On convergence of orthogonal expansion of a function from the class in the eigenfunctions of a differential operator of the third order

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Abstract— We consider a third-order ordinary differential operator with summable coefficients. The absolute and uniform convergence of the orthogonal expansion of a function from the class in the eigenfunctionsof this operator is studied and the rate of uniform convergence of these expansions on is estimated.

Keywords— eigenfunctions, third-order ordinary differential operator, orthogonal expansion

Received: May 17, 2021. Revised: August 19, 2021. Accepted: September 20, 2021. Published: October 9, 2021.

1. Introduction

IT is well known that any function in the domain of a self-adjoint ordinary differential operator can be expanded in a uniformly convergent series in the eigenfunctions of this operator [1. p. 90]. For functions that do not belong to the domain of self-adjoing Strum-Liouville operator, the problems of absolute and uniform convergence have been studied in [2-5] in [2,3] the Strum-Liouville operator

$Lu = -u'' + q(x)u, \quad x \in G = (0,1)$,

with two point self-adjoint boundary conditions (the coefficients in the boundary conditions are real) was considered, and under the condition $q(x) \in L_1(G)$, the absolute and uniform convergence on the interval $\bar{G}$ of the expansions of functions $f(x) \in W_1^1(G)$ $1 < p \le 2,$ $f(0) = f(1) = 0$, in orthonormal eigenfunctions of this operator was proved.

The operator L with a real potential $q(x) \in L_1(G)$ independent of the specific boundary conditions (in particular, self-adjoint boundary conditions with complex coefficients are also allowed) was consider in [4, 5]. The results obtained in [2-5] were generalized in [6] and [7] (for the one-dimensional Şchrödinger operator).

On the interval $G = (0,1)$, consider the differential operator

$$Lu = u^{(3)} + p_1(x)u^{(2)} + p_2(x)u^{(1)} + p_3(x)u, \qquad (1)$$

with coefficients

$$p_1(x) \in L_2(G), \ \ p_l(x) \in L_1(G), \ \ l = 2,3.$$

In the present paper, we study the problems of absolute and uniform convergence of expansions of functions of the class $W_1^1(G)$ in the eigenfunctions of a third-order differential operator (1) (see [8], [9]). Sufficient conditions for the absolute and uniform convergence of these expansions are obtained, and the rate of uniform convergence is estimated. This study are based on Ilins spectral method [10].

By $D(G)$ we denote the class of functions absolutely continuous together with their derivatives up to the second order, inclusively, on the segment $\bar{G} = [0,1]$.

An eigenfunctions of the operator L corresponding to the eigenvalue λ is understood as any function not identically equal to zero $u(x) \in D(G)$ and satisfying (almost everywhere in G) the equation (see [10])

$$Lu + \lambda u = 0.$$

We say that a function $f(x)$ belongs to $W_p^1(G),\ 1 \le p \le \infty$, if $f(x)$ is absolutely continuous on $\bar{G}$ and $f'(x)$ belongs to $L_p(G)$. The norm of the function $f(x) \in W_p^1(G)$ is given by the equality

$$\| f \|_{W_p^1(G)} = \| f \|_p + \| f' \|_p,$$

where $\| \cdot \|_p = \| \cdot \|_{L_p(G)}$.

Assume that $\{u_k(x)\}_{k=1}^{\infty}$ is the complete system of eigenfunctions of the operator L ortonormal in $L_2(G)$. By $\{\lambda_k\}_{k=1}^{\infty}$ we denote the corresponding system of eigenvalues. Moreover, we assume that $\operatorname{Re}\lambda_k = 0$. Parallel with the spectral parameter λ_k, we consider a parameter μ:

$$\mu_k = \begin{cases} (-i\lambda_k)^{1/3} & \text{for} \quad I_m\lambda_k \ge 0, \\ (i\lambda_k)^{1/3} & \text{for} \quad I_m\lambda_k < 0. \end{cases}$$

We now introduce a partial sum of the orthogonal expansion of the function $f(x) \in W_1^1(G)$ in the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x,f)=\sum_{\mu_k\le\nu} f_k u_k(x), \quad \nu>0,$$

where

$$f_k=(f,u_k)=\int_G f(x)\overline{u_k(x)}\,dx,$$

and the difference

$$R_\nu(x,f)=f(x)-\sigma_\nu(x,f).$$

In the present paper, we prove the following statements:

Theorem 1. Suppose that $f(x)\in W_p^1(G),\ p_1(x)\in L_2(G),\ p_l(x)\in L_1(G),\ l=2,3$ and following conditions are satisfied:

$$|f(x)\overline{u^{(2)}(x)}|\le C_1(f)\mu_k^\alpha\|u_k\|_\infty, \quad 0\le\alpha<2,\quad \mu_k\ge 1; \tag{2}$$

$$\sum_{k=2}^{\infty}k^{-1}\omega_1(f',k^{-1})<\infty. \tag{3}$$

Then the spectral expansion of the function $f(x)$ in the system $\{u_k(x)\}_{k=1}^\infty$ absolutely and uniformly converges on the segment $\overline{G}=[0,1]$ and the following estimate is true:

$$\|R_\nu(\cdot,\infty)\|_{C[0,1]}\le C\Big\{C_1(f)\nu^{\alpha-2}+\nu^{-\frac12}\|p_1 f\|_2+ +(1+\|p_1\|_1)\Big[\sum_{k=[\nu]}^{\infty}k^{-1}\omega_1(f',k^{-1})+\omega_1(f',\nu^{-1})\Big] +(1+\|p_1\|_1)\|f'\|_1\nu^{-1}+ +\nu^{-1}(\|f\|_\infty+\|f'\|_1)\sum_{r=2}^{\infty}\nu^{2-r}\|p_r\|_1\Big\}, \quad \nu\ge 8\pi, \tag{4}$$

where $\omega_1(g,\delta)$ is the integral modulus of continuity of the function $g(x)\in L_1(G)$, and the constant C is independent of $f(x)$.

Corollary 1. If the function $f(x)\in W_1^1(G)$ in the Theorem 1 satisfies the conditions $f(0)=f(1)=0$, then condition (2) is necessarily satisfied (with the constant $C_1(f)=0$), its spectral expansion in the system $\{u_k(x)\}_{k=1}^\infty$ converges absolutely and uniformly on the segment $\overline{G}=[0,1]$, and the following estimate holds:

$$\|R_\nu(\cdot,\infty)\|_{C[0,1]}\le \le \text{const}\ \Big\{\nu^{-\frac12}\|p_1 f\|_2+\Big[\omega_1(f,\nu^{-1})+\sum_{k=[\nu]}^{\infty}k^{-1}\omega_1(f',k^{-1})\Big] (1+\|p_1\|_1)+\nu^{-1}\Big[1+\|p_1\|_1+2\sum_{r=2}^{\infty}\nu^{2-r}\|p_r\|_1\Big]\|f'\|_1\Big\}, \quad \nu\ge 8\pi.$$

Corollary 2. If the function in the Theorem 1 satisfies the relations

$$f(0)=f(1)=0$$

and

$$f'(x)\in H_1^\beta(G),\ 0<\beta\le 1,$$

($H_1^\beta(G)$ is the Nikolski class), then conditions (2) and (3) are necessarily satisfied, its spectral expansion converges absolutely and uniformly on the segment $\overline{G}=[0,1]$, and the following estimate holds:

$$\|R_\nu(\cdot,\infty)\|_{C[0,1]}\le \le\text{const}\ \Big\{\nu^{-\frac12}\|p_1 f\|_2+\nu^{-\beta}\|f'\|_1^\beta\Big\},\ \nu\ge 8\pi,$$

where

$$\|f'\|_1^\beta=\|f'\|_1+\delta^{-\beta}\omega_1(f',\delta).$$

Theorem 2. Suppose that

$$f(x)\in W_1^1(G),$$
$$p_1(x)\in L_2(G),$$
$$p_l(x)\in L_1(G),\ l=2,3;$$

conditions (2), (3) and

$$\sum_{k=2}^{\infty}k^{-1}\omega_1(p_1 f,k^{-1})<\infty \tag{5}$$

are satisfied. Then the spectral expansion of the function $f(x)$ in the system $\{u_k(x)\}_{k=1}^\infty$ absolutely and uniformly converges on $\overline{G}=[0,1]$ and the following estimate is true:

$$\|R_\nu(\cdot,\infty)\|_{C[0,1]}\le C\Big\{C_1(f)\nu^{\alpha-2}+ +\Big[\sum_{k=[\nu]}^{\infty}k^{-1}\omega_1(\overline{p}_1 f,k^{-1})+\sum_{k=[\nu]}^{\infty}k^{-1}\omega_1(f',k^{-1})+\omega_1(\overline{p}_1 f,\nu^{-1})+\omega_1(f',\nu^{-1})+ +\nu^{-1}(\|p_1 f\|_1+\|f'\|_1)\Big]+\nu^{-1}(\|p_1 f\|_1+\|f\|_\infty+\|f'\|_1)\sum_{r=2}^{\infty}\nu^{2-r}\|p_r\|_1\Big\}, \quad \nu\ge 8\pi. \tag{6}$$

Corollary 3. If the function $f(x)\in W_1^1(G)$ in the Theorem 2 satisfies the relations $f(0)=f(1)=0$ and

$$f'(x)\in H_1^{\beta}(G),\ 0<\beta\le 1,$$
$$\bar{p}_1 f\in H_1^{\gamma}(G),\ \ 0<\gamma\le 1,$$

then condition (2) and (3) are necessarily satisfied, its spectral expansion converges absolutely and uniformly on the segment $\overline{G}=[0,1]$, and the following estimate holds:

$$\left\|R_{\nu}(\cdot,\infty)\right\|_{C[0,1]}\le$$
$$\le \text{const}\ \left\{\nu^{-\beta}\left\|f'\right\|_1^{\beta}+\nu^{-\gamma}\left\|\bar{p}_1 f\right\|_1^{\gamma}\right\},\ \ \nu\ge 8\pi$$

where constant is independent of the function $f(x)$.

2. Some auxiliary lemmas

To prove the results, we must estimate the Fourier coefficients f_k of the function $f(x)\in W_1^1(G)$. To this end, we use representation of the eigenfunction $u_k(x)$. Let as introduce

$$x_j^{\pm}\equiv x_{jk}^{\pm}(0)=\frac{1}{3\mu_k^2}\sum_{r=0}^{2}(i\mu_k)^r\,\omega_j^{r+1}u^{2-r}(0);$$
$$\mu(\xi,u_k)=\frac{-1}{3\mu_k^2}\sum_{e=1}^{3}p_l(\zeta)\omega_j^{r+1}u^{(3-l)}(\xi),$$
$$i=\sqrt{-1},$$

where

$$\omega_1=-1,$$
$$\omega_2=\exp(-i\pi/3),$$
$$\omega_3=\exp(i\pi/3).$$

Lemma1. (see [8,9]). If $\lambda_k\ne 0$, then the following representation is valid for the eigenfunction $u_k(x)$:

$$\mu_k^{-l}u_k^{(l)}(t)=$$
$$=\sum_{j=1}^{2}(-i\omega_j)^l\,x_j^{-}(0)\exp(-i\omega_j\mu_k t)+(-i\omega_j)^l\,B_{3k}^{-}\exp(i\omega_3\mu_k(1-t))-$$
$$-\sum_{j=1}^{2}(-i)^l\,\omega_j^{l+1}\int_0^t M(\xi,u_k)\exp(i\omega_j\mu_k(\xi-t))d\xi+$$
$$+(-i)^l\,\omega_j^{l+1}\int_t^1 M(\xi,u_k)\exp(i\omega_3\mu_k(\xi-t))d\xi \qquad (7)$$

for $\operatorname{Im}\lambda_k>0$ and

$$\mu_k^{-l}u_k^{(l)}(t)=$$
$$=\sum_{j=1,\ j\ne 2}^{3}(i\omega_j)^l\,x_j^{+}(0)\exp(i\omega_j\mu_k t)+(i\omega_2)^l\,B_{2k}^{+}\exp(-i\omega_2\mu_k(1-t))-$$
$$-\sum_{j=1,\ j\ne 2}^{3}(i)^l\,\omega_j^{l+1}\int_0^t M(\xi,u_k)\exp(-i\omega_j\mu_k(\xi-t))d\xi+$$
$$+(i)^l\,\omega_2^{l+1}\int_t^1 M(\xi,u_k)\exp(-i\omega_2\mu_k(\xi-t))d\xi \qquad (8)$$

for $\operatorname{Im}\lambda_k<0$ and. Moreover,

$$B_3^{-}=x_3^{-}(0)\exp(-i\omega_3\mu_k)-$$
$$-\omega_3\int_0^1 M(\xi,u_k)\exp(-i\omega_3\mu_k(\xi-1))d\xi,$$
$$B_2^{+}=x_2^{+}(0)\exp(i\omega_2\mu_k)-$$
$$-\omega_2\int_0^1 M(\xi,u_k)\exp(i\omega_2\mu_k(\xi-1))d\xi,$$

the coefficients in relations (7) and (8) satisfy the inequalities:

$$|\,x_1^{\pm}(0)|\le C\left\|u_k\right\|_2\le C;\quad |\,x_j^{\pm}(0)|\le C\left\|u_k\right\|_{\infty}$$

for

$j=2,3;\ \ |\,B_{2k}^{+}|\le C\left\|u_k\right\|_{\infty};\ \ |\,B_{3k}^{-}|\le C\left\|u_k\right\|_{\infty}$ where C is a constant.

Lemma 2. Suppose that the function $f(x)\in W_1^1(G)$ and the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy condition (2). Then the Fourier coefficients f_k satisfy the inequalities $(\mu_k\ge 8\pi)$:

$$|\,f_k\,|\le C\{C_1(f)\mu_k^{\alpha-3}+$$
$$+\mu_k^{-1}(1+\left\|p_1\right\|_1)[\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\left\|f'\right\|_1]+$$
$$+\mu_k^{-2}\left(\left\|f'\right\|+\left\|f\right\|_{\infty}\right)\sum_{r=2}^{3}\mu_k^{2-r}\left\|p_r\right\|_1\Big\}\left\|u_k\right\|_{\infty}+$$
$$+C\mu_k^{-1}\,|\,(\bar{p}_1 f,\mu_k^{-2}u_k^{(2)})\,|\ ; \qquad (9)$$

$$|\,f_k\,|\le C\{C_1(f)\mu_k^{\alpha-3}+\mu_k^{-1}(1+\left\|p_1\right\|_1)[\omega_1(\bar{p}_1 f,\mu_k^{-1})+$$
$$+\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\left\|p_1 f\right\|_1+$$
$$+\mu_k^{-1}\left\|f'\right\|_1]+\mu_k^{-2}\left(\left\|f'\right\|_1+\left\|f\right\|_{\infty}+\left\|p_1 f\right\|_1\right)\sum_{r=2}^{3}\mu_k^{2-r}\left\|p_r\right\|_1\Big\}\left\|u_k\right\|_{\infty};\ (9`)$$

where C is a a constant independent of $f(x)$.

Proof. Since the eigenfunction $u_k(x)$ is a solution of the equation $Lu_k=-\lambda_k u_k$, we represent the Fourier coefficient

f_k of $\mu_k \neq 0$ to the form

$$f_k = (f, u_k) = (f, -\lambda_k^{-1} L u_k) =$$

$$= -\bar{\lambda}_k^{-1}(f, u_k^{(3)}) - \bar{\lambda}_k^{-1} \sum_{r=1}^{3} (f, p_r u_k^{(3-r)}) =$$

$$= -\bar{\lambda}_k^{-1}(f, u_k^{(3)}) - \bar{\lambda}_k^{-1}(f, p_1 u_k^{(2)}) - \bar{\lambda}_k^{-1} \sum_{r=2}^{3} (f, p_r u^{(3-r)}) . \quad (10)$$

By virtue of the estimate (see [11])

$$\left\| u_k^s \right\|_\infty \le const \left(1+\mu\right)^{s+\frac{1}{p}} \left\| u_k \right\|_p , \quad p \ge 1, \quad s = \overline{0,2} \quad (11)$$

we obtain the following estimate of the third term of the right-hand side in (10):

$$\left| \bar{\lambda}_k^{-1} \sum_{r=2}^{3} (f, p_r u^{(3-r)}) \right| \le \mu_k^{-3} \left\| f \right\|_\infty \sum_{r=2}^{3} \left\| p_r \right\|_1 \left\| u_k^{(3-r)} \right\|_\infty \le$$

$$\le const\, \mu_k^{-3} \left\| f \right\|_\infty \left(\sum_{r=2}^{3} \left\| p_r \right\|_1 \mu_k^{3-r} \right) \left\| u_k \right\|_\infty \le$$

$$\le const\, \mu_k^{-2} \left\| f \right\|_\infty \left\| u_k \right\|_\infty \sum_{r=2}^{3} \mu_k^{2-r} \left\| p_r \right\|_1 . \quad (12)$$

Integrating the first term on the right-hand side of equality (10) by parts and using condition (2), we get

$$|\lambda_k|^{-1} |(f, u_k^3)| \le |\lambda_k|^{-1} \left| f(t) \overline{u_k^{(2)}(t)} \Big|_0^1 \right| +$$

$$+ |\lambda_k|^{-1} \left| \int_0^1 f'(t) u_k^{(2)}(t) \right| \le$$

$$\le C_1(f) \mu_k^{\alpha-3} \left\| u_k \right\|_\infty + \mu_k^{-3} \left| \left(f', u_k^{(2)} \right) \right| . \quad (13)$$

We now estimate the expression $\mu_k^{-3} |(f', u_k^2)|$ on the right-hand side of inequality (13). For that we use formulas (7) and (8) subject to the sign of $\operatorname{Im} \lambda_k$. For definiteness consider the case $\operatorname{Im} \lambda_k < 0$ and apply relation (8) with $l = 2$.

$$\mu_k^{-3} \left(f', u_k^{(2)} \right) = \mu_k^{-1} \left(f', \mu_k^{-2} u^{(2)} \right) =$$

$$= \mu_k^{-1} \sum_{j=1,\ j \neq 2}^{3} (f' x_j^+(0)(i\omega_j)^2 \exp(i\omega_j \mu_k t)) +$$

$$+ \mu_k^{-1} \overline{B_{2k}^+ (i\omega_2)^2} (f', \exp(-i\omega_2 \mu_k (1-t)) -$$

$$- \mu_k^{-1} \sum_{j=1,\ j \neq 2}^{3} (f', \int_0^t M(\xi, u_k) \exp(-i\omega_j \mu_k (\xi - t)) d\xi) +$$

$$- \mu_k^{-1} (f', \int_t^1 M(\xi, u_k) \exp(-i\omega_2 \mu_k (\xi - t)) d\xi) . \quad (14)$$

Estimate each term in this equality. Obviously

$$(f', x_j^+(0)(i\omega_j)^2 \exp(i\omega_j \mu_k t)) =$$

$$= \overline{x_j^+(0)(i\omega_j)^2} (f, \exp(i\omega_j \mu_k t)), \quad j = 1,3 .$$

Taking into account the inequality

$$|x_j^+(0)| \le const \ \left\| u_k \right\|_\infty , \quad j = 1,3 , \quad (15)$$

That follows from estimation (11), and using the estimation (see [12], [13])

$$\left| \int_0^1 \overline{f'(t)} \exp(i\omega_j \mu_k t) dt \right| \le$$

$$\le const \ \left\{ \omega_1 (f', \mu_k^{-1}) + \mu_k^{-1} \left\| f' \right\|_1 \right\}, \quad j = 1,3$$

we have

$$\left| (f', x_j^+(0)(i\omega_j)^2 \exp(i\omega_j \mu_k t)) \right| \le$$

$$\le const \ \left\{ \omega_1 (f', \mu_k^{-1}) + \mu_k^{-1} \left\| f' \right\|_1 \right\} \left\| u_k \right\|_\infty , \quad j = 1,3 . \quad (16)$$

Apply the estimation $|\beta_{2k}^+| \le const \ \left\| u_k \right\|_\infty$ in the second term of equality (14). As a result we have

$$\left| B_{2k}^+ (iw_2)^2 (f', \exp(i\omega_2 \mu_k (1-t))) \right| \le$$

$$\le const \ \left\{ \omega_1 (f', \mu_k^{-1}) + \mu_k^{-1} \left\| f' \right\|_1 \right\} \left\| u_k \right\|_\infty . \quad (17)$$

The third and fourth terms in equality (14) are estimated by the same scheme. Therefore we estimate the third term. For that we use the representation

$$M(\xi, u_k) = -\frac{1}{3\mu_k^2} p_1(\xi) u_k^{(2)}(\xi) - \frac{1}{3\mu_k^2} \sum_{r=2}^{3} p_r(\xi) .$$

$u_k^{(3-r)}(\xi)$ and the inequality

$$\left| \frac{1}{3\mu_k^2} \sum_{r=2}^{3} p_r(\xi) u_k^{(3-r)} \right| \le$$

$$\le const \ \mu_k^{-1} \left[\sum_{r=2}^{3} |p_r(\xi)| \ \mu_k^{2-r} \right] \left\| u_k \right\|_\infty .$$

Then we have

$$\left| \mu_k^{-1} \sum_{j=1,\ j \neq 2}^{3} (f', \int_0^t M(\xi, u_k) \exp(-i\omega_j \mu_k (\xi - t)) d\xi) \right| \le$$

$$\le \frac{1}{3\mu_k^3} \sum_{j=1,\ j \neq 2}^{3} \left| (f', \int_0^t p_1(\xi) u_k^{(2)}(\xi) \exp(-i\omega_j \mu_k (\xi - t)) d\xi) \right| +$$

$$+ \frac{const}{\mu_k^2} \left[\sum_{r=2}^{3} \left\| p_r \right\|_1 \mu_k^{2-r} \right] \left\| f' \right\|_1 \left\| u_k \right\|_\infty . \quad (18)$$

After changing the integration order in the first term, we get that it doesn`t exceed the quantity

$$\frac{const}{\mu_k}\sum_{j=1,\ j\neq 2}^{3}\int_0^1 |p_1(\xi)|\left|\int_\xi^1 \overline{f'(t)}\ \exp(-i\omega_j\mu_k(\xi-t))dt\right| d\xi \|u_k\|_\infty , \quad (19)$$

$$j=1,3$$

Taking into account the following chain of inequalities (see [5], [6])

$$\left|\int_\xi^1 \overline{f'(t)}\ \exp(-i\omega_j\mu_k(\xi-t))dt\right| \le$$

$$\le const\ \left\{\omega_1(g_\xi,\mu_k^{-1})+\mu_k^{-1}\|g_\xi\|_1\right\}\le$$

$$\le const\ \left\{\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\|f'\|_1+\mu_k^{-1}\|f'\|_1\right\}\le$$

$$\le const\ \left\{\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\|f'\|_1\right\},\quad j=1,3,$$

where

$$g_\xi(z)=\begin{cases} f'(\xi+z) & \text{for}\quad 0\le z\le 1-\xi \\ 0 & \text{for}\quad 1-\xi<z\le 1,\end{cases}\qquad \xi\in[0,1],$$

we prove that expression (19) is bounded from above by the quantity

$$\frac{const}{\mu_k}\|p_1\|_1\ \left\{\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\|f'\|_1\right\}\ \|u_k\|_\infty .$$

Consequently, the left side of (18) doesn`t exceed the quantity

$$\frac{const}{\mu_k}\|p_1\|_1\ \left\{\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\|f'\|_1\right\}\ \|u_k\|_\infty +$$

$$+\frac{const}{\mu_k^2}\ \sum_{r=2}^{3}\|p_r\|_1\,\mu_k^{2-r}\ \ \|f'\|\ \|u_k\|_\infty .$$

Hence and from estimations (16), (17) and relation (14) we get

$$\mu_k^{-3}\,|(f',u_k^{(2)})|\le$$

$$\le\frac{const}{\mu_k}\Big\{(1+\|p_1\|_1)[\omega_1(f',\mu_k^{-1})+\mu_k^{-1}\|f'\|_1]+$$

$$+\mu_k^{-1}\|f'\|_1\sum_{r=2}^{3}\|p_r\|_1\,\mu_k^{2-r}\Big\}\ \|u_k\|_\infty \qquad (20)$$

Estimate now the term $\bar\lambda_k^{-1}(f,p_1u_k^2)$ in equality (10).

Obviously

$$\left|\frac{1}{\bar\lambda_k}(f,p_1u_k^2)\right|=\frac{1}{\mu_k^3}\left|(\bar p_1 f,\,u_k^2)\right|. \quad (21)$$

By estimations (12), (13), (20) and equality (21) from equality (10) we get inequality (9).

Since the function $\bar p_1(x)f(x)$ belongs to the class $L_1(G)$, we can apply estimation (20) with substitution of $p_1 f$ for f'. As a result, we have

$$\left|\frac{1}{\bar\lambda_k}(f,p_1u_k^2)\right|=\frac{1}{\mu_k^3}\left|(\bar p_1 f,\,u_k^2)\right|\le$$

$$\le\frac{const}{\mu_k}\Big\{(1+\|p_1\|_1)[\omega_1(\bar p_1 f,\mu_k^{-1})+\mu_k^{-1}\|p_1 f\|_1]+$$

$$+\mu_k^{-1}\|p_1 f\|_1\sum_{r=2}^{3}\|p_r\|_1\,\mu_k^{2-r}\Big\}\ \|u_k\|_\infty \qquad (22)$$

Consequently, by estimations (12), (13), (20) and (22) from equality (10) we have

$$|f_k|\le const\Big\{C_1(f)\mu_k^{\alpha-3}+\mu_k^{-1}(1+\|p_1\|_1)[\omega_1(f',\mu_k^{-1})+\omega_1(\bar p_1 f,\mu_k^{-1})+$$

$$+\mu_k^{-1}\|f'\|_1+\mu_k^{-1}\|p_1 f\|_1]+\ \mu_k^{-2}\Big(\ \|f'\|_1+\|f\|_\infty+\|p_1 f\|_1\Big)\sum_{r=2}^{3}\mu_k^{2-r}\|p_r\|_1\Big\}\|u_k\|_\infty$$

The case $\operatorname{Im}\lambda_k>0$ is considered in the same way. The lemma 2 is proved.

Lemma 3. (see [11]) Assume that $p_1(x)\in L_2(G),\ p_l(x)\in L_1(G),\ l=2,3$. Then for the orthonormal system of eigenfunctions $\{u_k(x)\}_{k=1}^{\infty}$ and the sequence $\{\mu_k\}_{k=1}^{\infty}$, the following estimates are true:

$$\sum_{\tau\le\mu_k\le\tau+1}' 1\le C\quad\text{for any}\quad \tau\ge 0 \qquad (23)$$

$$\sum_{\tau\le\mu_k\le\tau}\|u_k\|_\infty^2\le C(1+\tau)\quad\text{for any}\quad\tau\ge 0. \qquad (24)$$

Lemma 4. (see [14]). If the conditions of Lemma 3 a satisfies, then

$$\{\mu_k^{-2}\,u_k^{(2)}(x)\}_{k=1}^{\infty},\quad \mu_k\ne 0$$

is a Bessel system, i.e., for any function $f(x)\in L_2(G)$, the following inequality a true:

$$\left(\sum_{\mu_k>0}|(f,\mu_k^{2-r}\,u_k^{(2)})|^2\right)^{1/2}\le const\ \|f\|_2 . \quad (25)$$

Lemma 5. Suppose that the conditions of Lemma 3 are satisfied. Then the following estimate hold for the system $\{u_k(x)\}_{k=1}^{\infty}$ for any $\mu\ge 2$

$$\sum_{\mu_k\ge\mu}\mu_k^{-(1+\delta)}\|u_k\|_\infty^2\ \le C(\delta),\quad \delta>0, \qquad (26)$$

where $C(\delta)$ is positive constant.

Proof. Take a positive integer n_0. By the estimates (23) and (24), using the Abel transformation, we obtain the chain of inequalities

$$\sum_{\mu\le\mu_k\le[\mu]+n_0}\mu_k^{-(1+\delta)}\|u_k\|_\infty^2\le\sum_{[\mu]\le\mu_k\le[\mu]+n_0}\mu_k^{-(1+\delta)}\|u_k\|_\infty^2\le$$
$$\le\sum_{n=[\mu]}^{[\mu]+n_0}n^{-(1+\delta)}\|u_k\|_\infty^2\left(\sum_{n\le\mu_k<n+1}\|u_k\|_\infty^2\right)\le$$
$$\le\sum_{n=[\mu]}^{[\mu]+n_0-1}\left(\sum_{n\le\mu_k<n+1}\|u_k\|_\infty^2\right)\left(n^{-(1+\delta)}-(n+1)^{-(1+\delta)}\right)+$$
$$+\left(\sum_{1\le\mu_k<[\mu]+n_0+1}\|u_k\|_\infty^2\right)\left([\mu]+n_0\right)^{-(1+\delta)}+$$
$$+\left(\sum_{1\le\mu_k<[\mu]}\|u_k\|_\infty^2\right)[\mu]^{-(1+\delta)}\le$$
$$\le const\sum_{n=[\mu]}^{[\mu]+n_0-1}(n+1)\frac{(1+\delta)\,(1+n)^\delta}{(n(n+1))^{1+\delta}}+$$
$$+const\ (n_0+[\mu])^{-(1+\delta)}(n_0+[\mu]+1)+$$
$$+const\ [\mu]^{-(1+\delta)}(1+[\mu])\le$$
$$\le const\ \left\{(1+\delta)\sum_{n=[\mu]}^{\infty}(n)^{-(1+\delta)}+[\mu]^{-\delta}\right\}\le C(\delta)\mu^{-\delta},$$

whence, since the number n_0 is arbitrary, we obtain the estimate (26).

Lemma 6. Assume that

$$p_1(x)\in L_2(G),\ \ p_l(x)\in L_1(G),\ \ l=2,3;$$

and a $g(x)\in L_1(G)$ function satisfies condition

$$\sum_{k=2}^{\infty}k^{-1}\omega_1(g,k^{-1})<\infty\ . \qquad (27)$$

Then the estimate

$$\sum_{\mu_k\ge\mu}\mu_k^{-1}\|u_k\|_\infty^2\,\omega_1(g,\mu_k^{-1})\le$$
$$\le C\left\{\omega_1(g,\mu^{-1})+\sum_{k=[\mu]}^{\infty}k^{-1}\omega_1(g,k^{-1})\right\} \qquad (28)$$

holds, where $\mu\ge 8\pi$ and C is a positive constant independent of μ and the function $f(x)$

Proof. Take a positive integer m. By the estimate, (24) using the Abel transformation, we obtain the chain of inequalities

$$\sum_{\mu\le\mu_k\ge[\mu]+m}\mu_k^{-1}\|u_k\|_\infty^2\,\omega_1(g,\mu_k^{-1})\le$$
$$\le\sum_{n=[\mu]}^{[\mu]+m}n^{-1}\omega_1(g,n^{-1})\left\{\sum_{n\le\mu_k<n+1}^{\infty}\|u_k\|_\infty^2\right\}\le$$
$$\le\sum_{n=[\mu]}^{[\mu]+m-1}\left(\sum_{1\le\mu_k<n+1}^{\infty}\|u_k\|_\infty^2\right)[n^{-1}\omega_1(g,n^{-1})-(n+1)^{-1}\omega_1(g,(n+1)^{-1})]+$$
$$+\left(\sum_{1\le\mu_k<[\mu]+m-1}^{\infty}\|u_k\|_\infty^2\right)([\mu]+m)^{-1}\omega_1(g,([\mu]+m)^{-1})+$$
$$+\left(\sum_{1\le\mu_k\le[\mu]}^{\infty}\|u_k\|_\infty^2\right)[\mu]^{-1}\omega_1(g,[\mu]^{-1})\le C\sum_{n=[\mu]}^{[\mu]+m-1}(n+1)[n^{-1}\omega_1(g,n^{-1})-$$
$$-(n+1)^{-1}\omega_1(g,(n+1)^{-1})]+$$
$$+C([\mu]+m)\ \ ([\mu]+m)^{-1}\omega_1(g,([\mu]+m)^{-1})+$$
$$+C[\mu]\,[\mu]^{-1}\,\omega_1(g,[\mu]^{-1})\le$$
$$\le C\left\{\sum_{n=[\mu]}^{[\mu]+m-1}n^{-1}\omega_1(g,n^{-1})+\omega_1(g,[\mu]^{-1})-\omega_1(g,([\mu]+m)^{-1})\right\}+$$
$$+C\,\omega_1(g,([\mu]+m)^{-1})+C\omega_1(g,[\mu]^{-1})\le$$
$$\le C\left\{\sum_{n=[\mu]}^{[\mu]+m-1}n^{-1}\omega_1(g,n^{-1})+\omega_1(g,[\mu]^{-1})+\omega_1(g,([\mu]+m)^{-1})\right\}$$

Since the number m is arbitrary, this together with inequality (27), implies the estimate (28).

3. Proof of the results

We the uniform convergence of the series $\sum_{k=1}^{\infty}|\ f_k\ ||\ u_k(x)|$ on the segment $\bar{G}=[0,1]$. To this end, we represent this series as

$$\sum_{k=1}^{\infty}|\ f_k\ ||\ u_k(x)|=$$
$$=\sum_{0\le\mu_k<8\pi}|\ f_k\ ||\ u_k(x)|+\sum_{\mu_k\ge 8\pi}|\ f_k\ ||\ u_k(x)|\ . \qquad (29)$$

To estimate the first sum on the right-hands side in (28), we apply the estimate (24) in Lemma 3 and inquality $|\ f_k\ |\le\|\ f\|_1\|u_k\|_\infty$. As a result we have

$$\sum_{0\le\mu_k<8\pi}' |f_k||u_k(x)| \le$$

$$\sum_{0\le\mu_k<8\pi} \|f_1\| \|u_k\|_\infty^2 = \|f_1\| \sum_{0\le\mu_k\le 8\pi} \|u_k\|_\infty^2 =$$

$$= C(1+8\pi)\|f_1\| \le const \|f_1\|.$$

To estimate the second sum in (29), we use the estimate (9) in Lemma 2:

$$\sum_{\mu_k\ge 8\pi}' |f_k||u_k(x)| \le$$

$$\le const \left\{ C_1(f) \sum_{\mu_k\ge 8\pi} \mu_k^{\alpha-3} \|u_k\|_\infty^2 + (1+\|p_1\|_1)\times \right.$$

$$\times \sum_{\mu_k\ge 8\pi} \mu_k^{-1}\omega_1(f',\mu_k^{-1}) \|u_k\|_\infty^2 + \|f'\|_1 (1+\|p_1\|_1) \sum_{\mu_k\ge 8\pi} \mu_k^{-2} \|u_k\|_\infty^2 +$$

$$+(\|f'\|_1 + \|f\|_\infty) \sum_{r=2}^{3} \|p_r\|_1 \left(\sum_{\mu_k\ge 8\pi} \mu_k^{-r} \|u_k\|_\infty^2 \right) +$$

$$\left. + \sum_{\mu_k\ge 8\pi} \mu_k^{-1} \|u_k\|_\infty^2 \left| (\bar{p}_1 f, \mu_k^{-2} u_k^{(2)}) \right| \right\}.$$

Since $\bar{p} f \in L_2(G)$ and $\{\mu_k^{-2} u_k^{(2)}(x)\}_{\mu_k>0}$ is a Bessel system (see Lemma 4), we apply Bessel inequality (25), Lemma 5 and Lemma 6. As a result we get

$$\sum_{\mu_k\ge 8\pi} |f_k||u_k(x)| \le const \left\{ C_1(f)(8\pi)^{\alpha-2} + (1+\|p_1\|_1)\cdot \right.$$

$$\cdot \left[\sum_{n=[8\pi]}^{\infty} n^{-1}\omega_1(f',n^{-1}) + \omega_1(f',(8\pi)^{-1}) \right] + \|f'\|_1 (1+\|p_1\|_1)[8\pi]^{-1} +$$

$$\left. +(\|f'\|_1 + \|f\|_\infty) \sum_{r=2}^{3} \|p_r\|_1 [8\pi]^{1-r} + \|p_1 f\|_2 [8\pi]^{-1/2} \right\} < \infty$$

.

Thus, the series (29) convergence uniformly on the segment $\bar{G}=[0,1]$. Therefore, the expansion $\sum_{k=1}^{\infty} f_k u_k(x)$ converges absolutely and uniformly on this interval. By the completeness of the system $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$ and the absolute continuity of the function $f(x)$, we have the identity

$$f(x) = \sum_{k=1}^{\infty} f_k u_k(x) \qquad x\in\bar{G} \tag{30}$$

The prove the estimate (4) we use lemma 2, 4, 5 and 6.

$$\|R_\nu(\cdot,f)\|_{C[0,1]} = \|f - \sigma_\nu(\cdot,f)\|_{C[0,1]} =$$

$$= \left\| \sum_{k=1}^{\infty} f_k u_k(\cdot) - \sum_{\mu_k\le\nu} f_k u_k(\cdot) \right\|_{C[0,1]} =$$

$$= \left\| \sum_{\mu_k>\nu} f_k u_k(\cdot) \right\|_{C[0,1]} \le$$

$$\le \sum_{\mu_k\ge\nu} |f_k| \|u_k\|_\infty \le const \sum_{\mu_k\ge\nu} \left\{ C_1(f)\mu_k^{\alpha-3} + (1+\|p_1\|_1)\cdot \right.$$

$$\left. \cdot\, \mu_k^{-1}\omega_1(f',\mu_k^{-1}) + \|f'\|_1 (1+\|p_1\|_1)\mu_k^{-2} + (\|f'\|_1 + \|f\|_\infty) \sum_{r=2}^{3} \|p_r\|_1 \mu_k^{-r} \right\} \|u_k\|_\infty^2 +$$

$$+const \sum_{\mu_k\ge\nu} \mu_k^{-1} \|u_k\|_\infty \left| (\bar{p}_1 f, \mu_k^{-2} u_k^{(2)}) \right| \le$$

$$\le const \left\{ C_1(f)\nu^{\alpha-2} + (1+\|p_1\|_1)\cdot \right.$$

$$\cdot \left[\sum_{n=[\nu]}^{\infty} n^{-1}\omega_1(f',n^{-1}) + \omega_1(f',\nu^{-1}) \right] +$$

$$+ \|f'\|_1 (1+\|p_1\|_1)\nu^{-1} + (\|f'\|_1 + \|f\|_\infty)\cdot$$

$$\left. \cdot \sum_{r=2}^{3} \|p_r\|_1 \nu^{1-r} + \nu^{-\frac{1}{2}} \|p_1 f\|_2 \right\}.$$

The proof of Theorem 1 is complete.

Proof of the Theorem 2. We prove the uniform convergence of the series $\sum_{\mu_k\ge 8\pi} |f_k| \|u_k(x)\||$ on the segment $\bar{G}=[0,1]$.

To estimate this series, we use the estimate (9`) in Lemma 2:

$$\sum_{\mu_k\ge 8\pi}' |f_k||u_k(x)| \le$$

$$\le const \left\{ C_1(f) \sum_{\mu_k\ge 8\pi} \mu_k^{\alpha-3} \|u_k\|_\infty^2 + (1+\|p_1\|_1)\cdot \right.$$

$$\cdot \sum_{\mu_k\ge 8\pi} \mu_k^{-1}\omega_1(\bar{p}_1 f,\mu_k^{-1}) \|u_k\|_\infty^2 +$$

$$+(1+\|p_1\|_1) \sum_{\mu_k\ge 8\pi} \mu_k^{-1}\omega_1(f',\mu_k^{-1}) \|u_k\|_\infty^2 +$$

$$+\|p_1 f\|_1 (1+\|p_1\|_1) \sum_{\mu_k\ge 8\pi} \mu_k^{-2} \|u_k\|_\infty^2 + \|f'\|_1 (1+\|p_1\|_1) \sum_{\mu_k\ge 8\pi} \mu_k^{-2} \|u_k\|_\infty^2 +$$

$$\left. +(\|f'\|_1 + \|f\|_\infty + \|p_1 f\|_1) \sum_{r=2}^{3} \|p_r\|_1 \left(\sum_{\mu_k\ge 8\pi} \mu_k^{-r} \|u_k\|_\infty^2 \right) \right\}$$

.

Since $\bar{p}_1 f \in L_2(G) \subset L_1(G)$, we apply Lemmas 5 and 6. As a result we have

$$\sum_{\mu_k \geq 8\pi} |f_k|\,|u_k(x)| \leq const\Big\{ C_1(f)[8\pi]^{\alpha-2} + (1+\|p_1\|_1)\cdot$$

$$\cdot\Bigg[\sum_{n=[8\pi]}^{\infty} n^{-1}\omega_1(\bar{p}_1 f, n^{-1}) + \omega_1(\bar{p}_1 f, [8\pi]^{-1}) +$$

$$+\sum_{n=[8\pi]}^{\infty} n^{-1}\omega_1(f', n^{-1}) + \omega_1(f', [8\pi]^{-1})\Bigg]$$

$$+\|p_1 f\|_1 (1+\|p_1\|_1)[8\pi]^{-1} +$$

$$+\|f'\|_1 (1+\|p_1\|_1)[8\pi]^{-1} +$$

$$+(\|f'\|_1 + \|f\|_\infty + \|p_1 f\|_1)\sum_{r=2}^{3}\|p_r\|_1 [8\pi]^{1-r}\Big\} < \infty .$$

Thus, the expansion $\sum_{k=1}^{\infty} f_k u_k(x)$ converges absolutely and uniformly on $\bar{G}$. From the completeness of the system $\{u_k(x)\}_{k=1}^{\infty}$ $L_2(G)$ the given expansion uniformly converges exactly to the function. Consequently, the identity (30) is true.

Estimate now difference $R_\nu(x,f)$. for that we use equality (30), Lemmas 2, 5 and 6.

$$\|R_\nu(\cdot,f)\|_{C[0,1]} = \|f - \sigma_\nu(\cdot,f)\|_{C[0,1]}$$

$$= \Big\|\sum_{\mu_k > \nu} f_k u_k(\cdot)\Big\|_{C[0,1]} \leq$$

$$\leq \sum_{\mu_k \geq \nu} |f_k|\,\|u_k\|_\infty \leq$$

$$\leq const\Big\{C_1(f)\sum_{\mu_k \geq \nu}\mu_k^{\alpha-3}\|u_k\|_\infty^2 + (1+\|p_1\|_1)\cdot$$

$$+(1+\|p_1\|_1)\sum_{\mu_k \geq \nu}\mu_k^{-1}\omega_1(\bar{p}_1 f, \mu_k^{-1})\|u_k\|_\infty^2 + \|p_1 f\|_1 (1+\|p_1\|_1)\sum_{\mu_k \geq \nu}\mu_k^{-2}\|u_k\|_\infty^2 +$$

$$+\|f'\|_1 (1+\|p_1\|_1)\sum_{\mu_k \geq \nu}\mu_k^{-2}\|u_k\|_\infty +$$

$$+(\|f'\|_1 + \|f\|_\infty + \|p_1 f\|_1)\sum_{r=2}^{3}\|p_r\|_1\Big(\sum_{\mu_k \geq \nu}\mu_k^{-r}\|u_k\|_\infty^2\Big) \leq$$

$$\leq const\Big\{C_1(f)\nu^{\alpha-2} + (1+\|p_1\|_1)$$

$$\cdot\Bigg[\sum_{k=[\nu]}^{\infty} k^{-1}\omega_1(\bar{p}_1 f, k^{-1}) + \sum_{k=[\nu]}^{\infty} k^{-1}\omega_1(f', k^{-1}) + \omega_1(\bar{p}_1 f, \nu^{-1}) +$$

$$+\omega_1(f',\nu^{-1}) + \nu^{-1}(\|p_1 f\|_1 + \|f'\|_1)\Bigg] + (\|p_1 f\|_1 + \|f\|_\infty + \|f'\|_1)\sum_{r=2}^{3}\nu^{2-r}\|p_r\|_1\Big\}$$

.

The estimation (6) is proved. The proof of Theorem 2 is complete.

Corollary 2 follows from the definition of norm, on the space $H_1^\beta(G)$ and Theorem 1 with regard to the inequality $\|f\|_\infty \leq \|f'\|_1$, which holds for any function $f(x) \in W_1^1(G)$, satisfying the relations $f(0) = f(1) = 0$. Indeed, if $f(0) = f(1) = 0$ and $f'(x) \in H_1^\beta(G)$, then we have $C_1(f) = 0$, and the following chain of inequalities is satisfied $(\nu \geq 8\pi)$.

$$C_1(f)\nu^{\alpha-2} + \nu^{-\frac{1}{2}}\|p_1 f\|_2 + (1+\|p_1\|_1)\times$$

$$\times\Bigg[\sum_{k=[\nu]}^{\infty} k^{-1}\omega_1(f, k^{-1}) + \omega_1(f', \nu^{-1})\Bigg] +$$

$$+(1+\|p_1\|_1)\|f'\|_1 \nu^{-1} + \nu^{-1}(\|f\|_\infty + \|f'\|_1)\times$$

$$\times\sum_{r=2}^{3}\nu^{2-r}\|p_r\|_1 \leq \nu^{-\frac{1}{2}}\|p_1 f\|_2 +$$

$$+(1+\|p_1\|_1)\Big\{\sup_{\delta>0}(\delta^{-\beta}\omega_1(f',\delta))\Big[\sum_{k=[\nu]}^{\infty} k^{-(1+\beta)} + \nu^{-\beta}\Big] + \nu^{-1}\|f'\|_1\Big\} +$$

$$+2\nu^{-1}\|f'\|_1\sum_{r=2}^{3}\nu^{2-r}\|p_r\|_1 \leq \nu^{-\frac{1}{2}}\|p_1 f\|_2 +$$

$$+const\Big\{\|f'\|_1 + \sup_{\delta>0}(\delta^{-\beta}\omega_1(f',\delta))\Big\}[\nu]^{-\beta} \leq$$

$$+\nu^{-\frac{1}{2}}\|p_1 f\|_2 + const\;\nu^{-\beta}\|f'\|_1^\beta .$$

Acknowledgment

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan **– Grant № EİF-ETL-2020-2(36)-16/08/1-M-08**

References

[1] Il'in V.A. On the unconditional basicity of systems of eigen and adjoint functions of second order differential operator on a closed interval Doklady AN SSSR 1983 Vol. 273 No.5 pp.1048- 1053.
[2] Kerimov N.B. On the unconditional basicity of systems of eigen and adjoint functions of the fourth order differential operator Doklady AN SSSR 1986, Vol. 286 No.4 pp.803-808.
[3] Budaev V.D. The Bessel property and Riesz basicity of the system of root functions of differential operators l, ll Differential Equations 1991, Vol.27 No.12 2033-2044 Vol.28 No.1 pp.23- 33.
[4] Lomov I.S. The Bessel inequality, Riesz theorem and unconditional basicity for root vectors of ordinary differential operators Vestnik Moskow 1992, No.5 pp. 42-52.
[5] Kurbanov V.M. On Hausedorff-Young inequality of root vector-functions of n-th order differential operator Differential Equations 1997, Vol.33 No.3 pp.356- 367.
[6] Kurbanov V.M. On distribution of eigen values and Bessel criterion of root functions of differential operator l,ll Differential Equations, 2005, Vol.41 No.4 464-478 Vol.41 No.5 pp. 623-631
[7] Kurbanov V.M., Garaeva A.T. Absolute and uniform convergence of expansions in the root function of the Schrodinger operator with a matrix potential. Doklady Mathematics. 2013, vol. 87 № 3, p. 304-306.
[8] Kurbanov V.M. A theorem on equivalent bases for a differential operator Doklady RAN, 2006, Vol.406 No.7 pp.17-20
[9] Abbasova Yu. G., Kurbanov V.M. Convergence of the spectral decomposition of a function from the class $W^1_{p,m}(G),\ p>1$, in the vector eigenfunctions of a differential operator of the third order. Ukrainian, Mathematical Jornal, 2017, vol. 69, № 6, p. 839-856.
[10] Kritskov L.V. Bessel property of the system of root functions of a second-order singular operator on an interval Differential Equations 2018, Vol. 54 No.8 pp. 1032–1048.
[11] [10] Kerimov N.R. Some properties of eigen and associate functions of ordinary differential operators Doklady AN SSSR, 1986, Vol.291 No.5 pp.1054-1055.
[12] Kurbanov V.M. Equiconvergence of biorthogonal expansions in root functions of differential operators: I, Differ. Equ., 35 (1999), № 12, p. 1619-1633.
[13] Kurbanov V.M. II, Differ. Equ., 36 (2000), № 3, p. 358-376.
[14] Kurbanov V.M. On an analog of the Riesz theorem and the basis property of the system of root functions of a differential operator in L_p: I, Differ. Equ., 49 (1), 2013, 7-19.