymbol{r}) = \iiint \rho(\boldsymbol{r}')\,(\boldsymbol{r}-\boldsymbol{r}')\,d\upsilon/\,|\boldsymbol{r}-\boldsymbol{r}'|^3$. This formula is also useful for distributed electric charges around a supposed point charge.

This electric field can be related to the electric potential V defined above via

$$\vec{E} = -\mathrm{grad}V \tag{5}$$

So far, the electric field is just a sort of computational aid (like the potential V). From now on, we pass to the *field energy*.

2.2 Field energy: A mathematical approach

The concept of magnetic field energy was introduced by Maxwell (1856), based on Poisson's model of magnetic potential energy together with the magnetic field equations $\vec{\nabla}\cdot\vec{\boldsymbol{B}} = 4\pi\rho_m$ and $\vec{\boldsymbol{B}} = -\vec{\boldsymbol{\nabla}}V_m$. James Clerk Maxwell, followed Faraday in supposing that the electric (and magnetic) field have a 'dynamical' significance in storing energy. Within this context, in 1873, Maxwell gave the analogous relation for the electric field U_f:

$$\begin{aligned} U_f &= \frac{1}{2}\int_{Volume} \rho V\,\mathrm{d}\upsilon = \frac{1}{2}\varepsilon_0 \int_{Volume} E^2\,\mathrm{d}\upsilon \\ &= \int_{Volume} \frac{E^2}{8\pi k_e}\mathrm{d}\upsilon \end{aligned} \tag{6}$$

which suggests that $u_E = \frac{1}{2}\varepsilon_0 E(\vec{\mathbf{r}})^2 = E(\vec{\mathbf{r}})^2/(8\pi k_e)$ is the physical density of electric field energy in the vicinity of point $\vec{\mathbf{r}}$.

However, there is a problem with Maxwell's formula for the field energy if we consider a point charge: the integral is infinite! Maxwell worked in a vision of the "ether" which filled all space. Today we say that the "ether" is not a 'mechanical' entity (with mass), but just the electric field itself (plus other fields as well, magnetic, gravitational, Higgs, and so on).

Henceforth, we focus on Eq. (6) and particularly on the term E^2. As also it was mentioned in the 'Introduction' (Section 1), the two components of the electric intensity (due to two electric charges only) form the total field $\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2$, of which the main interest is the magnitude $E = |\vec{\boldsymbol{E}}|^2$.

Here, we make use of the well known vector identity:

$$|\vec{\boldsymbol{E}}|^2 = \vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}} = |\vec{\boldsymbol{E}}_1|^2 + |\vec{\boldsymbol{E}}_2|^2 + 2\vec{\boldsymbol{E}}_1\cdot\vec{\boldsymbol{E}}_2 \tag{7}$$

Substituting Eq. (7) into Eq. (6), we have:

$$\begin{aligned} U_f &= \int_{Volume} \frac{E^2}{8\pi k_e}\mathrm{d}\upsilon = \int_{Volume} \frac{|\vec{E}_1|^2}{8\pi k_e}\mathrm{d}\upsilon \\ &+ \int_{Volume} \frac{|\vec{E}_2|^2}{8\pi k_e}\mathrm{d}\upsilon + \int_{Volume} \frac{\vec{E}_1\cdot\vec{E}_2}{4\pi k_e}\mathrm{d}\upsilon \end{aligned} \tag{8}$$

For point-like charges, the first two integrals in Eq. (8) (with integrands $|\vec{E}_i|^2, i = 1,2$), represent the *self energy* of the point charges, and are *infinite*. What to do? We just ignore these infinite integrals, and say what matters physically is the third integral, which we call the "*interaction energy*". This is a kind of "classical renormalization" (see [5]-[7]).

Below we distinguish two cases, the former being for point-like charges and the latter for charged spheres of radius R_1 and R_2.

2.2.1 Point-like Charges

When the electric charges are isolated at infinity they have their own self-fields $\vec{\boldsymbol{E}}_1$ and $\vec{\boldsymbol{E}}_2$, and obviously do not interact at all. The electric potential of point-like charge i is $k_e q_i/r_i$, where $\vec{\boldsymbol{r}}_i$ points from the point i to the observation point P. The electric field $\vec{\boldsymbol{E}}_i$ equals $k_e q_i \vec{\boldsymbol{r}}_i/r_i^2$, according to Eq. (4).

Obviously, by virtue of Eq. (4) the volume integrals of $\vec{\boldsymbol{E}}_1$ and $\vec{\boldsymbol{E}}_2$ (self energies in Eq. (8)) are infinite terms which are successively written as:

$$U_i = \int_{Volume} \frac{E_i^2}{8\pi k_e} \mathrm{d}\upsilon = \int_{r=0}^{\infty} \frac{\left(k_e \frac{q_i}{r^2}\right)^2}{8\pi k_e} \left(4\pi r^2\right) \mathrm{d}r \tag{9a}$$
$$= \int_{r=0}^{\infty} \frac{k_e q_i^2}{2r^2} \mathrm{d}r = \frac{k_e q_i^2}{2} \left[-\frac{1}{r}\right]_0^{\infty}, \quad i = 1,2.$$

In contrast, the volume integral of $\left(2\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2\right)$ which is the third term in the right-hand side of Eq. (8), not only is bounded but also equal to the potential energy $U = k_e \frac{q_1 q_2}{d}$, i.e.,

$$U_{12} = \frac{1}{2}\varepsilon_0 \int_{Volume} \left(2\vec{E}_1 \cdot \vec{E}_2\right) \mathrm{d}\upsilon = k_e \frac{q_1 q_2}{d} \tag{10}$$

Thus, the "interaction energy" corresponds to the "potential energy" of Coulomb and Poisson, while the total energy includes "something more" that does not make much sense. This is perhaps the main reason to be skeptical about field energy (Hilborn, [1]), and many people remain more comfortable with the view of Coulomb and Poisson that emphasizes scalar potentials. But this is a static view, and Maxwell's view is much more powerful in time-dependent problems that constitute the real world.

The proof of Eq. (10) has been highlighted by [1] and [4]. A closely related but complete mathematical proof of Eq. (10), without any gaps of thought, is given in *Appendix A* (see, Eq. (A.13) therein). Another proof, based on physical principles, is given in Section 3.

2.2.2 Two Charges of Radii R_1 and R_2 Separated by Distance d

In reality, a point-like charge may be equivalently substituted by many point-like charges uniformly distributed over the surface of a small sphere of radius R, which logically should not exceed more than half the distance d (no charge penetration, thus should be $0 < R_1 + R_2 \leq d$). The electric field $\vec{\boldsymbol{E}}_i$ is zero inside the sphere i, and equal to $k_e q_i \vec{\boldsymbol{r}}_i / r_i^2$ outside. Taking the limits of the integral in r between R_i and infinity (∞), Eq. (9a) changes to:

$$U_i = \frac{k_e q_i^2}{2R_i} = \frac{q_i V_i(r_i = R_i)}{2}, \quad i = 1,2. \tag{9b}$$

In the case of uniformly distributed charges over two spheres, the interaction energy is difficult to calculate using the fields $\vec{\boldsymbol{E}}_i$. McDonald, [9], has used the electric potentials of the two equal spheres in contact to compute, via surface integrals over the two spheres, and eventually has obtained again Eq. (10). In the present paper (see *Appendix C*), we follow a different technique working in conjunction with spherical coordinates, inspired by [1], and generalize the proof for any radius R_i ($i = 1,2$), regardless of whether the two spheres are in contact or not. In other words, the interaction energy of the two spheres of radius R is the same as if the charge were concentrated at the centers of the spheres. This result is also sustained by the findings of *Appendix B* (in the form of a theorem), in which it was found that the action of two point-like charges within any sphere centered at one of them has a *null effect*. In other words, either the spheres of radius R are real (thus with zero electric intensity per se) or they are fictitious, the volume integral of the dot-product $\left(2\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2\right)$ over them *vanishes*.

Therefore, in the case of distributed charges over the surface of two corresponding spheres of radius R_1 and R_2, respectively, by virtue of Eq. (10), Eq. (8) becomes:

$$U_f = U_1 + U_2 + U_{12} = k_e \left(\frac{q_1^2}{2R_1} + \frac{q_2^2}{2R_2} + \frac{q_1 q_2}{d}\right), \tag{11}$$

Remark-1: It is worthy to mention that for any isolated spherical charge of radius R and charge q, the field energy outside an imaginary sphere of radius r which surrounds it is given in an analogous way with Eq. (9b), by $U_R(r) = k_e q^2/(2r)$, with $r \geq R$. Therefore, if $U_{R0} = U_R(R) = k_e q^2/(2R)$ is the maximum possible field energy which corresponds to the entire exterior of the capacitor (of radius R), the previous expression can be also written as $U_R(r) = \left[\frac{k_e q^2}{2R}\right](R/r) = U_{R0}(R/r)$, and the corresponding graph is illustrated in Fig. 2. One may observe that 99% of the total field energy is trapped into a sphere of radius $r_{99} = 100R$. In other words, one hundredth of the maximum field energy value exists in the exterior of a sphere with $r/R =$

100, i.e., at a distance of 100 times the capacitor's radius R.

Remark-2: Regarding the same isolated spherical charge of radius R and uniform charge q, it is worth-mentioning that the field energy trapped between two radii r_1 and r_2 (with $R \leq r_1 < r_2$) does not depend on the particular value of capacitor's radius R. Actually, applying Eq. (9b) twice, one time for each radius we have: $\Delta U = U_R(r_2) - U_R(r_1) = \left[\frac{k_e q^2}{2r_2}\right] - \left[\frac{k_e q^2}{2r_1}\right] = \frac{k_e q^2}{2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) < 0$. Obviously, the latter expression does not depend on R but only on the radii r_1 and r_2 as well as the electric charge q. This is a consequence of the fact that the uniform distribution of electric charge on the surface of the sphere is equivalent to the field which creates a concentrated (point-like) charge at its center.

Remark-3: Quite similarly, for a specific sphere (O, r), of radius r and centered at the middle O of the two spherical charges, the self-energy *outside* the sphere depends only on the value r and the charge q (according to $U = k_e q^2/(2r)$), while the energy *inside* the sphere highly depends on the radii R_i $(i = 1,2)$ of the two charged spheres as well. Again, in case of point-like charges the self-energy inside the sphere (O, r) is infinite, whereas in case of charged spheres it is bounded given by $k_e q^2(1/2R_i - 1/2r)$; obviously the latter occurs because the field is limited in the interval $[R_i, r]$.

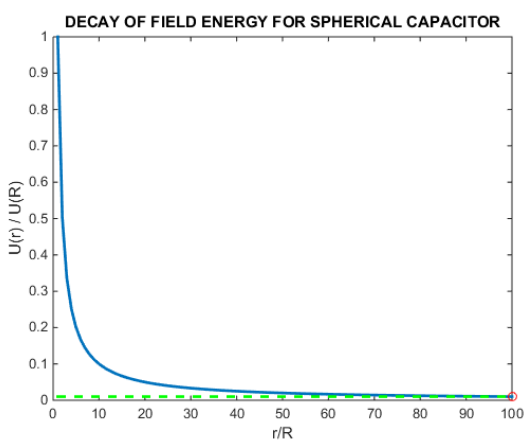

Fig. 2: Decay of field energy in a charged sphere of radius R.

3 Interpretation of the physics involved

3.1 Basic discussion

In this section we deal again with Eq. (11) but now using the energy foundations of physics and not the mathematical physics. To make this point quite clear, we can assume that the electric charge q_1 is found at the $-\infty$ whereas the charge q_2 at $+\infty$, as shown in Fig. 3(a). Thus, none of them influences the other one; simply each of them possesses its self-energy U_i $(i = 1,2)$ given by Eq. (9b). From the aforementioned state, we keep q_1 fixed while we slightly push q_2 towards q_1. Then, due to Coulomb's law according to Eq. (3), although the distance r is very large the Coulomb interaction force will give a small force along the line 12 of the two point charges. Below we distinguish two cases.

If the sign of the charges is the same (i.e., $q_1 q_2 > 0$), the interaction force is repulsive thus the charge q_2 will continue staying at $+\infty$. Then we need to offer a positive amount of work on q_2, which by definition equals to the (here) positive potential energy, to move q_2 at a distance d from the immobile charge q_1. At the end of the process, a positive work $U_{12} = q_1 q_2/d$ will be added to the initial energy $U_\infty = (U_1 + U_2) > 0$, thus eventually leading to total field energy equal to $U_{total} = U_1 + U_2 + U_{12} > 0$.

In contrast, if the sign of the charges is different (i.e., $q_1 q_2 < 0$), the interaction force is attractive thus the charge q_2 will be accelerated (according to Newton's Second Law) until an obstacle permanently keeps it at a distance d from q_1. By keeping the distance d for ever, this kinetic energy is lost thus the total field energy decreases by the negative potential energy, leading to $U_{total} = U_1 + U_2 - |U_{12}| > 0$. Alternatively, if we do not like the scenario of the accelerated charge we can resort to an external force which cancels the attraction of q_2 thus absorbing the negative energy $k_e q_1 q_2/d$ from the initial state of $U_\infty = (U_1 + U_2) > 0$, and therefore eventually leading to total field energy equal to $U_{total} = U_1 + U_2 - |U_{12}| > 0$.

Therefore, whatever the sign of the charges is, we have shown that Eq. (8) is valid, thus we have $U_{total}^{field} = U_1 + U_2 + U_{12} > 0$.

The above analysis is valid regardless the two changes are point-like or charged spheres of radii R_1 and R_2, respectively. The only change for the point-like charges is that then we may assume that $U_1 = U_2 = 0$.

3.2 Exercise: From infinity to contact and then separation to the final state

The contents of subsection 3.1 are sufficient for a student to understand the energy conservation and the field energy of the final configuration of the two electric charges separated by distance d. However, it is interesting to extend it by changing the virtual path, starting again from infinity but now performing a by-pass as follows.

3.2.1 General electric charges

Although the overall idea is very similar, it is suitable to focus on the particular case of different-sign charges (i.e., $q_1 q_2 < 0$) where an attractive Coulomb force dominates. When the charge q_2 is on the right-hand side half-space at $+\infty$ and is accelerated by attraction to the left charge q_1 at $(-\infty)$, if we wish we may not consider an obstacle at distance d, but we may imagine of an artificial collision between the two charged spheres (where, due to the attraction, they remain in contact for ever as shown in Fig. 3(b)). Then we may imagine that a virtual retouching of them is imposed until their centers are eventually separated by distance $(d > R_1 + R_2)$ as shown in Fig. 3(c). In this thought-experiment, the initial energy at infinities is $U_\infty = (U_1 + U_2) > 0$, while at the collision (where the distance of centers is $d' = R_1 + R_2$) the interaction energy is

$$U_{total}^{contact} = \left[\frac{k_e q_1^2}{2R_1} + \frac{k_e q_2^2}{2R_2}\right] + \left(\frac{k_e q_1 q_2}{d'}\right), \; d' = R_1 + R_2 \tag{12}$$

Moreover, since we are concerned with charges separated by a distance, d, we can start from the above virtual contact state described by total field energy given by Eq. (12), and then perform additional work to increase the distance between the centers from $d' = R_1 + R_2$ to the final value d, thus external forces must perform a work equal to:

$$W_{d'\to d} = -\int_{d'}^{d} F dr = -k_e q_1 q_2 \int_{d'}^{d} \frac{1}{r^2} dr = -k_e q_1 q_2 \left[-\frac{1}{r}\right]_{d'}^{d} = k_e q_1 q_2 \left(\frac{1}{d} - \frac{1}{d'}\right). \tag{13}$$

Summing up the abovementioned work $W_{d'\to d}$ [Eq. (13)] to the initial energy $U_{total}^{contact}$ [Eq. (12)], the terms related to the distance $d' = R_1 + R_2$ are cancelled, thus the final field energy of the two electric charges with opposite charges separated at distance d, is given by:

$$U_{total}^{contact} + W_{d'\to d} = k_e \left(\frac{q_1^2}{2R_1} + \frac{q_2^2}{2R_2} + \frac{q_1 q_2}{d}\right) \tag{14}$$

Comparing Eq. (14) with Eq. (11), one may observe that they coincide. This factoid was anticipated because the field is conservative and we simply follow different energy-exchange paths in our thought experiment. Therefore, the total field energy has to be preserved.

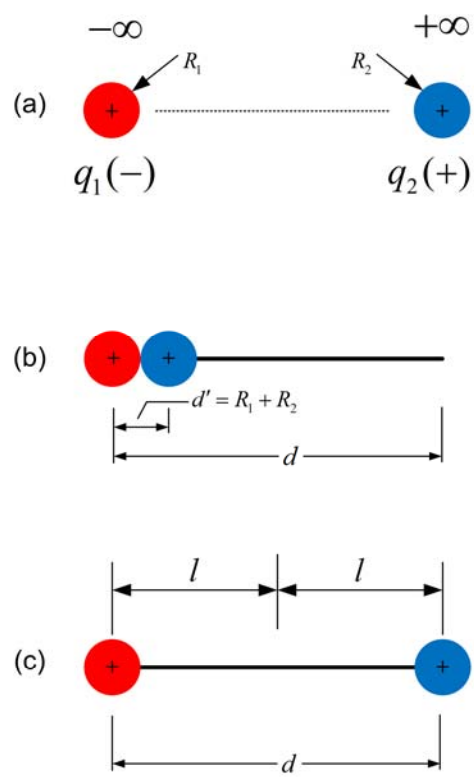

Fig. 3: Through-experiment: (a) Infinity (b) Contact (c) After separation by distance d.

4 Spatial breakdown of field energy components

4.1 General

The field energy is divided in two parts, the interaction energy and the self-energies. For each of them we distinguish two cases, the former being the point-like charges and the latter of two charged spheres of radii (R_1, R_2). As has been previously explained, the type of charges plays a role only when they belong to the area under consideration, otherwise not. First we refer to the *dashed line sphere* $(O, d/2)$ (to comply with Hilborn's, [1], terminology) as shown in Fig. 4a.

At every point P of the electric field made of two charges, q_1 and q_2, we can easily determine the magnitude of electric intensity $E = |\vec{E}|$ as shown in Fig. 1, thus the energy density $u_E = \frac{1}{2}\varepsilon_0 E^2 = E^2/(8\pi k_e)$ is known. By integrating the quantity u_E in a certain volume, we can obtain the corresponding electric field energy.

The aim of this section is to determine the tree components (U_1, U_2, and U_{12}) [see, Eq. (8)] of the field energy in characteristic areas of the infinite field which surrounds the two electric charges.

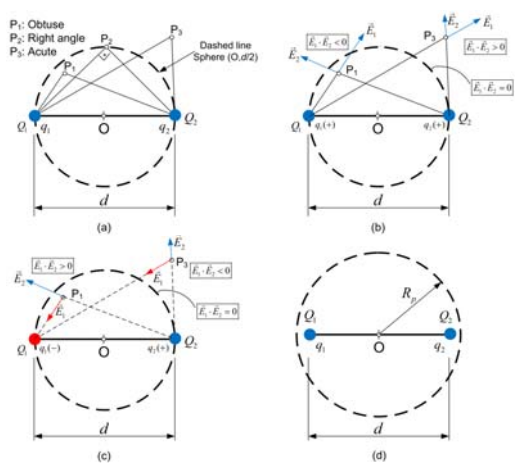

Fig. 4: Characteristic spheres and positions: (a) Dashed line sphere $(O, d/2)$, (b) Charge of the same sign $(q_1 q_2 > 0)$, (c) Charges of opposite sign $(q_1 q_2 < 0)$, (d) Fully surrounding sphere (O, R_p).

4.2 The Interaction Field Energy

One of these characteristic areas is the dashed line sphere $(O, d/2)$ which is centered at the middle O of the line $Q_1 Q_2$ connecting either the point-like charges or the centers of the correspondent spheres R_1 and R_2. The reason we select this sphere is because for each point P on it the angle $\widehat{Q_1 P Q_2}$ is 90 right angle (subtended from the diameter d), thus the dot-product $(\vec{E}_1 \cdot \vec{E}_2)$ vanishes on it. Moreover, when the point P lies inside this dashed line sphere, obviously the angle $\widehat{Q_1 P Q_2}$ is obtuse (> 90° deg), while when P is outside the same sphere the angle $\widehat{Q_1 P Q_2}$ is acute (<90° deg). This in turn determines the sign of the dot-product $(\vec{E}_1 \cdot \vec{E}_2)$, which is simply depended on the sign of the product $(q_1 q_2)$, accordingly.

In more detail, if $(q_1 q_2 > 0)$, the dot-product $(\vec{E}_1 \cdot \vec{E}_2)$ is negative inside the dashed line sphere $(O, d/2)$ and positive outside it (Fig. 4b). In contrast, if $(q_1 q_2 < 0)$, the dot-product $(\vec{E}_1 \cdot \vec{E}_2)$ is positive inside the dashed line sphere $(O, d/2)$ and negative outside it (Fig. 4c).

4.2.1 Point-like charges

As shown in *Appendix A*, the total amount of the interaction energy U_{12} is given by:

$$U_{12} = \frac{k_e q_1 q_2}{d} \tag{15}$$

Moreover, the accurate portion of the interaction energy inside the dashed line sphere $(O, d/2)$ is given by:

$$\begin{aligned} (U_{12})_{\substack{Inside \\ sphere(O,d/2)}} &= \frac{(2-\pi)}{4} \cdot \frac{k_e q_1 q_2}{d} \\ &\approx -0.2854 \frac{k_e q_1 q_2}{d}, \end{aligned} \tag{16a}$$

while outside the dashed line sphere $(O, d/2)$ is:

$$\begin{aligned} (U_{12})_{\substack{Outside \\ sphere(O,d/2)}} &= \frac{(2+\pi)}{4} \cdot \frac{k_e q_1 q_2}{d} \\ &\approx 1.2854 \frac{k_e q_1 q_2}{d} \end{aligned} . \tag{17a}$$

In other words, the most amount of the interaction field energy is outside the small dashed line sphere $(O, d/2)$ which marginally includes both charges. This may be physically explained by the fact that as one electric charge proceeds to the other, it is the far-field potential lines which will interact with each other.

4.2.2 Charged spheres

The setup is shown in Fig. 5(a). Interestingly, Eq. (15) holds also for the case that the charges are

uniformly distributed along the surfaces of two spheres of radii R_1 and R_2, respectively. In contrast, according to *Appendix C*, Eq. (16a) and Eq. (17a) have to be replaced by Eq. (16b) and Eq. (17b), respectively, while the sum of the interaction energy remains *unaltered* despite the differences.

$$\begin{aligned}\left(U_{12}\right)_{\substack{Inside\\ sphere(O,d/2)}} &= \frac{k_e q_1 q_2}{d}\frac{(2-\pi)}{4} \\ &+\frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_1}{d}\right)}{d}+\frac{1}{R_1}\left(\sqrt{1-\left(\frac{R_1}{d}\right)^2}-1\right)\right] \\ &+\frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d}+\frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2}-1\right)\right]\end{aligned} \tag{16b}$$

$$\begin{aligned}\left(U_{12}\right)_{\substack{Outside\\ sphere(O,d/2)}} &= \frac{k_e q_1 q_2}{d}\frac{(2+\pi)}{4} \\ &-\frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_1}{d}\right)}{d}+\frac{1}{R_1}\left(\sqrt{1-\left(\frac{R_1}{d}\right)^2}-1\right)\right] \\ &-\frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d}+\frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2}-1\right)\right]\end{aligned} \tag{17b}$$

Actually, first of all, we have to mention that the field outside each charged sphere is identical with that of a point-like charge. But this notice does not prove the coincidence between the two fields (point-like charges versus charged spheres). This is so because the electric filed inside each charged sphere of radius R_i vanishes ($\vec{E}_i = \vec{0}$) thus the relevant integral of the previous point-like field is not encountered in the case of the charged spheres. This in turn induces a nonzero change $(\Delta U_{12})_{in}$ inside and another nonzero change $(\Delta U_{12})_{out}$ outside the dashed line sphere $(O, d/2)$. The reason that for point-like charges the sum is preserved invariant is given in *Appendix B*, where it is shown that $(\Delta U_{12})_{in} + (\Delta U_{12})_{out} = 0$. Clearly, it is also shown that within each such 'imaginary/virtual' sphere R_i, of which a certain part is inside the bigger sphere $(O, d/2)$ and the rest outside it, for the particular case of *point-like* charges the total volume integral of $(\vec{E}_1 \cdot \vec{E}_2)$ *vanishes* as well, thus it consists of two equal but opposite nonzero terms.

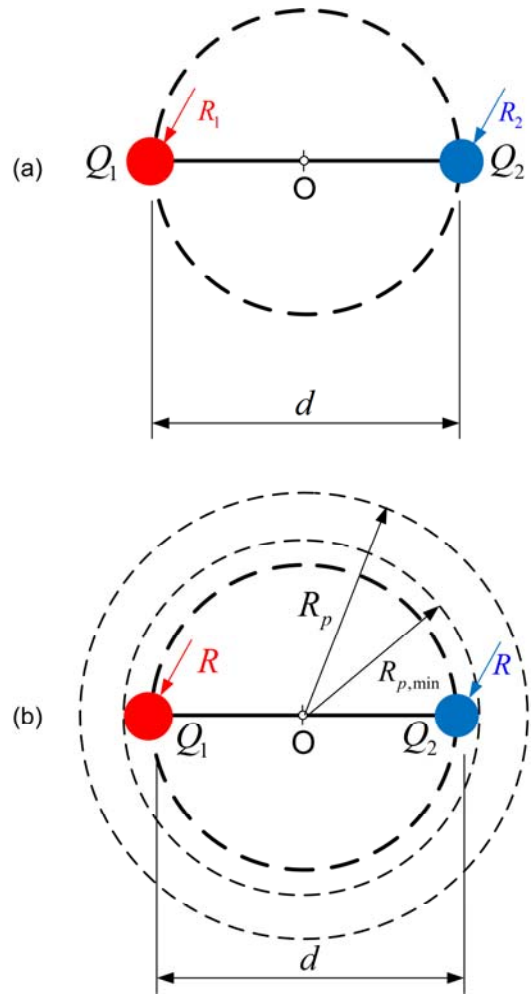

Fig. 5: (a) Dashed line sphere $(O, d/2)$ and charged spheres of radii (R_1, R_2), (b) Sphere (O, R_p) fully surrounding the two charges.

4.3 The self-energies

Following the methodology used in Section 4.1, and considering that the point Q_1 (q_1) is on the right while Q_2 (q_2) is on the left, in *Appendix D* it is shown that the self-energy of each charged sphere (R_1, R_2) can be analytically found as follows.

For the charge q_1 we have:

Inside the sphere $(O, d/2)$:

$$\left(U_1\right)_{Inside} = \frac{k_e q_1^2}{4}\left[\frac{1}{d}\ln\left(\frac{R_1}{d}\right)-\left(\frac{1}{d}-\frac{1}{R_1}\right)\right] \tag{18}$$

Outside the sphere $(O, d/2)$ Right:

$$(U_1)_{Outside-Right} = \frac{k_e q_1^2}{4}\left[\left(\frac{1}{R_1}+\frac{1}{d}\right)\right] \tag{19a}$$

Outside the sphere $(O, d/2)$ Left:

$$(U_1)_{Outside-Left} = \frac{k_e q_1^2}{4}\left[-\frac{1}{d}\ln\left(\frac{R_1}{d}\right)\right] \tag{19b}$$

Outside the sphere $(O, d/2)$ Total:

$$(U_1)_{Outside-Total} = \frac{k_e q_1^2}{4}\left[\left(\frac{1}{R_1}+\frac{1}{d}\right)-\frac{1}{d}\ln\left(\frac{R_1}{d}\right)\right]. \tag{19c}$$

As a first check, the sum of the above three terms [i.e., Eq. (18) as well as Eq. (19a) and Eq. (19b)] equals to the anticipated value $k_e q_1^2/(2R_1)$:

$$\begin{aligned}(U_1)_{Total} &= \frac{k_e q_1^2}{4}\cdot\left[\frac{1}{d}\ln\left(\frac{R_1}{d}\right)-\left(\frac{1}{d}-\frac{1}{R_1}\right)\right]_{Inside} \\ &+ \frac{k_e q_1^2}{4}\cdot\left[\left(\frac{1}{R_1}+\frac{1}{d}\right)\right]_{Outside-Right} \\ &+ \frac{k_e q_1^2}{4}\cdot\left[-\frac{1}{d}\ln\left(\frac{R_1}{d}\right)\right]_{Outside-Left} \\ &= \frac{k_e q_1^2}{2R_1}\end{aligned} \tag{20}$$

Similarly, for the charge q_2 we can write the analogous field energy expressions:

Inside the sphere $(O, d/2)$:

$$(U_2)_{Inside} = \frac{k_e q_2^2}{4}\left[\frac{1}{d}\ln\left(\frac{R_2}{d}\right)-\left(\frac{1}{d}-\frac{1}{R_2}\right)\right] \tag{21}$$

Outside the sphere $(O, d/2)$ Right:

$$(U_2)_{Outside-Right} = -\frac{k_e q_2^2}{4}\frac{1}{d}\ln\left(\frac{R_2}{d}\right) \tag{22a}$$

Outside the sphere $(O, d/2)$ Left:

$$(U_2)_{Outside-Left} = \frac{k_e q_2^2}{4}\left(\frac{1}{R_2}+\frac{1}{d}\right) \tag{22b}$$

Outside the sphere $(O, d/2)$ Total:

$$(U_2)_{Outside-Total} = \frac{k_e q_2^2}{4}\left[\left(\frac{1}{R_2}+\frac{1}{d}\right)-\frac{1}{d}\ln\left(\frac{R_2}{d}\right)\right] \tag{22c}$$

As a second check, it can be verified that the sum of the three portions of U_2-self energy is equal to the anticipated formula $k_e q_2^2/(2R_2)$:

$$\begin{aligned}(U_2)_{Total} &= \frac{k_e q_2^2}{4}\cdot\left[\frac{1}{d}\ln\left(\frac{R_2}{d}\right)-\left(\frac{1}{d}-\frac{1}{R_2}\right)\right]_{Inside} \\ &+ \frac{k_e q_2^2}{4}\cdot\left[-\frac{1}{d}\ln\left(\frac{R_2}{d}\right)\right]_{Outside-Right} \\ &+ \frac{k_e q_2^2}{4}\cdot\left[\left(\frac{1}{R_2}+\frac{1}{d}\right)\right]_{Outside-Left} \\ &= \frac{k_e q_2^2}{2R_2}\end{aligned} \tag{23}$$

Therefore, the total field energy with respect to the dashed line sphere $(O, d/2)$ is as follows.

- Inside the sphere $(O, d/2)$:

$$\begin{aligned}U_{Total-Field}^{Inside\,(O,d/2)} &= \frac{k_e q_1^2}{4}\left[\frac{1}{d}\ln\left(\frac{R_1}{d}\right)-\left(\frac{1}{d}-\frac{1}{R_1}\right)\right] \\ &+ \frac{k_e q_2^2}{4}\left[\frac{1}{d}\ln\left(\frac{R_2}{d}\right)-\left(\frac{1}{d}-\frac{1}{R_2}\right)\right] \\ &+ \frac{(2-\pi)}{4}\cdot\frac{k_e q_1 q_2}{d} \\ &+ \frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_1}{d}\right)}{d}+\frac{1}{R_1}\left(\sqrt{1-\left(\frac{R_1}{d}\right)^2}-1\right)\right] \\ &+ \frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d}+\frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2}-1\right)\right]\end{aligned} \tag{24}$$

- Outside the sphere $(O, d/2)$:

$$U_{Total-Field}^{Outside(O,d/2)} = \frac{k_e q_1^2}{4}\left[-\frac{1}{d}\ln\left(\frac{R_1}{d}\right)+\left(\frac{1}{d}+\frac{1}{R_1}\right)\right] + \frac{k_e q_2^2}{4}\left[-\frac{1}{d}\ln\left(\frac{R_2}{d}\right)+\left(\frac{1}{d}+\frac{1}{R_2}\right)\right] + \frac{(2+\pi)}{4}\cdot\frac{k_e q_1 q_2}{d} - \frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_1}{d}\right)}{d}+\frac{1}{R_1}\left(\sqrt{1-\left(\frac{R_1}{d}\right)^2}-1\right)\right] - \frac{k_e q_1 q_2}{2}\left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d}+\frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2}-1\right)\right] \quad (25)$$

5 Field energies in a sphere (O, R_p) fully surrounding the two charges

In this section we consider another sphere (O, R_p), again centered at the midpoint O of the connecting line Q_1Q_2, but now with a larger radius so as to entirely include the two electric charges. The case of the point-like charges is shown in Fig. 4(d). In case that the radii of the charged spheres are equal one another (i.e., $R_1 = R_2 = R$), it should obviously be $R_p \geq \frac{d}{2} + R = R_{p,min}$ (see, Fig. 5(b)).

5.1 Outside the sphere (O, R_p), i.e., for $r \geq R_{p,min}$

This is shown either in Fig. 4d (point-like) or in Fig. 5b (charged spheres).

According to *Appendix E*, the self-energy of each charge in the exterior of the sphere (O, R_p) is given by:

$$(U_i)_{\substack{Outside\\(r>R_p)}} = \frac{k_e q_i^2}{4}\left[\frac{4R_p}{4R_p^2-d^2}+\frac{1}{d}\ln\left(\frac{2R_p+d}{2R_p-d}\right)\right], \quad (i=1,2) \quad (26)$$

thus the observer *cannot* distinguish whether the charges are point-like of not.

Moreover, regarding the interaction energy $(U_{12})_{r>R_p}$ in the infinite space outside the sphere (O, R_p), based on variables $(\theta = \theta_1, r = r_1)$, Eqs. (E2) to (E.4) of *Appendix E* show that:

$$(U_{12})_{r>R_p} = \frac{k_e q_1 q_2}{2}\cdot \int_{\theta=0}^{\pi/2}\left(\frac{\sin\theta}{\left[R_p^2+\frac{3}{4}d^2-\frac{d^2}{2}\cos^2\theta - d\cos\theta\left(R_p^2-\frac{d^2}{4}\sin^2\theta\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}\right)\mathrm{d}\theta. \quad (27)$$

In the context of this paper, it was not possible to derive a closed–form analytical expression for the integral in Eq. (27), despite the fact that a commercial symbol manipulation code was used. For the moment, it is proposed to calculate it numerically, for example applying Simpson's trapezoidal rule. A short program in MATLAB® is given at the end of *Appendix E*.

5.2 Inside the sphere (O, R_p), i.e., for $r < R_p$

The field energies in the interior of the sphere (O, R_p) are easily determined by subtracting Eq. (26) and Eq. (27) from the corresponding total energies.

First we recall that in case of charged spheres the total self-energy is $(U_1 + U_2)_{Total} = k_2\left(\frac{q_1^2}{2R_1}+\frac{q_2^2}{2R_2}\right)$, while it becomes infinity for point-like charges. Obviously, regarding the self-energies inside the sphere (O, R_p) it makes sense to talk about only for the former case (of charged spheres), for which we have:

$$(U_i)_{\substack{Inside\\(r<R_p)}} = \frac{k_e q_i^2}{2R_i} - \frac{k_e q_i^2}{4}\left[\frac{4R_p}{4R_p^2-d^2}+\frac{1}{d}\ln\left(\frac{2R_p+d}{2R_p-d}\right)\right], \quad (i=1,2). \quad (28)$$

Second, regarding the interaction energy $(U_{12})_{\substack{Inside\\(r<R_p)}}$, we have:

$$(U_{12})_{\substack{Inside\\(r<R_p)}} = \frac{k_e q_1 q_2}{d} - (U_{12})_{r>R_p}, \quad (29)$$

where $(U_{12})_{r>R_p}$ is given by Eq. (27).

Note: All the formulas of Section 5 concern the case that the sphere (O, R_p) does *not* intersect the charged spheres. Therefore, for the particular case

that $R_1 = R_2 = R$, Eqs. (26) to (29) are valid onlywhen $R_p \geq \frac{d}{2} + R = R_{p,min}$, as shown in Fig. 5(b). Again, the case that $R_p = d/2$ is fully covered in Section 4. Therefore, the very special case according to which $d/2 < R_p < d/2 + R$ is not covered in this paper, and is left as an exercise for the interested reader.

6 Electric Dipole

6.1 General

The topic of electric dipoles is very interesting in electrostatics and electrodynamics. The energetics of a magnetic dipole of moment **M** in an external field **B**, [10], as well as the 'hidden momentum forces', [11], and numerical methods such as the multipole expansion method, [12], have been a matter of intensive research to better understand the behavior of dipoles. The mathematical complexity and the involved simplifications sometimes is a reason for debate, [13], particularly in electrodynamics. Differences between electrostatics and magnetostatics regarding the force have been commented, [14], while in the past the magnetic force has been discussed in detail, [15]. Not only theoretical, [16], but also practical topics have appeared in literature, [17], among many others. For details the reader is referred to standard textbooks such as [18] and [19].

Two equal charges, q, of opposite sign, separated by a distance $d = 2l$, constitute an electric dipole. Therefore, the electric dipole is the special case of the previous sections simply setting ($q_1 = -q_2 = q$). The *electric moment* of a dipole $\vec{p}$ is defined to have the magnitude $p = qd$ and points from the negative charge to the positive charge.

A point P is preferably specified by the coordinates r_m and θ_m illustrated in Fig. 1. As previously, the electric potential at P will be given by the exact relationship:

$$V = V_1 + V_2 = k_e \left[\frac{q}{r_1} - \frac{q}{r_2} \right] \tag{30a}$$

Assuming that $r \gg d$, we can write $r_2 - r_1 \cong d\,cos\theta$ and $r_1 r_2 = r_m^2$, so Eq. (30a) changes into the approximation:

$$V \cong k_e q \frac{d \cos\theta_m}{r_m^2} = k_e \frac{p \cos\theta_m}{r_m^2} \tag{30b}$$

Then, based on Eq. (5) [$\vec{\boldsymbol{E}} = -\operatorname{grad} V$], we can determine the electric intensity, which is presented later in Eq. (35). Alternatively, the electric intensity may be produced by synthesizing the two vectors, $\vec{E}_1$ and $\vec{E}_2$, due to the separated actions of the two opposite charges.

The condition $q_1 = -q_2 = q$ induces some simplifications in the previous formulas which are outlined below.

Total interaction energy:

$$(U_{12})_{Dipole} = \frac{1}{2}\varepsilon_0 \int_{All-Volume} \left(2\vec{E}_1 \cdot \vec{E}_2\right) \mathrm{d}\upsilon = -k_e \frac{q^2}{d} \tag{31}$$

Total self-energy:

$$(U_1 + U_2)_{Dipole} = k_e \left(\frac{q_1^2}{2R} + \frac{q_2^2}{2R} \right) = k_e \frac{q^2}{R} \tag{32}$$

Total Field energy:

$$(U_1 + U_2 + U_{12})_{Dipole} = k_e q^2 \left(\frac{1}{R} - \frac{1}{d} \right) \tag{33}$$

In addition to Eq. (8) which creates no doubt, Eq. (33) also shows that in case of charged spheres (of radii $R_1 = R_2 = R$), the total field energy is strictly positive. Actually, the condition of maximum allowable radius equal to $R_{max} = d/2$ (the *equality* holds for charged spheres *in contact*) gives the inequality $R \leq d/2$ (no penetration of charges), which algebraically leads to $1/R \geq 2/d > 1/d$, which implies $(1/R - 1/d) > 0$, which proves that the total field energy of the dipole is positive.

Note: When the two equal spherical charges are in contact, we have $R = d/2$ and consequently $U_1 = U_2 = -U_{12} = +k_e q^2/d$, thus Eq. (33) becomes:

Total Field energy for spherical charges *in contact*:

$$\left(U_1+U_2+U_{12}\right)_{\substack{Total\ Field\\ Dipole}} = k_e q^2\left(\frac{2}{d}-\frac{1}{d}\right) = \frac{k_e q^2}{d} = -\left(U_{12}\right)_{Dipole} \tag{34}$$

6.2 The semi-analytical approximate model

Let us suppose that we are interested in determining the radius R_p of a sphere centered at the middle O of a dipole of length d, with $R_p > d/2$, in the exterior of which the total field energy equals to a certain given value. This task could be solved analytically in a single step using Eq. (26) and Eq. (27), but unfortunately we were not able to derive a closed-form analytical expression for the portion of the interaction energy $(U_{12})_{r>R_p}$ given by Eq. (27). Below we present a semi-analytical way, where the first analytical guess can be iteratively improved easily in about five to six steps.It is worthy to mention that when established textbooks refer to electric dipoles, they usually present only far-field analytical expressions in terms of the dipole moment $\vec{\boldsymbol{p}}$ (with measure $p = |\vec{\boldsymbol{p}}| = qd$) and the dipole's length $d = 2l$ (see, [18-20]). In more detail, at distant points the magnitude of the vector sum $\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2$ is written in terms of (r_m, θ_m) as follows (see, [18], p. 158):

$$E = \left|\vec{E}\right| \cong \frac{k_e p}{r_m^3}\left(1+3\cos^2\theta_m\right)^{1/2}, \quad r_m \gg d, \tag{35}$$

where r_m is the distance with respect to the middle of the dipole, θ_m is the polar angle formed between the dipole and the position vector as shown in Fig. 1b.

One may observe that in Eq. (35) there is a fictitious singularity at the middle O of the dipole ($r_m = 0$), which of course is not true because these formulas are valid only when $r_m \gg d$ (thus not applicable at the point O). While the field $\vec{\boldsymbol{E}}$ has been extensively presented in many manuscripts and textbooks such as, [18-20], the same is not the case for the *field energy* of a dipole.

Under the condition $r_m \gg d$, the associated field energy in the infinite space (for $r \geq R_p$) will be produced by substituting Eq. (35) into Eq. (6) thus receiving the following *approximate* (because it is valid for $r_m \gg d$, practically when $r_m > 10d$) analytical formula:

$$\begin{aligned} U_{\substack{approx\\(r_m>R_p)}} &\approx \frac{1}{8\pi k_e}\iiint_V \left[\frac{k_0 p}{r_m^3}\left(1+3\cos^2\theta_m\right)^{1/2}\right]^2 (r_m^2 \sin\theta_m d\theta_m d\phi dr_m) \\ &= \frac{k_e}{8\pi} p^2 \left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi}\left(1+3\cos^2\theta_m\right)\sin\theta_m d\theta_m\right)\left(\int_{R_p}^{\infty}\frac{dr_m}{r_m^4}\right) \\ &= \frac{k_e}{8\pi} p^2 \times 2\pi \times 4 \times \left[-\frac{1}{3r_m^3}\right]_{R_p}^{\infty} \\ &= \frac{k_e p^2}{3R_p^3} = \frac{k_e (q\cdot d)^2}{3R_p^3} = \frac{k_e q^2 d^2}{3R_p^3}. \end{aligned} \tag{36}$$

In Eq. (36), the quantity R_p represents the radius of a certain sphere centered at the middle point O of the dipole (see, Fig. 6) which has to be subtracted from the integral, otherwise it would be singular. Obviously, the smaller the radius R_p the greater the field energy U_{approx}. It should become clear that the field energy U_{approx} is influenced by all the three intensity components, i.e., $\vec{\boldsymbol{E}}_1$, $\vec{\boldsymbol{E}}_2$, and $\vec{2}\left(\vec{\boldsymbol{E}}_1 \cdot \vec{\boldsymbol{E}}_2\right)$ which contribute to the two self-energies plus the interaction energy. Equation (36) includes all the three terms, i.e., (U_1, U_2, and U_{12}) but not the accurate values which were earlier found (see, Eqs. (26) to (29)). In other words, it is a rough approximation when the radius R_p is close to the dipole whereas it is very accurate for large values of R_p. In any case it is sufficient to give us a first rapid estimation of R_p when we wish to include a certain amount of field energy outside it ($r > R_p$).

As an example, if we temporarily assume that the total field energy U_{approx} of the dipole outside the sphere (O, R_p) (given by Eq. (36)) equals to the absolute value ($|U| = |U_{12}|$) of the potential energy (given by Eq. (2)), i.e., $U = -k_e q^2/d$, then we will have:

$$\left|U\right| = k_e q^2/d = U_{approx} = k_e q^2 d^2/(3R_{p1}^3), \tag{37a}$$

whence a first-order estimation R_{p1} of the radius R_p will be (see, Fig. 6):

$$R_{p1} = \frac{d}{\sqrt[3]{3}} \cong 0.6934\, d\,. \tag{37b}$$

Therefore, if Eq. (35) were correct everywhere in the space, then the field energy trapped between the surface of the sphere of radius R_{p1} given by (37b) and the infinity would be equal to the potential energy of the electric dipole with charges separated at distance d. In other words, in this particular example the energy *inside* the sphere of radius R_{p1} (if were correct) is in *excess* of the potential energy $(U_1 + U_2 + U_{12} - |U_{12}|)$.

Moreover, in order to draw definite conclusions for the accurate boundary of the sphere (O, R_p) the interior of which includes the sum $(U_1 + U_2 + U_{12} - |U_{12}|)$ and its exterior includes $|U_{12}|$, we have to replace the approximate expression (37b), R_{p1}, by the accurate one, R_{p2}. To this purpose, we perform the following iterative scheme:

Step-1: Start with the initial guess R_{p1} of Eq. (37b) and then calculate the accurate field energies using Eqs. (26) to (29). Then the initial pair of values (R_{p1}, U_{total1}) is known.

Step-2: Increase the initial guess by (say) 10 percent $(\tilde{R}_{p2} = 1.1R_{p1})$ and repeat Step-1 but now for this new value $\tilde{R}_{p2}$. Now, the updated pair of values $(\tilde{R}_{p2}, U_{total2})$ is known.

Step-3: Perform linear interpolation between the pairs (R_{p1}, U_{total1}) and $(\tilde{R}_{p2}, U_{total2})$ to obtain the desired value, here it is $\bar{U} = |U_{12}|$, using the simple formula: $R_{p3} = R_{p1} + (\tilde{R}_{p2} - R_{p1})(\bar{U} - U_{total1})/(U_{total2} - U_{total1})$. Based on this new value R_{p3}, repeat Step-1 and then calculate the accurate field energies using Eqs. (26) to (29).

Step-4: Repeat for a small number of iterations until the result for the updated value $R_{p,final}$ remains unchanged.

In this specific case in which the prescribed value of the field energy outside the sphere (O, R_p) was chosen as $\bar{U} = |U_{12}|$, convergence was achieved in only six steps, and the final result was found to be:

$$R_{p2} \cong 0.7412\, d \tag{37c}$$

The fact that the accurate radius R_{p2} (thick solid line) is greater than the initial guess (dashed line) is clearly shown in Fig. 6.

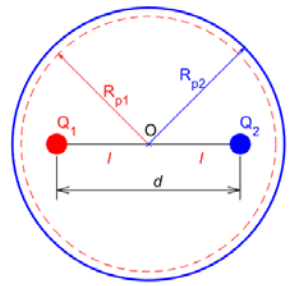

Fig. 6: The radius R_p (approximate R_{p1} and accurate R_{p2}) surrounding the dipole.

To make the results of this example neutral, we first notice that the potential energy $U = -k_e q^2/d$ is expressed as a unit multiple of the standard ratio $\mu = k_e q^2/d$, which could be also equivalently written in terms of the dipole moment p as $\mu = k_e p^2/d^3$. Therefore, instead of numbers it is more convenient to present the results as factors of the aforementioned ratio μ.

In this context, the initial guess given by Eq. (37b) corresponds to $U_{total,1} = 1.3132\,\mu$, while the next iterations are shown in Table 1. One may observe that while the improvement of the radius from the initial guess $R_{p1} = 0.6934\,d$ to the final value $R_{p2} = 0.7412\,d$ was only +6.9%, the decrease of the initially erroneous total field energy is about 31%. In other words, for this particular case the semi-analytical approximate formula which is given by Eq. (36) *overestimates* the true total energy by 31 percent. In general, five to six iterations are sufficient to give us the correct total field energy which is represented by the unity factor, i.e., in this example we have $U_{total}^{exact} = 1.000 \times \mu = k_e q^2/d = U$. Interestingly, the difference between 6.9% and 31% reflects the inaccuracy of the semi-analytical formula regarding the total field energy.

Table 1. Iterations for the derivation of the exact total field energy as a factor of $\mu = k_e q^2/d$

Iteration	Factor of μ
1 (initial guess)	1.3132
2	0.8959
3	0.9782
4	1.0020
5	1.0000
6	1.0000

6.3 The accuracy of the semi-analytical model

Section 6.1 was useful for the reader to understand that when the distance from the middle of the dipole is small the inaccuracy of the semi-analytical formula, given by Eq. (35), is substantial. A bulk of papers note that the critical point where Eq. (35) is accurate is about $r > 10d$ and the overall feeling is that the field energy vanishes for greater values. In this section we complete the established knowledge by drawing definite conclusions.

Having developed in the previous sections all those analytical formulas that can be easily applied either near or far away from the electric dipole, below we shall give some details concerning the comparison of the accurate field and the approximate semi-analytical model. For the sake of completeness, first the comparison is trivial and is concerned with the electric field itself, a task which can be easily performed by any student. Fig. 7 shows the comparison of the magnitude of electric intensity $E = |\vec{E}|$, where the results are normalized accordingly (are multiplied by $(d/2/p^2/q^2)$), in two following typical directions:

(i) Along the perpendicular bisector, where we have $\theta_m = \pi/2$ thus [according to Eq. (35)] $E \cong k_e p/r_m^3$ (Fig. 7a), and

(ii) Along the dipole axis, where we have $\theta_m = 0$ or π thus [according to Eq. (35)] $E \cong 2\, k_e p/r_m^3$ (Fig. 7b).

In a first glance, one may observe that, along the perpendicular bisector the difference between the exact and the semi-analytical form becomes small after a distance r_m of about d, while along the dipole axis we need a distance from the midpoint O at least $1.5d$. The latter is due to the singularity of the closest charge to the observer, which (singularity) is at a distance $d/2$ from the midpoint O.

(a)

(b)

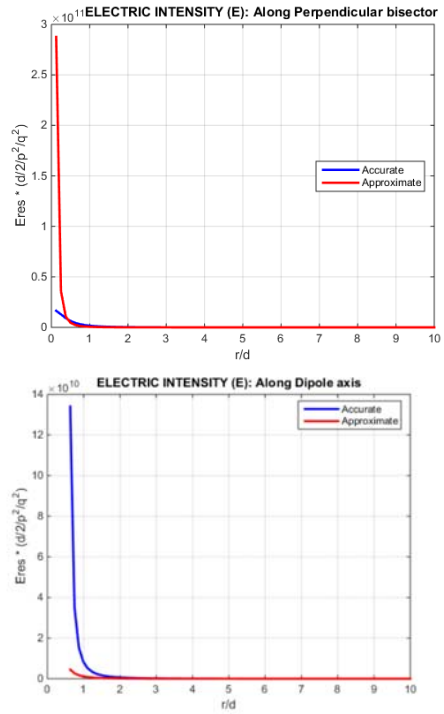

Fig. 7: Electric intensity E: (a) perpendicular bisector, (b) dipole axis.

In a second glance, a better impression is obtained when studying the ratio of the exact intensity over the semi-analytical one, as shown in Fig. 8, where the difference becomes small for $r_m > 8$ and $r_m > 4$, respectively.

The differences become a little larger when comparing the field energy densities in Fig. 9, where it is illustrated that the usually mentioned threshold of $r_m > 10d$ is applicable only along the dipole axis where it is rather better to consider $r_m > 12d$ along the perpendicular bisector.

(a)

(b)

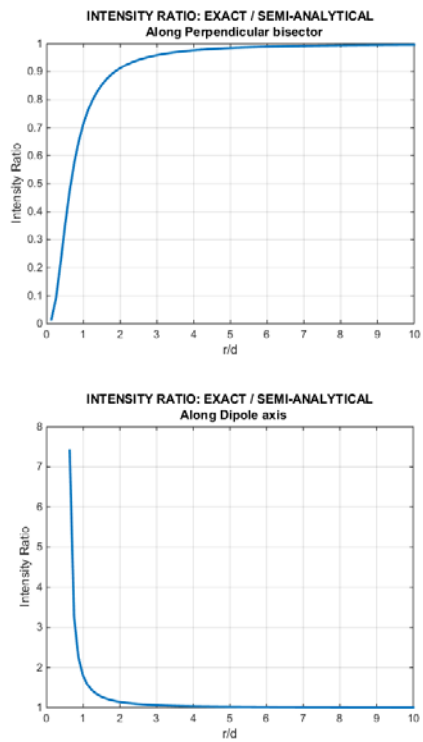

Fig. 8: Ratio of electric intensity ($E_{exact}/E_{semi-analytical}$): (a) perpendicular bisector, (b) dipole axis.

(a)

(b)

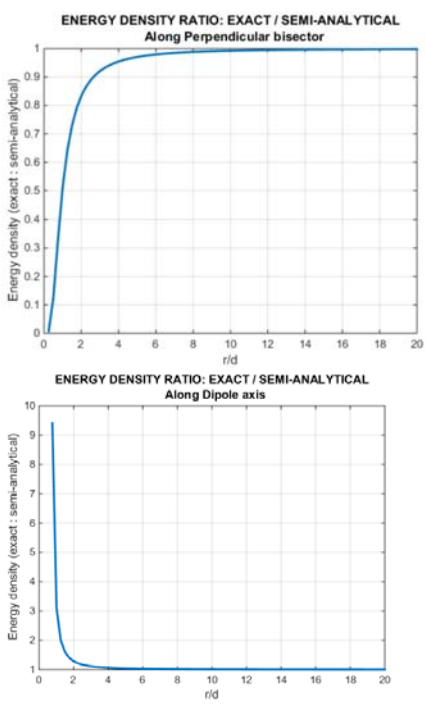

Fig. 9: Ratio of energy densities: (a) perpendicular bisector, (b) dipole axis.

(a)

(b)

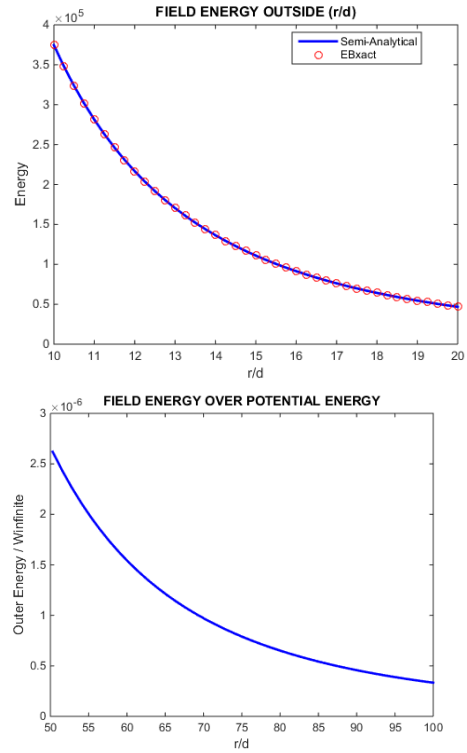

Fig. 10: Total field energy outside a sphere of radius r, (a) as factor of $2q^2M^2/q$, (b) as the ratio of actual energy over the absolute potential energy.

Now, it is interesting to see how the total field energy reduces outside the sphere (O, r) with increasing radius $r > 10d$. Actually, Fig. 10 shows that the total field energy outside a sphere of radius $R_p > 10d$ is accurately described by the semi-analytical formula (35). Also, Fig. 10(b) shows that although the total field energy is small when $r_m > 10d$, it does not entirely vanish even at $r_m/r = 100$. In more detail, if we ask to determine the critical radius $R_{p,99}$ which includes the 99% of the total field energy density, the answer depends on the chosen radius $R\ (= R_1 = R_2)$ of the charged spheres.

6.4 Numerical results for a dipole

Here we present some results regarding uniformly charged spheres separated by distance d. Considering that the infinite space is extended until $r_m = 10d$, the results of the field energy density are shown in Fig. 11, for two cases, i.e., $R_1 = R_2 = 0.125\ d$ and also $R_1 = R_2 = 0.250\ d$. To avoid any doubt, the energy density was calculated

synthesizing the two vectors E_+ and E_- due to the corresponding charges $+q$ and $-$q, respectively.

To test the analytical formulas which were presented in Section 4, first we present Table 2 which is related to the dashed line sphere $(O, d/2)$.

One may observe that as the radius of each charged sphere increases the amount of interactive energy inside the sphere $(O, d/2)$ increases as well, and the same occurs with the absolute value of the interactive energy outside it. Note that contact between the two spherical charges occurs when $R_1 = R_2 = 0.5\,d$, which corresponds to the last column of Table 2.

(a)

(b)

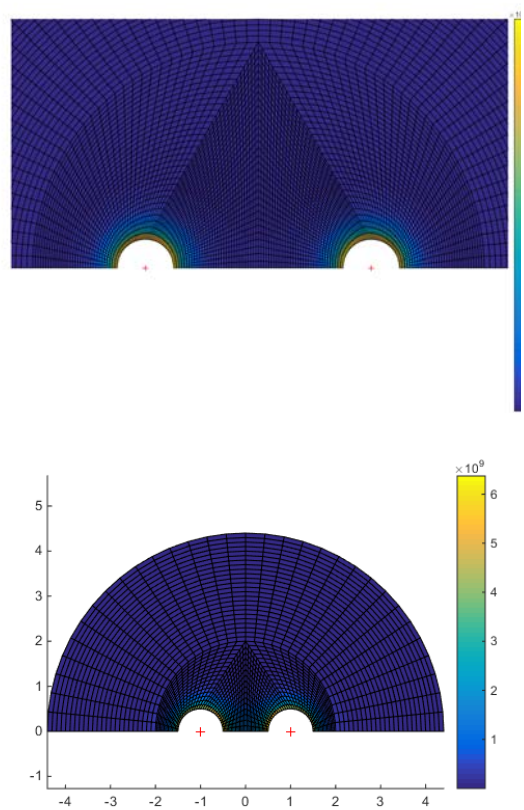

Fig. 11: Energy density for charged spheres of radius (a) $R_1 = R_2 = 0.125\,d$ (detail) and (b) $R_1 = R_2 = 0.250\,d$ (full mesh).

When spheres of charge, $q_1 = q$ and $q_2 = -q$, and each of radius R are considered, the minimum value of the radius R_p for which Eqs. (26) to (29) are applicable is $(R_p)_{min} = d/2 + R$. Based on the latter constructive element of the dipole, and setting $\boldsymbol{R = d/n}$ where n is a positive integer, after substitution in the formula $U = U_1 + U_2 + U_{12} = k_e q^2/(2R) + k_e q^2/(2R) - k_e q^2/d$, we obtain the following Eq. (38):

$$U = (n-1)\frac{k_e q^2}{d}. \tag{38}$$

For $n = 2, 4,$ and 8, the results are given in Table 3, Table 4 and Table 5, respectively. In all these three cases, the results are normalized to the absolute value of the interaction energy, $\frac{k_e q^2}{d} > 0$.

Note: Not only the numerical results in Table 2 to Table 5 are reasonable and show a monotonic behavior but also a part of them was validated by performing numerical integration (3×3 Gauss quadrature) on computational meshes such as that shown in Fig. 11. Increasing the mesh density the numerical results were found to coincide with the analytical ones.

Table 2. Breakdown of Field Energies for point-like and spherical charges inside and outside the characteristic *dashed line sphere* $(O, d/2)$ in the form of factors of the value $\frac{k_e q^2}{d} > 0$

	POINT-LIKE	**CHARGED SPHERES of radius $R_1 = R_2$**				
		$\mathbf{0.1\,d}$	$\mathbf{0.2\,d}$	$\mathbf{0.3\,d}$	$\mathbf{0.4\,d}$	$\mathbf{0.5\,d}$
$\boldsymbol{U_{12}}$: Inside the sphere $(O, d/2)$	0.2854	0.3855	0.4861	0.5877	0.6910	0.7967
$\boldsymbol{U_{12}}$: Outside the sphere $(O, d/2)$	–1.2854	–1.3855	–1.4861	–1.5877	–1.6910	–1.7967
$\boldsymbol{U_1}$: Inside the sphere $(O, d/2)$	∞	1.6744	0.5976	0.2823	0.1459	0.0767
$\boldsymbol{U_1}$: Outside the sphere $(O, d/2)$	∞	3.3256	1.9024	1.3843	1.1041	0.9233
$\boldsymbol{U_2}$: Inside the sphere $(O, d/2)$	∞	1.6744	0.5976	0.2823	0.1459	0.0767
$\boldsymbol{U_2}$: Outside the sphere $(O, d/2)$	∞	3.3256	1.9024	1.3843	1.1041	0.9233
SUM: $(\boldsymbol{U_1} + \boldsymbol{U_2} + \boldsymbol{U_{12}})$	∞	9.0000	4.0000	2.3333	1.5000	1.0000

Table 3. Breakdown of Field Energies for point-like and spherical charges *in contact* ($n = 2$, i.e., $R_1 = R_2 = d/2$) (as factors of the value $k_e q^2/d > 0$)

	Fully Including the Dipole SPHERE of Radius (R_p)					
	$\boldsymbol{d}$	$\mathbf{2\,d}$	$\mathbf{5\,d}$	$\mathbf{10\,d}$	$\mathbf{20\,d}$	$\mathbf{100\,d}$
$\boldsymbol{U_{12}}$: Inside the sphere (O, R_p)	-0.1364	-0.5197	-0.8013	-0.9002	-0.9500	-0.9900
$\boldsymbol{U_{12}}$: Outside the sphere (O, R_p)	-0.8636	-0.4803	-0.1987	-0.0998	-0.0500	-0.0100
$\boldsymbol{U_1}$: Inside the sphere (O, R_p)	0.3920	0.7390	0.8993	0.9499	0.9750	0.9950
$\boldsymbol{U_1}$: Outside the sphere (O, R_p)	0.6080	0.2610	0.1007	0.0501	0.0250	0.0050
$\boldsymbol{U_2}$: Inside the sphere (O, R_p)	0.3920	0.7390	0.8993	0.9499	0.9750	0.9950
$\boldsymbol{U_2}$: Outside the sphere (O, R_p)	0.6080	0.2610	0.1007	0.0501	0.0250	0.0050
SUM: $(\boldsymbol{U_1} + \boldsymbol{U_2} + \boldsymbol{U_{12}})$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 4. Breakdown of Field Energies for point-like and spherical charges with $n = 4$, i.e., $R_1 = R_2 = d/4$ (as factors of the value $\frac{k_e q^2}{d} > 0$)

	Fully Including the Dipole SPHERE of Radius (R_p)					
	$\boldsymbol{3d/4}$	$\boldsymbol{d}$	$\boldsymbol{2\,d}$	$\boldsymbol{5\,d}$	$\boldsymbol{10\,d}$	$\boldsymbol{100\,d}$
$\boldsymbol{U_{12}}$: Inside the sphere (O, R_p)	0.0495	-0.1364	-0.5197	-0.8013	-0.9002	-0.9900
$\boldsymbol{U_{12}}$: Outside the sphere (O, R_p)	-1.0495	-0.8636	-0.4803	-0.1987	-0.0998	-0.0100
$\boldsymbol{U_1}$: Inside the sphere (O, R_p)	0.9976	1.3920	1.7390	1.8993	1.9499	1.9950
$\boldsymbol{U_1}$: Outside the sphere (O, R_p)	1.0024	0.6080	0.2610	0.1007	0.0501	0.0050
$\boldsymbol{U_2}$: Inside the sphere (O, R_p)	0.9976	1.3920	1.7390	1.8993	1.9499	1.9950
$\boldsymbol{U_2}$: Outside the sphere (O, R_p)	1.0024	0.6080	0.2610	0.1007	0.0501	0.0050
SUM: $(\boldsymbol{U_1 + U_2 + U_{12}})$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000

Table 5. Breakdown of Field Energies for point-like and spherical charges with $n = 8$, i.e., $R_1 = R_2 = d/8$ (as factors of the value $\frac{k_e q^2}{d} > 0$)

	Fully Including the Dipole SPHERE of Radius (R_p)					
	$\boldsymbol{5d/8}$	$\boldsymbol{d}$	$\boldsymbol{2\,d}$	$\boldsymbol{5\,d}$	$\boldsymbol{10\,d}$	$\boldsymbol{100\,d}$
$\boldsymbol{U_{12}}$: Inside the sphere (O, R_p)	0.1625	-0.1364	-0.5197	-0.8013	-0.9002	-0.9900
$\boldsymbol{U_{12}}$: Outside the sphere (O, R_p)	-1.1625	-0.8636	-0.4803	-0.1987	-0.0998	-0.0100
$\boldsymbol{U_1}$: Inside the sphere (O, R_p)	2.3396	3.3920	3.7390	3.8993	3.9499	3.9950
$\boldsymbol{U_1}$: Outside the sphere (O, R_p)	1.6604	0.6080	0.2610	0.1007	0.0501	0.0050
$\boldsymbol{U_2}$: Inside the sphere (O, R_p)	2.3396	3.3920	3.7390	3.8993	3.9499	3.9950
$\boldsymbol{U_2}$: Outside the sphere (O, R_p)	1.6604	0.6080	0.2610	0.1007	0.0501	0.0050
SUM: $(\boldsymbol{U_1 + U_2 + U_{12}})$	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000

7 Electric Charges in the Form of Conducting Charged Spheres

In the previous Sections, we have considered that the charged spheres of radii (R_1, and R_2) are *not* conductors. This means that in each of them the electric charge is uniformly distributed either they are found in the infinity or they are close one another. In such a case, that is for non-conductive charged spheres of radius R in contact ($d = 2R$), with total charge Q (each of them carries $q_1 = q_2 = Q/2$), if the charge were uniformly distributed, the "interaction energy" would take the usual value ($U = k_e q_1 q_2/d$, with $d = 2R$):

$$U = k_e \frac{(Q/2)(Q/2)}{2R} = k_e \frac{Q^2}{8R} \tag{39}$$

A slightly different problem concerns two *conducting* spheres, of charge q_1 and q_2. Here, the charge distribution is not uniform, and the "interaction energy" is *not* $q_1 q_2/d$. This can be computed via the so-called "method of images". A famous special case is considered in prob. 9 of Reference [21].

For the particular case that the conducting spheres of radius R are *in contact* (i.e., are separated by distance $d = 2R$), it can be shown that their capacitance is $C = 2R\, ln2$, and the "interaction energy" is

$$U' = k_e \frac{Q^2}{2C} = k_e \frac{Q^2}{2(2R\ln 2)} = \left(k_e \frac{Q^2}{8R}\right)\cdot\left(\frac{2}{\ln 2}\right) \tag{40}$$

Comparing (40) with (39), one may observe that in the case of the conductive spheres, the "interaction energy" is larger by a factor of $(2/ln\ 2) = 2.89$. In other words, the conducting capacitor is a better energy storage device than two separate non-conducting spheres.

8 Discussion

This study refers to the general case of two electric charges, q_1 and q_2 separated by distance d, either point-like or of finite size (in the form of charged spheres of radii R_1 and R_2, respectively). It was found that while the "interaction energy" is the same for finite or point-like spheres, the total field energy is quite different.

Of particular interest is the case of two spheres of equal radius R that *just touch* each other (spheres *in contact*), such that $d = 2R$. Then:

- For $q_1 = q_2 = q$, the interaction energy is $U_{12} = k_e q^2/2R$, and $U_f = U_{total} = k_e 3q^2/2R$,
- While for $q_1 = -q_2 = q$ (electric dipole), the interaction energy is $U_{12} = -k_e q^2/2R$, and $U_f = U_{total} = \frac{k_e q^2}{2R} = -U_{12}$.

But, clearly, the formulas that were presented in the previous sections cover all the combinations of charges and radii, for either the "interaction energy" or the self-energies.

Regarding the self-energies, the radii (R_1, R_2) of the spherical charges highly influence the field. For given electric charges (q_1, q_2), the smaller the radii the larger the self-energy. If the radii are given then there is no problem to estimate the self-energy thus the total field energy, otherwise we have to guess them.

Within this context, we recall that for a charged sphere of radius R_i, the uniform surface charge density is $\sigma_i = q_i/(4\pi R_i^2)$. Therefore, a quite theoretical lower limit for the radius R_i is probably Plack length ($1.61625502 \times 10^{-35}$ meters), and another more realistic lower bound comes from an upper bound set for the surface charge density σ_i, i.e., $R_{min} = \sqrt{\frac{q_i}{4\pi(\sigma_i)_{max}}}$. For example, one could consider a reasonable radius based on the fact that the maximum known charge density is probably the value 1003 μCm^{-2}, which is close to the limit of dielectric breakdown [22].

The interested reader could extend the same methodology but now considering a smaller sphere (O, R_p) than the dashed line sphere $(O, d/2)$, i.e., now with $R_p < d/2$. Then, he/she has to consider two cases, the former when this sphere intersects the charged spheres and the latter when it does not.

9 Conclusions

It was shown that the field energy due to two electric charges can be described in full detail using closed-form analytical expressions. This can be done either for point-like charges or for charged spheres of given radii. In the latter case, the charged spheres may be either in contact (extreme case) or not. In all these cases, a critical area of the infinite space is the sphere of which the diameter is the line segment that connects the centers of the two charges. It was found that the biggest part of the "interaction energy" is found in the exterior of the aforementioned sphere, while the algebraic sum (inside and outside it) is finite and always equals to the potential energy (that is, by definition, the work to bring one charge at a certain distance from the other). In the exterior space of a big sphere that surrounds the charges, the total energy does not depend on whether the electric charges are point-like or not. In contrast, the total energy in the interior of this big sphere highly depends on their type. Clearly, point-like charges are characterized by infinite self-energy whereas charged spheres are related to finite self-energies. In the case of an electric dipole of any type (point-like charges or charged spheres), the total field energy in the exterior of a sphere with diameter about one-and-a-half times the separation distance equals to the absolute value of the negative "interaction energy".

Acknowledgement:
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Appendix A
The interactive field energy

Theorem-1: Consider two electric point-like charges, q_1 and q_2, separated by distance d, and then a sphere of diameter d centered at the middle O of the distance d. With regards to the interaction field energy U_{12} of these point charges, we have:

(i) Inside the sphere, the field energy $(U_{12})_{Inside}$ has the *opposite* sign of the product $q_1 q_2$.

(ii) Outside the sphere (in the infinite space), the field energy $(U_{12})_{Outside}$ has the *same* sign of the product $q_1 q_2$.

(iii) On the surface of the sphere ($r = d/2$) the dot-product $(\vec{E}_1 \vec{E}_2)$ equals to zero, thus it is a transition state between the abovementioned (plus to minus) values of the interaction energy.

(iv) The biggest amount of the interaction field energy is outside the sphere $(O, d/2)$.

(v) The overall interaction energy, $(U_{12})_{All-space} = (U_{12})_{Inside} + (U_{12})_{Outside}$, in the infinite three-dimensional space is *bounded* and equals to the potential energy, which is given by $U = k_e q_1 q_2 / d$, thus is has the *same* sign of the product $q_1 q_2$.

Proof

Let us consider the point-like electric charges, q_1 at point Q_1 and q_2 at point Q_2, (of arbitrary values) separated by distance d. For the sake of simplicity, it is convenient to assume that both of them are positive ($q_1 > 0$ and $q_2 > 0$); however the conclusions are of more general applicability. On an axial (meridian) plane passing through the line segment $Q_1 Q_2$, and centered at the midpoint O of this segment, we consider a circle of diameter d thus passing through the two charges, as shown by the dotted line in Fig. A-1. We also consider an arbitrary point P on the same axial section and a positive unit charge at it. The electric intensity is produced by the resultant of two vectors, the former $(\vec{E}_1)$ connecting P with Q_1 and the latter $(\vec{E}_2)$ connecting P with Q_2, as shown in Fig. A-1.

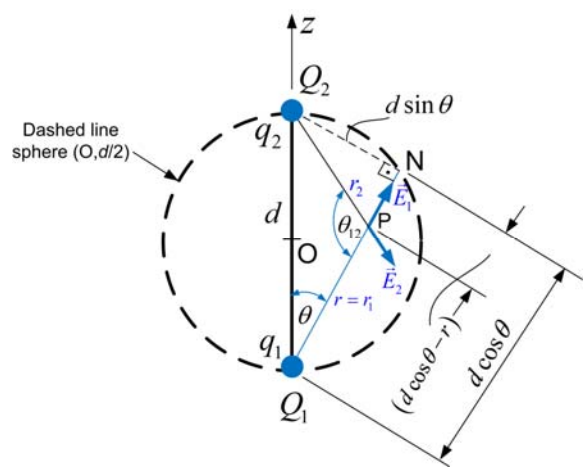

Fig. A-1: The hypothetical dashed line sphere $(O, d/2)$.

The interactive field energy is given by the third term in Eq. (8), thus it is:

$$\left(U_{12}\right)_{All-space} = \int_{Volume} \frac{\vec{E}_1 \cdot \vec{E}_2}{4\pi k_e} \mathrm{d}\upsilon \tag{A.1}$$

In order to calculate the volume integral of Eq. (A.1), it is convenient to use spherical coordinates, i.e., the azimuthal (longitudinal) φ, the latitude θ and the radius r measured from the point Q_1 at which the charge q_1 lies. Therefore, all the other geometrical quantities, such as distances and angles, can be calculated in terms of the aforementioned spherical coordinates (φ, θ, r). Obviously, by definition we have $r = r_1$, while the distance of P from q_2 is a secondary variable denoted by r_2.

By definition, the dot product $\left(\vec{E}_1 \cdot \vec{E}_2\right)$ in (A.1) is written as follows:

$$\vec{E}_1 \cdot \vec{E}_2 = \left|\vec{E}_1\right| \cdot \left|\vec{E}_2\right| \cdot \cos\theta_{12}, \tag{A.2}$$

where θ_{12} is the angle formed by the vectors $\vec{E}_1$ and $\vec{E}_2$.

From elementary trigonometry applied to the triangle (PNQ_2), with right angle at point N (see in Fig. A-1), we have:

$$\cos\theta_{12} = \frac{PN}{PQ_2} = \frac{r - d\cos\theta}{r_2}. \tag{A.3}$$

We distinguish two main parts, the former inside the dashed line sphere and the latter in all space.

In our proof, we start with last point (v) of the Theorem.

I. All-space interaction energy

First we proof the identity for the total space, where the total "interaction energy" is written as:

$$\left(U_{12}\right)_{All-space} = \frac{1}{4\pi k_e} \int_{\phi=0}^{2\pi} \mathrm{d}\phi \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \underbrace{\left(\vec{E}_1 \cdot \vec{E}_2\right)}_{F(r,\theta)} \sin\theta \mathrm{d}\theta r^2 dr \tag{A.4}$$

Substituting (A.2) and (A.3) in (A.4), and further considering that $\left|\vec{E}_1\right| = k_e q_1 / r_1^2$ (with $r_1 = r$) and $\left|\vec{E}_2\right| = k_e q_2 / r_2^2$, which both come from Eq. (4) of the main text, we obtain:

$$\begin{aligned}\left(U_{12}\right)_{All-space} &= \frac{1}{4\pi k_e}\left(\int_{\phi=0}^{2\pi} \mathrm{d}\phi\right) \\ &\cdot \left\{\int_{\theta=0}^{\pi}\left[\int_{r=0}^{\infty}\left(k_e \frac{q_1}{r^2}\right)\left(k_e \frac{q_2}{r_2^2}\right)\left(\frac{r - d\cos\theta}{r_2}\right) r^2 dr\right] \sin\theta \mathrm{d}\theta\right\}\end{aligned} \tag{A.5}$$

After simplifications, including the obvious substitution $\int_{\phi=0}^{2\pi} \mathrm{d}\phi = 2\pi$, (A.5) simplifies to:

$$\left(U_{12}\right)_{All-space} = \frac{k_e q_1 q_2}{2} \int_{\theta=0}^{\pi} \left[\int_{r=0}^{\infty} \frac{\left(r - d\cos\theta\right)}{r_2^3} dr\right] \sin\theta d\theta \tag{A.6}$$

Then applying the cosine-law in the triangle (Q_1PQ_2), focusing on the side $PQ_2 = r_2$, we receive:

$$r_2 = \left(d^2 + r^2 - 2d\, r\cos\theta\right)^{\frac{1}{2}} \tag{A.7}$$

Substituting (A.7) into (A.6), we receive:

$$\begin{aligned}\left(U_{12}\right)_{All-space} &= \frac{k_e q_1 q_2}{2} \\ &\cdot \int_{\theta=0}^{\pi}\left[\int_{r=0}^{\infty} \frac{\left(r - d\cos\theta\right)}{\left(d^2 + r^2 - 2d\, r\cos\theta\right)^{\frac{3}{2}}} dr\right] \sin\theta d\theta\end{aligned} \tag{A.8}$$

After elaboration, one can easily validate that the indefinite integral in r, of (A.8), is:

$$\begin{aligned}&\int \frac{\left(r - d\cos\theta\right)}{\left(d^2 + r^2 - 2d\, r\cos\theta\right)^{\frac{3}{2}}} dr \\ &= -\frac{1}{(d^2 + r^2 - 2d\, r\cos\theta)^{\frac{1}{2}}} = -\frac{1}{r_2}\end{aligned} \tag{A.9}$$

and therefore (A.8) becomes:

$$\begin{aligned}\left(U_{12}\right)_{All-space} &= \frac{k_e q_1 q_2}{2} \\ &\cdot \int_{\theta=0}^{\pi}\left[-\frac{1}{(d^2 + r^2 - 2d\, r\cos\theta)^{\frac{1}{2}}}\right]_0^{\infty} \sin\theta d\theta\end{aligned} \tag{A.10}$$

The lower limit ($r = 0$) of the definite integral in (A.10), is clearly the term $-1/d$, which after the subtraction becomes $1/d$. Moreover, for the upper limit ($r \to \infty$), since $-1 \le \cos\theta \le 1$, we have the inequality:

$$-\frac{1}{(r - d)} \le -\frac{1}{(d^2 + r^2 - 2d\, r\cos\theta)^{\frac{1}{2}}} \le -\frac{1}{(r + d)} \tag{A.11}$$

Since when $r \to \infty$ both bounds in (A.11) vanish, it turns out that the upper limit of the bracket in (A.10) vanishes as well, thus (A.10) becomes:

$$\left(U_{12}\right)_{All-space} = \frac{k_e q_1 q_2}{2}\left(\frac{1}{d}\right) \cdot \int_{\theta=0}^{\pi} \sin\theta d\theta, \tag{A.12}$$

Since $\int_{\theta=0}^{\pi} \sin\theta d\theta = 2$, (A.12) eventually gives:

$$(U_{12})_{All-space} = \frac{k_e q_1 q_2}{d}, \qquad \text{Q.E.D.} \tag{A.13}$$

I. Inside the dashed line sphere

The only difference with the above all-space analysis is the choice of the limits that have to be imposed for the integrals in θ (now $0 \le \theta \le \pi/2$) and r (now: $0 < r < d\cos\theta$), thus (A.5) is modified to the following expression:

$$(U_{12})_{Inside} = \frac{1}{4\pi k_e}\left(\int_{\phi=0}^{2\pi} \mathrm{d}\phi\right) \cdot \left\{\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{d\cos\theta}\left(k_e\frac{q_1}{r^2}\right)\left(k_e\frac{q_2}{r_2^2}\right)\left(\frac{r-d\cos\theta}{r_2}\right)r^2\,dr\right]\sin\theta\,\mathrm{d}\theta\right\} \tag{A.14}$$

By virtue of (A.9), now we receive:

$$(U_{12})_{inside} = \frac{k_e q_1 q_2}{2}\int_{\theta=0}^{\pi/2}\left(\frac{1}{d} - \frac{1}{d\sin\theta}\right)\sin\theta\, d\theta, \tag{A.15}$$

and eventually:

$$(U_{12})_{inside} = \frac{k_e q_1 q_2}{d}\frac{(2-\pi)}{4} \approx -0.2854 \times \frac{k_e q_1 q_2}{d} \tag{A.16}$$

Therefore, the part of the interaction field energy inside the sphere $(O, d/2)$, $(U_{12})_{Inside}$, has the *opposite* sign of the product $q_1 q_2$, and this completes the proof of point (i) of the Theorem.

Regarding the outer part, $(U_{12})_{Outside}$, we can simply consider the obvious identity,

$$(U_{12})_{All-space} = (U_{12})_{Inside} + (U_{12})_{Outside}, \tag{A.17}$$

which may be solved in $(U_{12})_{Outside}$:

$$(U_{12})_{Outside} = \frac{k_e q_1 q_2}{d}\left(\frac{2+\pi}{4}\right) \approx 1.2854 \times \frac{k_e q_1 q_2}{d}. \tag{A.18}$$

Therefore, the part of the interaction field energy outside the sphere, $(U_{12})_{Outside}$, has the *same* sign of the product $q_1 q_2$, and this completes the proof of point (ii) of the Theorem. Moreover, since $(U_{12})_{Outside}$ is larger than the $(U_{12})_{All-space}$, while $(U_{12})_{Inside}$ is only 28.5 percent of it (and even with a negative sign), it comes out that the most part of the interaction energy is outside the dashed line sphere, and this completes the proof of point (iv) of the Theorem.

Note: The Theorem of this Appendix-A was highly inspired by Hilborn (see [1], p. 69), but in the present paper there is no gap in the proof, which is of major assistance to the students and teachers. Repeating the criticism by Hilborn (see [1], p. 69) concerned with teaching guidelines in USA, "that result tells us that most of the interaction field energy is found outside the sphere, not "between the objects" as *A Framework* asserts in the quotations cited earlier or as one's (naive) intuition might lead one to believe", we would like to add an important factoid. In brief, we would like to advise teachers that when the electric charges are still at infinities ($-\infty$ and $+\infty$), as they approach each other to form the final structure (charges separated by distance d) in our thought-experiment, the first area in which they interact is the *far-field* first, and this factoid foreshadows that most interactive energy is anticipated to be in the outer space. In any case, the pair of equations (A.16) and (A.18) gives the definite answer and resolves any misunderstanding.

Appendix B
The interaction field energy within a fictitious sphere of radius *R* centered at one point-like charge

Theorem-2: Consider two electric point-like charges, q_1 at point Q_1 and q_2 at point Q_2, separated by distance d, and then a dashed line hypothetical sphere of diameter d centered at the middle O of the distance d. In addition, consider a fictitious second sphere (Q_2, R), i.e. centered at the point Q_2 where the point charge q_2 was concentrated (as was afore mentioned), but now of radius R.

For the point-like charges (q_1, q_2) show that the "interaction energy" $(U_{12})_{r<R}$ inside the sphere (Q_2, R) *vanishes*.

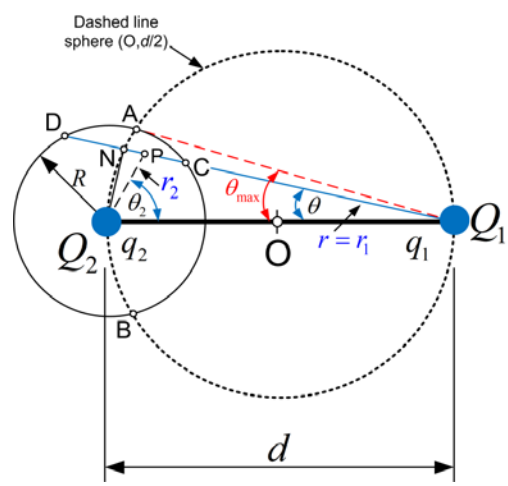

Fig. B-1: Intersection of the two spheres, $(O, d/2)$ and (Q_2, R).

Proof

On a median (axial) plane let the two spheres be intersected at the points A and B. For an arbitrary point P inside the sphere (Q_2, R), considering the axial plane through it, the line segment Q_1P intersects the aforementioned sphere at the points C and D, while the point N is the normal projection of Q_2 on the straight segment CD (since the angle $\widehat{Q_2NQ_1}$ subtends the diameter Q_2Q_1 of the dashed line circle), as shown in Fig. B-1.

In basic lines, we closely follow the procedure of *Appendix A* where axis origin of the spherical coordinates system is again Q_1 (with $r = r_1, \theta = \theta_1$), but now the most useful relationships are:

$$\cos\theta_{12} = \frac{PN}{Q_2P} = \frac{d\cos\theta - r}{r_2} \tag{B.1}$$

$$Q_1N = d\cos\theta \tag{B.2}$$

$$Q_2N = d\sin\theta \tag{B.3}$$

$$PN = Q_1N - PQ_1 = d\cos\theta - r \tag{B.4}$$

In an analogous way with (A.14), the interactive field energy under consideration will be given as:

$$\begin{aligned}(U_{12})_{r<R} &= \frac{k_e q_1 q_2}{4\pi}\left(\int_{\phi=0}^{2\pi} \mathrm{d}\phi\right) \\ &\cdot\left\{\int_{\theta=0}^{\theta_{\max}}\left[\int_{r=r_C}^{r_D}\left(\frac{d\cos\theta - r}{r_2^3}\right)dr\right]\sin\theta\,\mathrm{d}\theta\right\},\end{aligned} \tag{B.5}$$

where $\theta_{max} = \theta_{1,max}$ is the maximum latitudinal angle $(\widehat{Q_2Q_1A})$, and $(r_C = Q_1C, r_D = Q_1D)$ the minimum and maximum values of the variable r (position vector), which using elementary trigonometry are given by:

$$\begin{aligned} r_{\min} &= r_C = Q_1N - NC = d\cos\theta - \sqrt{R^2 - d^2\sin^2\theta} \\ r_{\max} &= r_D = Q_1N + ND = d\cos\theta + \sqrt{R^2 - d^2\sin^2\theta}\end{aligned} \tag{B.6}$$

Since (A.7) is still valid, the integral in r of (B.5) is described again by the opposite of (A.9)), thus is given by:

$$\begin{aligned}&\left[\frac{1}{(d^2 + r^2 - 2d\,r\cos\theta)^{\frac{1}{2}}}\right]_{r=r_R}^{r=r_S} = \left[\frac{1}{r_2}\right]_{r=r_C}^{r=r_D}, \\ &= \frac{1}{r_{2,D}} - \frac{1}{r_{2,C}} = \frac{1}{(Q_2D)} - \frac{1}{(Q_2C)} = \frac{1}{R} - \frac{1}{R} = 0\end{aligned} \tag{B.7}$$

and since $r_{2,D} = r_{2,C} = R$ (see Fig. B-1), it vanishes. Therefore, the entire integral $(U_{12})_{r<R}$ in (B.5) vanishes as well, and this completes the proof of *Theorem-2*.

Appendix C

Interaction field energy within and outside the sphere $(O, d/2)$ for uniformly charged spheres of radii (R_1, R_2)

Find the active part of the interaction field energy $(U_{12})_{Inside(O,\frac{d}{2})}$ within the dashed line sphere $(O, d/2)$, and then outside it, now by excluding the corresponding dead areas of the charges, of radii R_1 and R_2 (see, Fig. C-1).

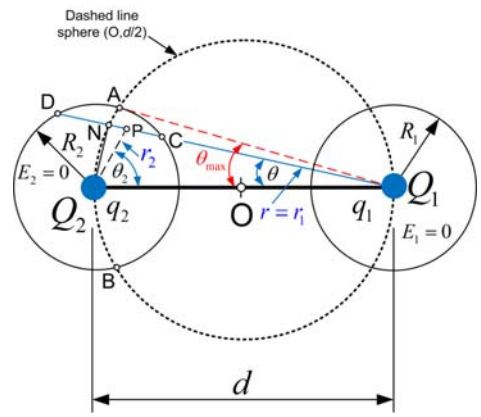

Fig. C-1: Interaction between two charged spheres separated by distance d, within and outside the dashed line sphere $(O, d/2)$.

Solution: Considering the entire dashed line sphere $(O, d/2)$, we have previously derived (in Appendix A) Eq. (A.16), which is valid inside the sphere $(O, d/2)$ for point-like charges. But now that we have electric charges with a radius (R_1 and R_2), we need to remove that part of the spherical charges which is inside the dashed line sphere (because in the volume Q_2BCANQ_2 for q_2, and similar for q_1, the electric intensity vanishes therein). Following the previous Appendices, when considering the charge (of radius R_2) at Q_2 the axis origin is taken at Q_1 as shown in Fig. C-1, the latitudinal angle is $\theta = \theta_1$ and the radius is $r = r_1$. In this case the limits in Eq. (B.7) now become $\left[-\frac{1}{r_2}\right]_{r=r_{2,R}}^{r=r_{2,N}} = -\frac{1}{r_{2,N}} + \frac{1}{r_{2,C}} = \frac{1}{d\sin\theta} - \frac{1}{R_2}$, with $\sin\theta_{1,max} = \sin\theta_{max} = Q_2A/Q_1Q_2 = R_2/d$ (see, Fig. C-1) thus $\cos\theta_{1,max} = \sqrt{1-(R_2/d)^2}$ (note that $\theta_{1,max} = Q_2Q_1A$ in Fig. B-1). Therefore, the final integral in $\theta = \theta_1$ will be:

$$
\begin{aligned}
(U_{12})_{\substack{Inside\\ r<R_2}} &= \frac{k_e q_1 q_2}{4\pi} 2\pi \left\{ \int_{\theta=0}^{\theta_{\max}} \left(-\frac{1}{d\sin\theta} + \frac{1}{R_2} \right) \sin\theta \, \mathrm{d}\theta \right\} \\
&= \frac{k_e q_1 q_2}{2} \left\{ \int_{\theta=0}^{\theta_{\max}} \left(-\frac{1}{d} + \frac{\sin\theta}{R_2} \right) \mathrm{d}\theta \right\} \\
&= -\frac{k_e q_1 q_2}{2} \left[\frac{1}{d}\theta_{\max} + \frac{1}{R_2}(\cos\theta_{\max} - 1) \right] \\
&= -\frac{k_e q_1 q_2}{2} \left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d} + \frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2} - 1 \right) \right]
\end{aligned} \tag{C.1}
$$

A similar expression may be derived for $(U_{12})_{\substack{Inside\\ r<R_1}}$ (inside the charged sphere of radius R_1 centered at Q_1), which is produced from (C.1) by simply replacing R_2 with R_1.

Then the clear portion of interaction energy within the dashed line sphere $(O, d/2)$ will be the algebraic sum of Eq. (A.16) and the term $\left\{ -\left[(U_{12})_{\substack{Inside\\ r<R_1}} + (U_{12})_{\substack{Inside\\ r<R_2}} \right] \right\}$, which eventually is:

$$
\begin{aligned}
(U_{12})_{\substack{Inside\\ sphere(O,d/2)}} &= \frac{k_e q_1 q_2}{d} \frac{(2-\pi)}{4} \\
&+ \frac{k_e q_1 q_2}{2} \left[\frac{\sin^{-1}\left(\frac{R_1}{d}\right)}{d} + \frac{1}{R_1}\left(\sqrt{1-\left(\frac{R_1}{d}\right)^2} - 1 \right) \right] \\
&+ \frac{k_e q_1 q_2}{2} \left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d} + \frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2} - 1 \right) \right]
\end{aligned} \tag{C.2}
$$

Regarding the net amount of the interaction energy *outside* the dashed line sphere, $(U_{12})_{\substack{outside\\ sphere(O,d/2)}}$, the opposite of the lost energy inside it (due to the fact that one of (E_1 or E_2) vanishes within the charges sphere) is added to the external term, thus it becomes:

$$
\begin{aligned}
(U_{12})_{\substack{Outside\\ sphere(O,d/2)}} &= \frac{k_e q_1 q_2}{d} \frac{(2+\pi)}{4} \\
&- \frac{k_e q_1 q_2}{2} \left[\frac{\sin^{-1}\left(\frac{R_1}{d}\right)}{d} + \frac{1}{R_1}\left(\sqrt{1-\left(\frac{R_1}{d}\right)^2} - 1 \right) \right] \\
&- \frac{k_e q_1 q_2}{2} \left[\frac{\sin^{-1}\left(\frac{R_2}{d}\right)}{d} + \frac{1}{R_2}\left(\sqrt{1-\left(\frac{R_2}{d}\right)^2} - 1 \right) \right]
\end{aligned} \tag{C.3}
$$

Adding (C.2) and (C.3) we derive the same amount, $(U_{12})_{All-space} = k_e q_1 q_2 / d$, as it were for point-like charges.

Appendix D

Self-field-energy with respect to the dashed line sphere $(O, d/2)$ for a uniformly charged sphere of radius $R = R_2$

We distinguish two areas, the former being inside the dashed line sphere $(O, d/2)$ while the latter outside it. In both cases the charged sphere of radius R is not included (because within it the electric intensity vanishes, i.e., $E_2 = 0$).

Here, it is convenient to take the axis origin at point Q_2, so it will be $r = r_2$. For the exterior of the sphere (Q_2, R) but within the sphere $(O, d/2)$, for the three spherical variables we have the following lower and upper limits: $0 \le \varphi \le 2\pi$, $R \le r = r_2 \le d\cos\theta_2$, and $0 \le \theta = \theta_2 \le \theta_{2,\max}$, with $\cos\theta_{2,max} = R/d$ (see Fig. D-1).

A. Inside the dashed line sphere $(O, d/2)$, not including the inner part of (Q_2, R)

We have $R \le r \le dcos\theta$, and $0 \le \theta \le \theta_{2,max}$, with $\cos\theta_{2,max} = R/r$.

Therefore, the field self-energy U_2 within this sphere becomes:

$$
\begin{aligned}
(U_2)_{Inside} &= \frac{k_e q_2^2}{8\pi} \int_0^{2\pi} d\phi \int_0^{\theta_{2,\max}} \sin\theta \int_{r=R}^{d\cos\theta} \frac{r^2}{r^4} dr \\
&= \frac{k_e q_2^2}{8\pi} \times (2\pi) \int_0^{\theta_{2,\max}} \sin\theta \int_{r=R}^{d\cos\theta} \frac{1}{r^2} dr \\
&= \frac{k_e q_2^2}{4} \int_0^{\theta_{2,\max}} \sin\theta \left[-\frac{1}{r} \right]_{r=R}^{d\cos\theta} \\
&= \frac{k_e q_2^2}{4} \int_0^{\theta_{2,\max}} \sin\theta \left[-\frac{1}{d\cos\theta} + \frac{1}{R} \right] \\
&= \frac{k_e q_2^2}{4} \int_0^{\theta_{2,\max}} \left[-\frac{1}{d}\tan\theta + \frac{\sin\theta}{R} \right] d\theta \\
&= \frac{k_e q_2^2}{4} \int_0^{\theta_{2,\max}} \left[-\frac{1}{d}\tan\theta + \frac{\sin\theta}{R} \right] d\theta \\
&= \frac{k_e q_2^2}{4} \left[\frac{1}{d}\ln\left(\frac{R}{d}\right) - \left(\frac{1}{d} - \frac{1}{R}\right) \right]
\end{aligned} \tag{D.1}
$$

B. Outside the dashed line sphere $(O, d/2)$, not including the outer part of (Q_2, R)

Outside the sphere $(O, d/2)$ we distinguish two areas, one at the left and one at the right, depending on the angle $\theta = \theta_2$, as follows.

B1. Outer space on the **right** of charge q_2: $0 \le \theta \le \theta_{2,max}$.

In this case we have:

$$
\begin{aligned}
(U_2)_{Outside-Right} &= \frac{k_e q_2^2}{8\pi}\int_0^{2\pi} d\phi \int_0^{\theta_{2,\max}} \sin\theta \int_{d\cos\theta}^{\infty} \frac{r^2}{r^4} dr \\
&= \frac{k_e q_2^2}{8\pi}\times(2\pi)\int_0^{\theta_{2,\max}} \sin\theta \int_{d\cos\theta}^{\infty} \frac{1}{r^2} dr \\
&= \frac{k_e q_2^2}{4}\int_0^{\theta_{2,\max}} \sin\theta \left[-\frac{1}{r}\right]_{d\cos\theta}^{\infty} \\
&= \frac{k_e q_2^2}{4}\int_0^{\theta_{2,\max}} \sin\theta \left(\frac{1}{d\cos\theta}\right) d\theta \\
&= \frac{k_e q_2^2}{4}\frac{1}{d}\int_0^{\theta_{2,\max}} \tan\theta d\theta \\
&= -\frac{k_e q_2^2}{4}\frac{1}{d}\ln\left(\frac{R}{d}\right)
\end{aligned}
\tag{D.2}
$$

B2. Outer space on the **left** of charge q_2: $\theta_{2,max} \le \theta \le \pi$.
In this case we have:

$$
\begin{aligned}
(U_2)_{Outside-Left} &= \frac{k_e q_2^2}{8\pi}\int_0^{2\pi} d\phi \int_{\theta_{2,\max}}^{\pi} \sin\theta \int_{r=R}^{\infty} \frac{r^2}{r^4} dr \\
&= \frac{k_e q_2^2}{8\pi}\times(2\pi)\int_0^{\theta_{2,\max}} \sin\theta \int_{r=R}^{\infty} \frac{1}{r^2} dr \\
&= \frac{k_e q_2^2}{4}\left[-\cos\theta\right]_{\theta_{2,\max}}^{\pi}\left[-\frac{1}{r}\right]_{r=R}^{\infty} \\
&= \frac{k_e q_2^2}{4}\left(1+\frac{R}{d}\right)\frac{1}{R} \\
&= \frac{k_e q_2^2}{4}\left(\frac{1}{R}+\frac{1}{d}\right)
\end{aligned}
\tag{D.3}
$$

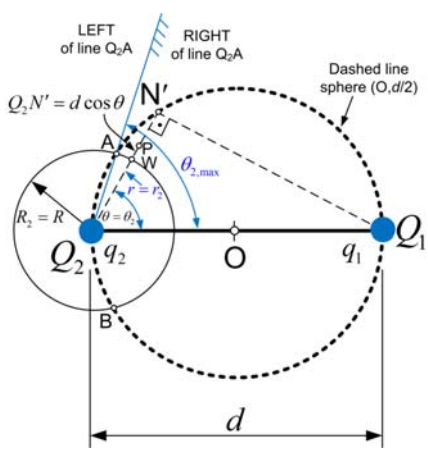

Fig. D-1**:** Overlapping between the spheres $(O, d/2)$ and (Q_2, R_2).

Appendix E

Energy breakdown outside a sphere $(\mathbf{O}, \mathbf{Rp})$ with $\mathbf{R_p} > \boldsymbol{d/2}$

The case of a sphere (O, R_p) with $R_p > d/2$ (see, Fig. E-1) is of great interest when we wish to determine the particular value of R_p so as the exterior of this sphere $(r > R_p)$ contains a prescribed amount of total field energy.

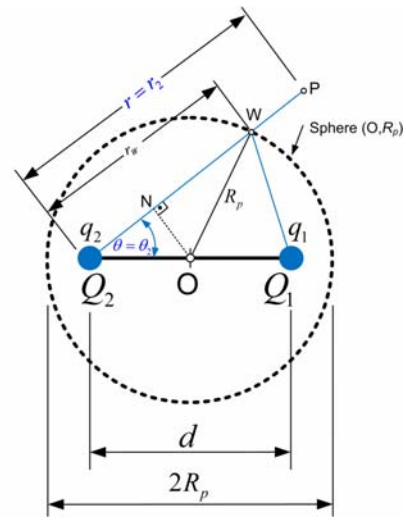

Fig. E-1: Field energy with respect to the surface of the sphere (O, R_p).

First we start with the electric charge q_2 located at the point Q_2, so it is convenient to select the main spherical variables $(r = r_2, \theta = \theta_2)$ with respect to axis origin at Q_2, as shown in Fig. E-1. For each point $P(r, \theta)$ in the exterior of the sphere (O, R_p), we bring the line Q_2P which intersects the sphere (O, R_p) at a point W, as shown in Fig. E-1. Then, the self-energy $(U_2)_{r>R_p}$ in the exterior of the sphere (O, R_p) becomes:

$$
\begin{aligned}
(U_2)_{r>R_p} &= \int_V \frac{E_2^2}{8\pi k_e} d\upsilon \\
&= \frac{1}{8\pi k_e}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi}\left[\int_{r_2=r_W}^{\infty}\left(\frac{k_e q_2}{r_2^2}\right)^2\right](r_2^2\sin\theta_2)dr_2 d\theta_2 \\
&\overset{(r=r_2)}{=} \frac{(k_e q_2)^2}{8\pi k_e}\times 2\pi \times \int_{\theta=0}^{\pi}\left[\int_{r=r_W}^{\infty}\left(\frac{1}{r^2}\right)dr\right]\sin\theta d\theta \\
&= \frac{k_e q_2^2}{4}\int_{\theta=0}^{\pi}\left[\int_{r=r_W}^{\infty} r^{-2} dr\right]\sin\theta d\theta \\
&= \frac{k_e q_2^2}{4}\int_{\theta=0}^{\pi}\left[-\frac{1}{r}\right]_{r_W}^{\infty}\sin\theta d\theta \\
&= \frac{k_e q_2^2}{4}\int_{\theta=0}^{\pi}\left[\frac{\sin\theta}{r_w}\right]d\theta
\end{aligned}
\tag{E.1}
$$

If we bring the normal projection N of the midpoint O onto the line Q_2WP (see, Fig. E-1), we have:

$$\begin{aligned} r_W &= Q_2W = Q_2N + NW \\ &= \frac{d}{2}\cos\theta + \sqrt{(OW)^2 - (ON)^2} \\ &= \frac{d}{2}\cos\theta + \sqrt{R_p{}^2 - (\frac{d}{2}\sin\theta)^2} \\ &= \frac{d}{2}\cos\theta + \sqrt{R_p{}^2 - \frac{d^2}{4} + \frac{d^2}{4}\cos\theta^2} \end{aligned} \tag{E.2}$$

Since $R_p > d/2$, the integrand in Eq. (E.1) will be always positive. The latter is symbolically ensured by setting the following auxiliary variable:

$$\bar{R}_p^2 = R_p^2 - \left(\frac{d}{2}\right)^2 \tag{E.3}$$

Substituting (E.3) into (E.2) and then into (E.1), the latter becomes:

$$(U_2)_{r>R_p} = \frac{k_e q_2^2}{4}\int_{\theta=0}^{\pi}\left[\frac{\sin\theta}{\frac{d}{2}\cos\theta + \sqrt{\bar{R}_p{}^2 + \frac{d^2}{4}\cos\theta^2}}\right]d\theta \tag{E.4}$$

After analytical integration and imposition of the limits, Eq. (E.4) progressively becomes:

$$\begin{aligned} (U_2)_{r>R_p} &= \frac{k_e q_2^2}{4}\left[\frac{d\cos^2\theta}{4\bar{R}_p^2} - \frac{\cos\theta\left(d^2\cos^2\theta + 4\bar{R}_p^2\right)^{1/2}}{4\bar{R}_p^2} - \frac{1}{d}\ln\left((d^2\cos^2\theta + 4\bar{R}_p^2)^{1/2} + d\cos\theta\right)\right]_0^{\pi} \\ &\frac{k_e q_2^2}{4}\left[\frac{\left(d^2 + 4\bar{R}_p^2\right)^{1/2}}{4\bar{R}_p^2} - \frac{1}{d}\ln\left((d^2 + 4\bar{R}_p^2)^{1/2} - d\right)\right]_{\theta=\pi} \\ &- \frac{k_e q_2^2}{4}\left[-\frac{\left(d^2 + 4\bar{R}_p^2\right)^{1/2}}{4\bar{R}_p^2} - \frac{1}{d}\ln\left((d^2 + 4\bar{R}_p^2)^{1/2} + d\right)\right]_{\theta=0} \\ &= \frac{k_e q_2^2}{4}\left[2\frac{\left(d^2 + 4\bar{R}_p^2\right)^{1/2}}{4\bar{R}_p^2} + \frac{1}{d}\ln\left((d^2 + 4\bar{R}_p^2)^{1/2} + d\right) - \frac{1}{d}\ln\left((d^2 + 4\bar{R}_p^2)^{1/2} - d\right)\right] \\ &= \frac{k_e q_2^2}{4}\left[\frac{\left(d^2 + 4\bar{R}_p^2\right)^{1/2}}{2\bar{R}_p^2} + \frac{1}{d}\ln\left(\frac{(d^2 + 4\bar{R}_p^2)^{1/2} + d}{(d^2 + 4\bar{R}_p^2)^{1/2} - d}\right)\right] \end{aligned} \tag{E.5a}$$

Then, substituting $\bar{R}_p$ by its definition, i.e. Eq. (E.3), Eq. (E.5a) eventually becomes:

$$(U_2)_{r>R_p} = \frac{k_e q_2^2}{4}\left[\frac{4R_p}{4R_p^2 - d^2} + \frac{1}{d}\ln\left(\frac{2R_p + d}{2R_p - d}\right)\right] \tag{E.5b}$$

A similar expression is immediately derived for $(U_1)_{r>R_p}$, simply by replacing in (E.5b), q_2 with q_1.

Finally, using now different than previously variables $(r = r_2, \theta = \theta_2)$ to comply with Appendix A (see, for example the integrand of Eq. (A.5)), the "interaction field energy" in the infinite space outside the sphere (O, R_p), i.e., with $r \geq R_p$, is given by:

$$\begin{aligned} (U_{12})_{r>R_p} &= \frac{1}{4\pi k_e}\times\left(\int_{\phi=0}^{2\pi} d\phi\right) \\ &\times\left\{\int_{\theta=0}^{\pi/2}\left[\int_{r=r_T}^{\infty}\left(k_e\frac{q_1}{r_1^2}\right)\left(k_e\frac{q_2}{r_2^2}\right)\left(\frac{r_1 - d\cos\theta_1}{r_2}\right)r_1^2\,dr_1\right]\sin\theta_1\,d\theta_1\right\} \\ &= \frac{1}{4\pi k_e}\times(2\pi)\left\{\int_{\theta=0}^{\pi/2}\left[\int_{r=r_T}^{\infty}\left(k_e\frac{q_1}{r_1^2}\right)\left(k_e\frac{q_2}{r_2^2}\right)\left(\frac{r - d\cos\theta}{r_2}\right)r_1^2\,dr\right]\sin\theta\,d\theta\right\} \\ &= \frac{k_e q_1 q_2}{2}\left\{\int_{\theta=0}^{\pi/2}\left[\int_{r=r_W}^{\infty}\left(\frac{r_1 - d\cos\theta_1}{r_2^3}\right)dr\right]\sin\theta\,d\theta\right\} \\ &\overset{(A.9)}{=} \frac{k_e q_1 q_2}{2}\left\{\int_{\theta=0}^{\pi/2}\left[\frac{1}{r_1}\right]_{r_W}^{\infty}\sin\theta_1\,d\theta_1\right\} = \frac{k_e q_1 q_2}{2}\underbrace{\int_{\theta_1=0}^{\pi/2}\left(\frac{\sin\theta_1}{r_W}\right)d\theta_1}_{I_\theta}, \end{aligned} \tag{E.6}$$

with

$$r_W = (Q_1W), \tag{E.7}$$

as shown in Fig. E-1.

Now we are seeking to express the edge $r_W = (Q_1W)$, which appears in the denominator of the integrand of Eq. (E.6), as a function of the angle θ_1. Actually, applying the cosine-rule to the triangle $(\widehat{OQ_1W})$ for the edge $(OW) = R_p$, after the solution of a quadratic polynomial in $Q_1W = r_{1,min}$, we obtain:

$$r_{1,\min} = (Q_1W) = \frac{d}{2}\cos\theta_1 + \left(R_p^2 - \left(\frac{d}{2}\sin\theta_1\right)^2\right)^{\frac{1}{2}} \tag{E.8}$$

Then applying for a second time the cosine-rule to the outer triangle $(\widehat{Q_1Q_2W})$ for the desired edge Q_1W, after the elimination of the common edge Q_1W, and using Eq. (E.8) again (it ensures a positive inner radicand in Eq. (E.8) and Eq. (E.9)) below, we eventually derive the lengthy expression:

$$\begin{aligned} r_W &= (Q_1W) \\ &= \left[\bar{R}_p^2 + d^2 - \frac{d^2}{2}\cos^2\theta_1 - d\cos\theta_1\left(\bar{R}_p^2 + \left(\frac{d}{2}\cos\theta_1\right)^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \end{aligned} \tag{E.9}$$

Unfortunately, the substitution of Eq. (E.9) into the denominator of the integrand in Eq. (E.6) does not lead to a well known closed-formed analytical expression (this claim was checked using the software MATLAB® and MATHEMATICA® as well). Therefore, it is proposed to calculate it numerically, for example applying Simpson's trapezoidal rule. A short program in MATLAB® which calculates the integral I_θ in (E.6), with $I_\theta = \int_{\theta_1=0}^{\pi/2}\left(\frac{\sin\theta_1}{r_W}\right)d\theta_1$, is given below.

Obviously, when working in the above numerical way, one may wish to skip the introduction of the auxiliary variable $\bar{R}_p$ in (E.9) which was used to ensure a positive base, thus instead it is proposed to directly apply Eq. (27) of the main text.

```
%% CALCULATE THE INTEGRAL USEFUL FOR INTERACTION ENERGY
U12
%-------------------------------------------------------
    clear all
    clc
%-------------------------------------------------------
    L=1;     %[m] Dipole's half-length
    d=2*L;   %[m] dipole's length (separation distance)
    Rcritical=1.5*d;%Rp[m]=critical distance (manually)
    Rp=Rcritical;
%-------------------------------------------------------
    Rpbar=(Rp^2-(d/2)^2)^(1/2);  %auxiliary variable
%-------------------------------------------------------
%---We  apply  Simpson's  integration  in  the  interval
[0,pi]:
%-------------------------------------------------------
    nseg=10*180;  %number of segments
    h=pi/nseg;    %step
    Ith=0;        %initialize integral
    for i=1:nseg+1
        th1=(i-1)*h;
        nominator = sin(th1);
        denominator=(Rpbar^2+d^2-d^2/2*    ...
                 cos(th1)^2-d*cos(th1)*    ...
            (Rpbar^2+d^2/4*cos(th1)^2)^(1/2))^(1/2);
        integrand = nominator / denominator;
        if( i==1 || i==(nseg+1) )
            Ith = Ith + integrand *(h/2);
        else
            Ith = Ith + integrand *h;
        end
    end
       fprintf('Ith=%20.15f\n',Ith);
```

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